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Asymptotics of Solutions of the Poisson Equation near Straight Edges

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Abstract. The paper deals with the Dirichlet problem for the Poisson equation $\Delta u = f$ in the domain $\mathcal{D} = K \times \mathbb{R}^{n-m}$, where K is a cone in $\mathbb{R}^m$. The author describes the singularities of the Green function near the edge of the domain. Using this result, he obtains the asymptotics of the solution of the boundary value problem for a right-hand side f belonging to a weighted L_p Sobolev space. Here, precise formulas for all coefficients in the asymptotics are given.

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1. Introduction

The paper is concerned with the first boundary value problem for the Poisson equation

$$\Delta u = f \quad \text{in } \mathcal{D} \tag{1}$$
$$u = 0 \quad \text{on } \partial\mathcal{D} \tag{2}$$

in the domain

$$\mathcal{D} = \{x = (x', x'') : x' \in K,\ x'' \in \mathbb{R}^{n-m}\},$$

where $K = \{x' \in \mathbb{R}^m : \frac{x'}{|x'|} \in \Omega\}$ is a cone in $\mathbb{R}^m$, $2 \le m \le n$, and Ω denotes a subdomain of the unit sphere with smooth (of class C^∞) boundary $\partial\Omega$.

The paper consists of two parts. In the first part (Section 2), we study the asymptotics of the Green function $G(x, y)$ for the problem (1), (2) near the edge $M = \{x = (x', x'') \in \mathbb{R}^n : x' = 0,\ x'' \in \mathbb{R}^{n-m}\}$ of $\mathcal{D}$. In the case $m = n$, the asymptotics of Green's function is even known for general elliptic boundary value problems. We refer here to the paper [9] by Maz'ya and Plamenevskiĭ

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and the monograph [14] of Nazarov and Plamenevskiĭ. The description of the singularities of Green's function near edges is more complicated. However, for many applications it is sufficient to employ point estimates of Green's function. Maz'ya and Plamenevskiĭ [7] obtained such estimates for a class of general elliptic problems in a dihedral angle $\mathcal{D}$ (see also the monograph of Maz'ya and Rossmann [12]). This class includes the Dirichlet problem for strongly elliptic operators. We give here the estimate for the Green function of the problem (1), (2). For this end, we introduce the following notation. Let $\{\Lambda_j\}_{j=1}^{\infty}$ be a nondecreasing sequence of eigenvalues of the Beltrami operator $-\delta$ on Ω (with Dirichlet boundary condition) counted with their multiplicities, and let $\{\phi_j\}_{j=1}^{\infty}$ be an orthonormal (in $L_2(\Omega)$) sequence of eigenfunctions corresponding to the eigenvalues Λ_j. Furthermore, we define

$$\lambda_j^{\pm} = \frac{2-m}{2} \pm \sqrt{(1-\frac{m}{2})^2 + \Lambda_j}.$$

This means that $\lambda_j^{\pm}$ are the solutions of the quadratic equation $\lambda(m{-}2{+}\lambda) = \Lambda_j$. Obviously, $\lambda_j^+ > 0$ and $\lambda_j^- < 2-m$ for $j = 1, 2, \ldots$. For an arbitrary point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $x' = (x_1, \ldots, x_m)$ and $x'' = (x_{m+1}, \ldots, x_n)$. Analogously, we set $\alpha' = (\alpha_1, \ldots, \alpha_m)$ and $\alpha'' = (\alpha_{m+1}, \ldots, \alpha_n)$ for an arbitrary multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then the following estimate holds for all multi-indices α and γ (cf. [12, Theorems 2.5.2 and 2.5.4]):

$$\begin{aligned} \left|\partial_x^\alpha \partial_y^\gamma G(x,y)\right| \le c_{\alpha,\gamma}\, |x-y|^{2-n-|\alpha|-|\gamma|} & \Big(\frac{|x'|}{|x'|+|x-y|}\Big)^{\lambda_1^+-|\alpha'|-\varepsilon} \\ \times \ & \Big(\frac{|y'|}{|y'|+|x-y|}\Big)^{\lambda_1^+-|\gamma'|-\varepsilon}. \end{aligned} \tag{3}$$

Here, ε is an arbitrarily small positive number. One of the goals of the present paper is to give a precise description of the singularities of $G(x,y)$ near the edge M. Let

$$c_j(y', x''-y'') = -\frac{\Gamma\big(\lambda_j^+ - 1 + \frac{n}{2}\big)}{2\pi^{\frac{n-m}{2}}\,\Gamma(\lambda_j^+ + \frac{m}{2})}\ \frac{|y'|^{\lambda_j^+}\,\phi_j(\omega_y)}{(|y'|^2 + |x''-y''|^2)^{\lambda_j^+-1+\frac{n}{2}}}$$

and let σ be an arbitrary real number such that

$$\sigma > \lambda_1^-, \quad \sigma \ne \lambda_j^+ \quad \text{for all } j. \tag{4}$$

Furthermore, we set

$$m_j = \Big[\frac{\sigma - \lambda_j^+}{2}\Big] \quad \text{for } j = 1, 2, \ldots,$$

where $[s]$ denotes the integral part of s. It is proved in Section 2 (see Theorem 2.2) that $G(x,y)$ admits the decomposition

$$G(x,y) = \sum_{\lambda_j^+<\sigma} \sum_{k=0}^{m_j} \frac{\Gamma(\lambda_j^+ + \frac{m}{2})\,(-\Delta_{x''})^k c_j(y', x''-y'')}{4^k\, k!\; \Gamma(\lambda_j^+ + k + \frac{m}{2})}\, |x'|^{\lambda_j^+ + 2k}\, \phi_j(\omega_x) + R_\sigma$$

with a remainder $R_\sigma(x,y)$ satisfying the estimate

$$\big|\partial_x^\alpha \partial_y^\gamma R_\sigma(x,y)\big| \le c_{\alpha\gamma} \frac{|x'|^{\sigma-|\alpha'|}\, |y'|^{\lambda_1^+ -|\gamma'|-\varepsilon}}{|x-y|^{\sigma+\lambda_1^+ + n-2+|\alpha''|+|\gamma''|-\varepsilon}} \quad \text{for } |x'| \le \frac{|x-y|}{2}. \tag{5}$$

Obviously, this result improves the estimate (3). Note that the method for constructing the asymptotics of Green's function in the present paper is similar to the approach for the heat equation in the paper [5] by Kozlov and Rossmann. It is also possible to obtain the asymptotics of the Green function $G(x,y)$ from the asymptotics of the Green function for the heat equation (by integration with respect to the time t). This approach is shortly discussed at the end of Section 2. However, the construction of the asymptotics for the Green function of the heat equation in a cone requires considerations similar to those of Section 2. Therefore, the proof given in Subsection 2.3 is a more direct method for finding the asymptotics of the Green function of the Poisson equation.

In the second part of the paper (Section 3), we apply the result of Section 2 in order to describe the asymptotics of the solution $u \in \overset{\circ}{W}{}_2^1(\mathcal{D})$ of the problem (1), (2) with the right-hand side $f \in W_2^{-1}(\mathcal{D}) \cap V_{p,\beta}^{l-2}(\mathcal{D})$. Here $V_{p,\beta}^l(\mathcal{D})$ is the weighted Sobolev space with the norm

$$\|u\|_{V_{p,\beta}^l(\mathcal{D}} = \Big(\int_{\mathcal{D}} \sum_{|\alpha|\le l} |x'|^{p(\beta-l+|\alpha|)} \big|\partial_x^\alpha u(x)\big|^p \, dx \Big)^{\frac{1}{p}}.$$

Asymptotic formulas for solutions near edges were given in a number of papers (see e.g. Kondrat'ev [3], Maz'ya and Rossmann [11], Dauge [2], Costabel and Dauge [1], Nazarov and Plamenevskiĭ [14]) even for general elliptic boundary value problems. In contrast to earlier results, Theorems 3.5 and 3.9 of the present paper contain precise formulas for all coefficients in the asymptotics. On the other hand, we are concerned here with a rather simple domain $\mathcal{D}$. The application to the boundary value problem in a bounded domain of $\mathbb{R}^n$ with smooth edges requires to construct some special solutions of the Laplace equation in this domain (see e.g. [10, Theorem 4.4], [14, Chapter 10, Theorem 3.2]). This is not subject of the present paper. The main result of Section 3 is given in Theorem 3.9. It is assumed that l, p, β satisfy the inequalities

$$\sigma \overset{def}{=} l - \beta - \frac{m}{p} > \lambda_1^-, \quad \sigma \ne \lambda_j^+ \quad \text{for all } j. \tag{6}$$

Under this condition, the solution u admits the decomposition

$$u(x) = \sum_{\lambda_j^+<\sigma} \sum_{k=0}^{m_j} \frac{\Gamma(\lambda_j^+ + \frac{m}{2})\ (-\Delta_{x''})^k (\mathcal{E}h_j)(x)}{4^k\, k!\ \Gamma(\lambda_j^+ + k + \frac{m}{2})}\, r^{\lambda_j^+ + 2k} \phi_j(\omega_x) + v(x), \tag{7}$$

with a remainder $v \in V^l_{p,\beta}(\mathcal{D})$. Here, $\mathcal{E}$ is the extension operator (42) introduced in Section 3.2, the functions h_j in (7) are given by the formula

$$h_j(x'') = \int_{\mathcal{D}} c_j(y', x'' - y'')\, f(y)\, dy \tag{8}$$

and belong to the Besov space $B_p^{\sigma-\lambda_j^+}(\mathbb{R}^{n-2})$. For the special case $p = 2$, $\lambda_1^+ < \sigma < \min(\lambda_2^+, \lambda_1^+ + 1)$, the formula (7) was essentially proved in [14, Chapter 10, Theorem 3.2].

2. The Green function of the Dirichlet problem for the Laplace equation

Let $G(x, y)$ be the Green function of the problem (1), (2). This means that

$$\Delta_x G(x, y) = \delta(x - y) \quad \text{for } x \in \mathcal{D},\ y \in \mathcal{D} \tag{9}$$

$$G(x, y) = 0 \quad \text{for } x \in \partial\mathcal{D}\backslash M,\ y \in \mathcal{D}. \tag{10}$$

If ζ is an arbitrary function with bounded derivatives of order $\le l$, $l \ge 1$, in $\mathcal{D}$, $\zeta = 1$ in a neighborhood of the point y, then the function $x \to \big(1-\zeta(x)\big)\, G(x, y)$ belongs to the space $\overset{\circ}{W}{}_2^1(\mathcal{D})$. Furthermore, we conclude from [12, Theorem 2.6.7] that $(1 - \zeta)\, G(\cdot, y) \in V^l_{p,\beta}(\mathcal{D})$ for $\lambda_1^- < l - \beta - \frac{m}{p} < \lambda_1^+$. The goal of this section is to describe the behavior of $G(x, y)$ for small $|x'|$ (for $|x'| < \frac{|x-y|}{2}$). We start with some elementary properties of this function.

2.1. Some properties of the Green function. First note that $G(x, y)$ is positively homogeneous of degree $2 - n$, i.e.

$$G(ax, ay) = a^{2-n}\, G(x, y) \quad \text{for all } x, y \in \mathcal{D},\ a > 0.$$

Obviously, it follows from (9), (10) that

$$\Delta_x G(x', x'' - y'', y', 0) = \delta(x' - y')\, \delta(x'' - y'') \quad \text{for } x, y \in \mathcal{D}$$

$$G(x', x'' - y'', y', 0) = 0 \quad \text{for } x \in \partial\mathcal{D}\backslash M,\ y \in \mathcal{D}.$$

This means that $G(x, y)$ depends only on x', y' and $x'' - y''$. And what is more, the Green function depends only on x', y' and $|x'' - y''|$, since the Laplace operator is invariant with respect to rotation. This means that the Green function has the representation

$$G(x',x'',y',y'') = \mathcal{G}(x',y',|x''-y''|).$$

We give some estimates which can be easily deduced from (3). If $\delta < 1$ and $|x'| \le \delta|x-y|$, then

$$c_1\,|x-y|^2 \le |y'|^2 + |x''-y''|^2 \le c_2\,|x-y|^2 \tag{11}$$

with certain positive constants c_1, c_2 depending on δ. Consequently, it follows from (3) that

$$\big|\partial_x^\alpha \partial_y^\gamma G(x,y)\big| \le c_{\alpha,\gamma} \frac{|x'|^{\lambda_1^+ - |\alpha'| - \varepsilon}\, |y'|^{\lambda_1^+ - |\gamma'| - \varepsilon}}{(|y'|^2 + |x''-y''|^2)^{\lambda_1^+ + \frac{n-2+|\alpha''|+|\gamma''|-2\varepsilon}{2}}} \tag{12}$$

for $|x'| \le \delta|x-y|$. An analogous estimate holds for $|y'| < \delta|x-y|$, while

$$\big|\partial_x^\alpha \partial_y^\gamma G(x,y)\big| \le c_{\alpha,\gamma}\,|x-y|^{2-n-|\alpha|-|\beta|} \quad \text{for } \min(|x'|,|y'|) \ge \delta|x-y|.$$

In the following, we use the notation

$$r = |x'|, \quad \rho = |y'|, \quad \omega_x = \frac{x'}{|x'|}, \quad \omega_y = \frac{y'}{|y'|}.$$

Lemma 2.1. *Let $G_j(r,\rho,|x''-y''|)$ be the Green function of the problem*

$$\Big(\Delta_{x''} + \partial_r^2 + \frac{m-1}{r}\,\partial_r - \frac{\Lambda_j}{r^2}\Big) U(r,x'') = F(r,x'') \quad \textit{for } r>0,\ x'' \in \mathbb{R}^{n-m}$$
$$U(0,x'') = 0 \quad \textit{for } x'' \in \mathbb{R}^{n-m}.$$

Then

$$\int_\Omega G(x',x'',y',y'')\,\phi_j(\omega_x)\,d\omega_x = \rho^{1-m}\,G_j(r,\rho,|x''-y''|)\,\phi_j(\omega_y). \tag{13}$$

Proof. Let $f \in W^0_{2,\beta}(\mathcal{D})$, $\lambda_1^- < 2-\beta-\frac{m}{2} < \lambda_1^+$, and let

$$u(x) = \int_{\mathcal{D}} G(x,y)\,f(y)\,dy$$

be the uniquely determined solution of the problem (1), (2). We define

$$U_j(r,x'') = \int_\Omega u(x)\,\phi_j(\omega_x)\,d\omega_x.$$

Then

$$\begin{aligned}
&\Big(\Delta_{x''} + \partial_r^2 + (m-1)r^{-1}\,\partial_r - \Lambda_j\, r^{-2}\Big) U_j(r,x'') \\
&= \Delta_x U_j(r,x'') - r^{-2}(\delta_\omega + \Lambda_j)\, U_j(r,x'') \\
&= \int_\Omega \Big(\Delta u - r^{-2}\big(\delta_\omega + \Lambda_j\big)\, u\Big)\, \phi_j(\omega_x)\, d\omega_x \\
&= \int_\Omega f(x)\, \phi_j(\omega_x)\, d\omega_x \\
&=: F_j(r,x'').
\end{aligned}$$

Hence, the function U_j admits the representation

$$\begin{aligned}
U_j(r,x'') &= \int_0^\infty \int_{\mathbb{R}^{n-m}} G_j(r,\rho,|x''-y''|)\, F_j(\rho,y'')\, dy''\, d\rho \\
&= \int_0^\infty \int_{\mathbb{R}^{n-m}} \int_\Omega G_j(r,\rho,|x''-y''|)\, \phi_j(\omega_y)\, f(y)\, d\omega_y\, dy''\, d\rho.
\end{aligned}$$

On the other hand, it follows from the definition of U_j that

$$\begin{aligned}
U_j(r,x'') &= \int_\Omega \int_{\mathcal{D}} G(x,y)\, \phi_j(\omega_x)\, f(y)\, dy\, d\omega_x \\
&= \int_0^\infty \int_{\mathbb{R}^{n-m}} \int_\Omega \int_\Omega G(x,y)\, \phi_j(\omega_x)\, d\omega_x\, \rho^{m-1}\, f(y)\, d\omega_y\, dy''\, d\rho.
\end{aligned}$$

Comparing the last two equalities, we obtain (13). □

2.2. Asymptotics of the Green function. In the sequel, let σ be an arbitrary real number satisfying the condition (4). Furthermore, let

$$\sigma_j = \lambda_j^+ - 1 + \frac{m}{2} = \sqrt{\Big(1-\frac{m}{2}\Big)^2 + \Lambda_j} \quad \text{for } j = 1,2,\dots$$

and

$$c_j(y) = c_j(y',y'') = -\frac{\Gamma\big(\sigma_j + \frac{n-m}{2}\big)}{2\pi^{\frac{n-m}{2}}\,\Gamma(\sigma_j+1)}\, |y|^{2-n-2\lambda_j^+}\, |y'|^{\lambda_j^+}\, \phi_j(\omega_y) \tag{14}$$

A simple calculation shows that

$$\Delta_y c_j(y) = 0 \quad \text{for } y \in \mathcal{D}. \tag{15}$$

We define $G_\sigma(x,y) = 0$ for $\sigma < \lambda_1^+$, while

$$G_\sigma(x,y) = \sum_{\lambda_j^+ < \sigma} \sum_{k=0}^{m_j} \frac{(-\Delta_{x''})^k c_j(y', x''-y'')}{4^k\, k!\, (\sigma_j + k)_{(k)}}\, r^{\lambda_j^+ + 2k}\, \phi_j(\omega_x) \quad \text{for } \sigma > \lambda_1^+. \tag{16}$$

Here, we used the notation $s_{(k)} = \frac{\Gamma(s+1)}{\Gamma(s-k+1)}$ for $s > k-1$, i.e.

$$s_{(k)} = s(s-1)\cdots(s-k+1) \quad \text{for } k = 1, 2, \ldots, \quad s_{(0)} = 1.$$

The remaining part of this section is concerned with the proof of the following theorem which was presented in the introduction.

Theorem 2.2. *Let σ be an arbitrary real number satisfying* (4). *Then*

$$G(x,y) = G_\sigma(x,y) + R_\sigma(x,y), \tag{17}$$

where $R_\sigma(x,y)$ satisfies the estimate (5) *for all α and γ.*

2.3. Proof of Theorem 2.2. Obviously, $G_\sigma(x,y) = G_{\sigma+\varepsilon}(x,y)$ for sufficiently small positive ε. Consequently, we may assume without loss of generality that $\frac{\sigma-\lambda_j^+}{2}$ is not integer for $\lambda_j^+ < \sigma$.

Let l be an arbitrary integer, $l \ge 2$, p an arbitrary real number, $p \in (1,\infty)$, and $\beta = l - \sigma - \frac{m}{p}$. We prove by induction in $m_1 = [\frac{\sigma-\lambda_1^+}{2}]$ that $G(x,y)$ admits the decomposition (17), where $R_\sigma(x,y)$ satisfies (5) and

$$\partial_{x''}^\alpha \partial_y^\gamma R_\sigma(\cdot, x'', y) \in V_{p,\beta}^l(K) \quad \text{for every } x'' \in \mathbb{R}^{n-m},\ y \in \mathcal{D},\ x'' \neq y''.$$

First let $m_1 < 0$, i.e. $\lambda_1^- < \sigma < \lambda_1^+$ and $G_\sigma = 0$. It follows from [8, Corollary 4.1 and Theorem 4.2] (see also [6, Theorem 3.2]) that $\partial_{x''}^\alpha \partial_y^\gamma G(\cdot, x'', y) \in V_{p,\beta}^l(K)$ for all $x'' \in \mathbb{R}^{n-m}$, $y \in \mathcal{D}$, $x'' \neq y''$. Moreover, $G(x,y)$ satisfies (12). Thus, the assertion of the theorem is true for $m_1 < 0$.

Suppose now that $m_1 = N$, i.e. $\lambda_1^+ + 2N < \sigma < \lambda_1^+ + 2(N+1)$, where N is a nonnegative integer, and that the theorem is proved for $\sigma < \lambda_1^+ + 2N$. We set $\sigma' = \sigma - 2$ if $N > 0$ and $\sigma' = \lambda_1^+ - \varepsilon$ if $N = 0$, where ε is a sufficiently small positive number. Then

$$\Big[\frac{\sigma' - \lambda_j^+}{2}\Big] = \Big[\frac{\sigma - \lambda_j^+}{2}\Big] - 1 = m_j - 1 \quad \text{for } \lambda_j^+ < \sigma'.$$

By the induction hypothesis, we have

$$G(x,y) = G_{\sigma'}(x,y) + R_{\sigma'}(x,y), \tag{18}$$

where $G_{\sigma'} = 0$ for $N = 0$,

$$G_{\sigma'}(x,y) = \sum_{\lambda_j^+ < \sigma'} \sum_{k=0}^{m_j - 1} \frac{(-\Delta_{x''})^k c_j(y', x''-y'')}{4^k\, k!\, (\sigma_j + k)_{(k)}}\, r^{\lambda_j^+ + 2k}\, \phi_j(\omega_x) \quad \text{for } N \ge 1$$

and $\partial_{x''}^\alpha \partial_y^\gamma R_{\sigma'}(\cdot, x'', y) \in V_{p,\beta'}^l(K)$, $\beta' = l - \sigma' - \frac{m}{p}$, for all $y' \in K$, $x'', y'' \in \mathbb{R}^{n-m}$, $x'' \neq y''$. Furthermore, $R_{\sigma'}$ satisfies the estimate (5) with σ' instead of σ. Since $\Delta_y c_j(y', x'' - y'') = 0$, it follows that $\Delta_y R_{\sigma'}(x, y) = 0$ for $y'' \neq x''$. Furthermore,

$$R_{\sigma'}(ax, ay) = a^{2-n}\, R_{\sigma'}(x, y) \quad \text{for all } x, y \in \mathcal{D},\ a > 0$$

and $R_{\sigma'}(x, y)$ depends only on $x', y', |x'' - y''|$, since the same is true for $G(x, y)$ and $G_{\sigma'}(x, y)$. The equality $\Delta_{x'} G(x, y) = -\Delta_{x''} G(x, y)$ for $x'' \neq y''$ implies

$$\Delta_{x'} R_{\sigma'}(x, y) = -\Delta_{x'} G_{\sigma'}(x, y) - \Delta_{x''} G_{\sigma'}(x, y) - \Delta_{x''} R_{\sigma'}(x, y) \tag{19}$$

for $x'' \neq y''$. Using the formula

$$\Delta_{x'} r^{\lambda_j^+ + 2k} \phi_j(\omega_x) = 4k(\sigma_j + k)\, r^{\lambda_j^+ + 2k - 2} \phi_j(\omega_x), \tag{20}$$

we get

$$\begin{aligned} -\Delta_{x''} G_{\sigma'}(x, y) &= \sum_{\lambda_j^+ < \sigma'} \sum_{k=1}^{m_j} \frac{(-\Delta_{x''})^k c_j(y', x'' - y'')}{4^{k-1}\, (k-1)!\, (\sigma_j + k - 1)_{(k-1)}}\, r^{\lambda_j^+ + 2k - 2}\, \phi_j(\omega_x) \\ &= \Delta_{x'} \sum_{\lambda_j^+ < \sigma'} \sum_{k=0}^{m_j} \frac{(-\Delta_{x''})^k c_j(y', x'' - y'')}{4^k\, k!\, (\sigma_j + k)_{(k)}}\, r^{\lambda_j^+ + 2k}\, \phi_j(\omega_x). \end{aligned}$$

This together with (19) yields

$$\Delta_{x'} R_{\sigma'}(x, y) = \Delta_{x'} \Sigma'(x, y) - \Delta_{x''} R_{\sigma'}(x, y) \quad \text{for } x'' \neq y'', \tag{21}$$

where

$$\Sigma'(x, y) = \sum_{\lambda_j^+ < \sigma'} \frac{(-\Delta_{x''})^{m_j} c_j(y', x'' - y'')}{4^{m_j}\, m_j!\, (\sigma_j + m_j)_{(m_j)}}\, r^{\lambda_j^+ + 2m_j}\, \phi_j(\omega_x)$$

($\Sigma' = 0$ for $N = 0$, i.e. for $\sigma' < \lambda_1^+$). Let ψ be a smooth function on $\mathbb{R}_+ = (0, \infty)$, $\psi(r) = 1$ for $r < \frac{1}{2}$, $\psi(r) = 0$ for $r > \frac{3}{4}$. Furthermore, let the function χ be defined as

$$\chi(x', x'', y'') = \psi\Big(\frac{2r}{|x'' - y''|}\Big).$$

Then by (21),

$$\Delta_{x'} \big(R_{\sigma'} - \chi \Sigma'\big) = \Delta_{x'} \big(1 - \chi\big) \Sigma' - \Delta_{x''} R_{\sigma'} \tag{22}$$

for $x'' \neq y''$. Here, by the induction hypotheses and by the definition of Σ',

$$\begin{gathered} \partial_{x''}^\alpha \partial_y^\gamma (R_{\sigma'} - \chi \Sigma')(\cdot, x'', y) \in V_{p,\beta'}^l(K), \\ \partial_{x''}^\alpha \partial_y^\gamma \big(\Delta_{x'}\, (1 - \chi) \Sigma'\big)(\cdot, x'', y) \in V_{p,\beta}^{l-2}(K) \end{gathered}$$

for arbitrary $x'' \in \mathbb{R}^{n-2}$, $y \in \mathcal{D}$, $x'' \neq y''$, and for all multi-indices α, γ. Furthermore,

$$\partial_{x''}^{\alpha} \partial_y^{\gamma} \Delta_{x''} R_{\sigma'}(\cdot, x'', y) \in V_{p,\beta}^{l-2}(K).$$

Indeed, if $N \geq 1$, then $\partial_{x''}^{\alpha} \partial_y^{\gamma} \Delta_{x''} R_{\sigma'}(\cdot, x'', y) \in V_{p,\beta'}^{l}(K) \subset V_{p,\beta}^{l-2}(K)$. If $N = 0$, then $\Sigma' = 0$ and $\partial_{x''}^{\alpha} \partial_y^{\gamma} \Delta_{x''} R_{\sigma'}(\cdot, x'', y) = -\partial_{x''}^{\alpha} \partial_y^{\gamma} \Delta_{x'} R_{\sigma'}(\cdot, x'', y) \in V_{p,\beta'}^{l}(K) \cap V_{p,\beta'}^{l-2}(K) \subset V_{p,\beta}^{l-2}(K)$. Thus, the x''- and y-derivatives of the right-hand sight of (22) belong to $V_{p,\beta}^{l-2}(K)$ for arbitrary $x'' \in \mathbb{R}^{n-2}$, $y \in \mathcal{D}$, $x'' \neq y''$. Applying [8, Theorem 4.2], we obtain

$$\partial_{x''}^{\alpha''} \partial_y^{\gamma} \big(R_{\sigma'}(x, y) - \chi\, \Sigma'(x, y)\big) = \sum_{\sigma' < \lambda_j^+ < \sigma} d_{j,\alpha'',\gamma}(x'', y)\, r^{\lambda_j^+}\, \phi_j(\omega_x) + v_{\alpha'',\gamma}(x, y), \quad (23)$$

where $v_{\alpha'',\gamma}(\cdot, x'', y) \in V_{p,\beta}^{l}(K)$. The coefficients $d_{j,\alpha'',\gamma}$ are given by the formula (cf. [6, Theorem 3.4])

$$d_{j,\alpha'',\gamma}(x'', y) = \int_K V_j(x')\, \partial_{x''}^{\alpha''} \partial_y^{\gamma} \Delta_{x'} \Big(R_{\sigma'}(x, y) - \chi \Sigma'(x, y)\Big)\, dx', \qquad (24)$$

where $V_j(x') = -\frac{1}{2\sigma_j}\, r^{\lambda_j^-}\, \phi_j(\omega_x)$. The integral in (24) is well-defined, since

$$\partial_{x''}^{\alpha''} \partial_y^{\gamma} \Delta_{x'} \big(R_{\sigma'} - \chi \Sigma'\big)(\cdot, x'', y) \in V_{p,\beta}^{l-2}(K) \cap V_{p,\beta'}^{l-2}(K)$$

and $V_j \in V_{p',-\beta}^{l-2}(K) + V_{p',-\beta'}^{l-2}(K)$, $p' = \frac{p}{p-1}$, for $\sigma' < \lambda_j^+ < \sigma$. Furthermore, the right-hand side of (24) is independent of the choice of the cut-off function χ, since $\Delta_{x'} V_j = 0$ in K and $V_j = \Sigma' = 0$ on $\partial K \backslash \{0\}$. The remainder $v_{\alpha'',\gamma}$ and the coefficients $d_{j,\alpha'',\gamma}$ in (23) satisfy the estimate

$$\begin{aligned} &\|v_{\alpha'',\gamma}(\cdot, x'', y)\|_{V_{p,\beta}^{l}(K)} + \sum_{\sigma' < \lambda_j^+ < \sigma} |d_{j,\alpha'',\gamma}(x'', y)| \\ &\leq c\, \|\partial_{x''}^{\alpha''} \partial_y^{\gamma} \Delta_{x'} (R_{\sigma'} - \chi \Sigma')(\cdot, x'', y)\|_{V_{p,\beta}^{l-2}(K) \cap V_{p,\beta'}^{l-2}(K)}. \end{aligned}$$

We set $d_j = d_{j,0,0}$. Then $d_{j,\alpha'',\gamma}(x'', y) = \partial_{x''}^{\alpha''} \partial_y^{\gamma} d_j(x'', y)$. Thus,

$$R_{\sigma'}(x, y) - \chi\, \Sigma'(x, y) = \sum_{\sigma' < \lambda_j^+ < \sigma} d_{\mu}(x'', y)\, r^{\lambda_j^+}\, \phi_j(\omega_x) + v(x, y),$$

where $\partial_{x''}^{\alpha''} \partial_y^{\gamma} v(\cdot, x'', y) = v_{\alpha'',\gamma}(\cdot, x'', y) \in V_{p,\beta}^{l}(K)$. This and (18) imply

$$G(x, y) = G_{\sigma}(x, y) + R_{\sigma}(x, y), \qquad (25)$$

where

$$G_{\sigma}(x, y) = G_{\sigma'}(x, y) + \Sigma'(x, y) + \sum_{\sigma' < \lambda_j^+ < \sigma} d_j(x'', y)\, r^{\lambda_j^+}\, \phi_j(\omega_x)$$

and

$$R_\sigma(x,y) = v(x,y) + \big(\chi - 1\big)\,\Sigma'(x,y).$$

We show that $d_j(x'',y)$ coincides with the function $c_j(y',x''-y'')$ defined by (14) if $\sigma' < \lambda_j^+ < \sigma$. For this, we need the following lemma.

Lemma 2.3. *Let $d_j = d_{j,0,0}$, where $d_{j,\alpha'',\gamma}$ is defined by (24). Then the following assertions hold.*

1) *The function d_j has the form*

$$d_j(x'',y) = \rho^{2-n-\lambda_j^+}\, h_j\Big(\frac{|x''-y''|}{\rho}\Big)\,\phi_j(\omega_y). \tag{26}$$

2) $\Delta_y d_j(x'',y) = 0$ *for $x'' \neq y''$.*

3) $\int_{\mathbb{R}^{n-m}} d_j(x'',y)\,dx'' = V_j(y') = -\frac{1}{2\sigma_j}\,\rho^{\lambda_j^-}\,\phi_j(\omega_y)$.

4) $|d_j(x'',y)| \le c\,|y'|^{\lambda_1^+-\varepsilon}\,|x''-y''|^{-(\lambda_1^++\lambda_j^++n-2-\varepsilon)}$ *for $|y'| \le |x''-y''|$,*

where c is a constant independent of x'',y and ε is an arbitrarily small positive real number.

Proof. 1) By (24), we have

$$d_j(x'',y) = \lim_{\varepsilon\to 0} \int\limits_{\substack{K\\ \varepsilon<r<\frac{1}{\varepsilon}}} V_j(x')\,\Delta_{x'}\big(G - G_{\sigma'} - \chi\Sigma'\big)\,dx'. \tag{27}$$

The homogeneity of the functions G, $G_{\sigma'}$, $\chi\Sigma'$ and V_j implies the equality

$$d_j(ax'',ay) = a^{2-n-\lambda_j^+}\,d_j(x'',y). \tag{28}$$

Using (20), the equality $\Delta_{x'}(\chi\Sigma') = (\Delta_{x'}\chi)\,\Sigma' + 2(\partial_r\chi)\,\partial_r\Sigma' + \chi\,\Delta_{x'}\Sigma'$, and the orthogonality of the eigenfunctions ϕ_j in $L_2(\Omega)$, we get

$$\begin{aligned}\int\limits_{\substack{K\\ \varepsilon<r<\frac{1}{\varepsilon}}} V_j(x')\Delta_{x'}\big(G - \sigma' - \chi\Sigma'\big)\,dx' &= \int\limits_{\substack{K\\ \varepsilon<r<\frac{1}{\varepsilon}}} V_j(x')\Delta_{x'}G(x,y)\,dx' \\ &= \int\limits_{\substack{K\\ \varepsilon<r<\frac{1}{\varepsilon}}} V_j(x')\big(\delta(x-y) - \Delta_{x''}G(x,y)\big)\,dx'.\end{aligned} \tag{29}$$

Applying Lemma 2.1, we get

$$\begin{aligned}\int\limits_{\substack{K\\ \varepsilon<r<\frac{1}{\varepsilon}}} V_j(x')\Delta_{x''}G(x,y)\,dx' &= -\Delta_{x''}\int_\varepsilon^{\frac{1}{\varepsilon}} \frac{r^{1-\lambda_j^+}}{2\sigma_j}\int_\Omega G(x,y)\phi_j(\omega_x)\,d\omega_x\,dr \\ &= -\frac{1}{2\sigma_j}\rho^{1-m}\phi_j(\omega_y)\Delta_{x''}\int_\varepsilon^{\frac{1}{\varepsilon}} r^{1-\lambda_j^+}G_j(r,\rho,|x''-y''|)\,dr.\end{aligned}$$

Consequently, the function d_j has the form $d_j(x'',y) = f_j(\rho,|x''-y''|)\,\phi_j(\omega_y)$. This together with (28) leads to the representation (26).

2) The second assertion follows immediately from (27) and from the equalities (cf. (15))

$$\Delta_y G(x,y) = \Delta_y G_{\sigma'}(x,y) = \Delta_y \Sigma'(x,y) = 0 \quad \text{for } x'' \neq y''.$$

3) By (27) and (29), we have

$$\begin{aligned}\int_{\mathbb{R}^{n-m}} d_j(x'',y)\,dx'' &= \lim_{\varepsilon\to 0} \int_{\substack{K\\ \varepsilon<r<\frac{1}{\varepsilon}}} V_j(x') \int_{\mathbb{R}^{n-m}} \big(\delta(x-y) - \Delta_{x''}G(x,y)\big)\,dx''\,dx' \\ &= \int_K \int_{\mathbb{R}^{n-m}} V_j(x')\,\delta(x-y)\,dx''\,dx' = V_j(y').\end{aligned}$$

4) Suppose that $|y'| < |x''-y''|$. Then

$$\frac{r^2}{2} + |x''-y''|^2 < 2|x-y|^2 < 4r^2 + 6|x''-y''|^2.$$

We introduce the sets $K_1 = \{x' \in K : 2r < |x''-y''|\}$ and $K_2 = K\backslash K_1$. Then $\chi = 0$ on K_2, and (21) yields

$$\begin{aligned}d_j(x'',y) &= \int_{K_1} V_j(x')\,\Delta_{x'}\big(R_{\sigma'} - \chi\Sigma'\big)\,dx' + \int_{K_2} V_j(x')\,\Delta_{x'}R_{\sigma'}\,dx' \\ &= \int_{K_1} V_j(x')\,\big(\Delta_{x'}(1-\chi)\Sigma' - \Delta_{x''}R_{\sigma'}\big)\,dx' + \int_{K_2} V_j(x')\,\Delta_{x'}(G - G_{\sigma'})\,dx'.\end{aligned}$$

By the induction hypothesis, we have

$$\big|\Delta_{x''}R_{\sigma'}(x,y)\big| \le c\,\frac{r^{\sigma'}\,\rho^{\lambda_1^+-\varepsilon}}{|x''-y''|^{\sigma'+\lambda_1^++n-\varepsilon}}$$

for $x' \in K_1$, $\rho = |y'| < |x''-y''|$. Furthermore, one can easily derive the estimate

$$\big|\Delta_{x'}(1-\chi)\Sigma'\big| \le c\,\frac{r^{\sigma'-2}\,\rho^{\lambda_1^+-\varepsilon}}{|x''-y''|^{\sigma'+\lambda_1^++n-2-\varepsilon}}$$

from the definition of Σ'. The last two estimates yield

$$\Big|\int_{K_1} V_j\,\big(\Delta_{x'}(1-\chi)\Sigma' - \Delta_{x''}R_{\sigma'}\big)\,dx'\Big| \le c\,|y'|^{\lambda_1^+-\varepsilon}\,|x''-y''|^{-(\lambda_1^++\lambda_j^++n-2-\varepsilon)}.$$

If $\rho < |x''-y''| < 2r$, then $r^2 < 2|x-y|^2 < 28r^2$. Thus, it follows from (3) that $\big|\Delta_{x'}G(x,y)\big| \le c\,r^{-n-\lambda_1^++\varepsilon}\,\rho^{\lambda_1^+-\varepsilon}$ for $x' \in K_2$. Moreover, the estimate

$\big|\Delta_{x'} G_{\sigma'}(x,y)\big| \le c\, \frac{r^{\sigma'-2}\,\rho^{\lambda_1^+-\varepsilon}}{|x''-y''|^{\sigma'+\lambda_1^++n-2-\varepsilon}}$ holds for $\rho < |x''-y''| < 2r$. Using these estimates, we obtain

$$\Big| \int_{K_2} V_j(x')\, \Delta_{x'}(G - G_{\sigma'})\, dx' \Big| \le c\, |y'|^{\lambda_1^+-\varepsilon}\, |x''-y''|^{-(\lambda_1^++\lambda_j^++n-2-\varepsilon)}.$$

The lemma ist proved. □

We continue the proof of Theorem 2.2. By Lemma 2.3, the function d_j has the form (26) and satisfies the equation $\Delta_y d_j(x'',y) = 0$. With the notation $s = |x''-y''|$ we get

$$\Big(\partial_\rho^2 + \frac{m-1}{\rho}\,\partial_\rho + \frac{1}{\rho^2}\,\delta_{\omega_y} + \partial_s^2 + \frac{n-m-1}{s}\,\partial_s\Big)\, \rho^{2-n-\lambda_j^+}\, h_j\Big(\frac{s}{\rho}\Big)\, \phi_j(\omega_y) = 0$$

for $s > 0$. This leads to the differential equation

$$(1+t^2)h_j''(t) + \Big((a+2b+2)t + \frac{a}{t}\Big)\, h_j'(t) + 2(a+1)b\, h_j(t) = 0,$$

where $a = n-m-1$ and $b = \lambda_j^+ - 1 + \frac{n}{2}$. The substitution $h_j(t) = (1+t^2)^{-b}\, w(t)$ yields

$$(1+t^2)\, w''(t) + \Big((a-2b+2)\, t + \frac{a}{t}\Big)\, w'(t) = 0.$$

The last equation has the solution $w(t) = c + d\int_1^t (1+s^2)^{b-1}\, s^{-a}\, ds$ with arbitrary constants c and d. Thus,

$$d_j(x'',y) = \frac{\rho^{\lambda_j^+}\, \phi_j(\omega_y)}{(\rho^2 + |x''-y''|^2)^b}\, \left(C_j + D_j \int_1^{\frac{|x''-y''|}{\rho}} s^{-a}\, (s^2+1)^{b-1}\, ds \right).$$

By item 4) of Lemma 2.3, the constant D_j must be equal to zero, since

$$\left| \int_1^{\frac{|x''-y''|}{\rho}} (s^2+1)^{\lambda_j^+-2+\frac{n}{2}}\, s^{m+1-n}\, ds \right| \ge c\, \left(\left(\frac{|x''-y''|}{\rho}\right)^{2\lambda_j^++m-2} - 1 \right)$$

for $\rho < |x''-y''|$, where c is a constant independent of $|x''-y''|$ and ρ. Applying item 3) of Lemma 2.3, we obtain

$$\begin{aligned} -\frac{\rho^{\lambda_j^-}\, \phi_j(\omega_y)}{2\sigma_j} &= \int_{\mathbb{R}^{n-m}} d_j(x'',y)\, dx'' \\ &= C_j \int_{\mathbb{R}^{n-m}} \frac{\rho^{\lambda_j^+}\, \phi_j(\omega_y)\, dx''}{(\rho^2 + |x''-y''|^2)^{\lambda_j^+-1+\frac{n}{2}}} \\ &= C_j\, \rho^{\lambda_j^+}\, \phi_j(\omega_y)\, \omega_{n-m} \int_0^\infty \frac{s^{n-m-1}}{(\rho^2+s^2)^{\lambda_j^+-1+\frac{n}{2}}}\, ds, \end{aligned}$$

where $\omega_{n-m} = 2\pi^{\frac{n-m}{2}} \Big(\Gamma\big(\frac{n-m}{2}\big)\Big)^{-1}$ is the measure of the $(n-m-1)$-dimensional unit sphere. The integral on the right-hand side of the last equality is equal to

$$\frac{1}{2}\,\rho^{2-m-2\lambda_j^+}\,\frac{\Gamma\big(\frac{n-m}{2}\big)\,\Gamma(\sigma_j)}{\Gamma\big(\sigma_j+\frac{n-m}{2}\big)}.$$

Thus, $C_j = -\frac{\Gamma\left(\sigma_j+\frac{n-m}{2}\right)}{2\pi^{\frac{n-m}{2}}\,\Gamma(\sigma_j+1)}$, and we obtain the formula $d_j(x'',y) = c_j(y',x''-y'')$. This proves that the function $G_\sigma(x,y)$ in (25) has the form (16), where the coefficients c_j are defined by (14).

It remains to prove the estimate (5) for the remainder R_σ. First note that the validity of (5) for $2|x'| < |x-y|$ implies the validity of this estimate for $|x'| < \frac{3}{4}\,|x-y|$. Indeed, if $\frac{1}{2}\,|x-y| < |x'| < \frac{3}{4}\,|x-y|$, then it follows from (3), (11) and from the representation of G_σ that

$$|\partial_x^\alpha \partial_y^\gamma G(x,y)| + |\partial_x^\alpha \partial_y^\gamma G_\sigma(x,y)| \le c\,\frac{|x'|^{\sigma-|\alpha'|}\,|y'|^{\lambda_1^+-|\gamma'|-\varepsilon}}{|x-y|^{\sigma+\lambda_1^++n-2+|\alpha''|+|\gamma''|-\varepsilon}}.$$

Let ψ be the same cut-off function as above, and let

$$\zeta(x,y) = \psi\left(\frac{|x'|}{|x-y|}\right). \tag{30}$$

Then $\zeta(x,y) = 1$ for $|x'| < \frac{1}{2}\,|x-y|$ and $\zeta(x,y) = 0$ for $|x'| > \frac{3}{4}\,|x-y|$. We estimate the $V^l_{p,\beta}(K)$-norm of $\partial^{\alpha''}_{x''}\partial^\gamma_y(\zeta\, R_\sigma)$. Obviously, $R_\sigma = R_{\sigma'} + G_{\sigma'} - G_\sigma = R_{\sigma'} - \Sigma$, where

$$\Sigma(x,y) = \sum_{\lambda_j^+<\sigma} \frac{(-\Delta_{x''})^{m_j}\,c_j(y',x''-y'')}{4^{m_j}\,m_j!\,(\sigma_j+m_j)_{(m_j)}}\,r^{\lambda_j^++2m_j}\,\phi_j(\omega_x).$$

Using (21) and the fact that $\Delta_{x'}(\Sigma-\Sigma')=0$, we obtain

$$\Delta_{x'} R_\sigma(x,y) = \Delta_{x'}\big(R_{\sigma'}(x,y) - \Sigma'(x,y)\big) = -\Delta_{x''} R_{\sigma'}(x,y).$$

This implies $\Delta_{x'}(\zeta R_\sigma) = f$, where

$$f = -\zeta\Delta_{x''}R_{\sigma'} + 2\nabla_{x'}\zeta\cdot\nabla_{x'}(R_{\sigma'}-\Sigma) + (\Delta_{x'}\zeta)\,(R_{\sigma'}-\Sigma).$$

Thus by [8, Theorem 4.2] (see also [6, Theorem 3.1]),

$$\|\partial^{\alpha''}_{x''}\partial^\gamma_y(\zeta R_\sigma)(\cdot,x'',y)\|_{V^l_{p,\beta}(K)} \le c\,\|\partial^{\alpha''}_{x''}\partial^\gamma_y f(\cdot,x'',y)\|_{V^{l-2}_{p,\beta}(K)} \tag{31}$$

with a constant c independent of x'' and y. By the induction hypothesis,

$$\big|\partial_x^\alpha\partial_y^\gamma(\zeta\Delta_{x''}R_{\sigma'})\big| \le c\,\frac{|x'|^{\sigma'-|\alpha'|+\varepsilon}\,|y'|^{\lambda_1^+-|\gamma'|-\varepsilon}}{(|y'|^2+|x''-y''|^2)^{\frac{1}{2}(\sigma'+\lambda_1^++n+|\alpha''|+|\gamma''|)}}.$$

Here, we used the equality $R_{\sigma'} = R_{\sigma'+\varepsilon}$ for sufficiently small ε and the inequality (11). Consequently,

$$\int_K r^{p(\beta-l+2+|\alpha'|)} \left|\partial_x^\alpha \partial_y^\gamma (\zeta \Delta_{x''} R_{\sigma'})\right|^p dx' \\ \le \frac{c\,|y'|^{p(\lambda_1^+ - |\gamma'| - \varepsilon)}}{(|y'|^2 + |x''-y''|^2)^{\frac{p}{2}(\sigma'+\lambda_1^+ + n + |\alpha''| + |\gamma''|)}} \int |x'|^{p(\beta-l+2+\sigma'+\varepsilon)}\, dx',$$

where the integration is extended over the set $|x'|^2 < c(|y'|^2 + |x''-y''|^2)$. Since $p(\beta - l + 2 + \sigma' + \varepsilon) = p(\sigma' - \sigma + 2 + \varepsilon) - m > -m$, it follows that

$$\|\partial_{x''}^{\alpha''} \partial_y^\gamma (\zeta \Delta_{x''} R_{\sigma'})(\cdot, x'', y)\|_{V^{l-2}_{p,\beta}(K)} \\ \le c\, \frac{|y'|^{\lambda_1^+ - |\gamma'| - \varepsilon}}{(|y'|^2 + |x''-y''|^2)^{\frac12(\sigma+\lambda_1^+ + n - 2 + |\alpha''| + |\gamma''| - \varepsilon)}} \tag{32}$$

In the same way, the norms of the $\partial_{x''}^{\alpha''} \partial_y^\gamma$-derivatives of $\nabla_{x'}\zeta \cdot \nabla_{x'} R_{\sigma'}$, $R_{\sigma'} \Delta_{x'}\zeta$, $\nabla_{x'}\zeta \cdot \nabla_{x'}\Sigma$ and $\Sigma \Delta_{x'}\zeta$ in $V^{l-2}_{p,\beta}(K)$ can be estimated by the right-hand side of (32). Here one can use the fact that $\nabla_{x'}\zeta$ and $\Delta_{x'}\zeta$ are zero outside the region $\frac12 |x-y| < |x'| < \frac34 |x-y|$. Consequently by (31),

$$\|\partial_{x''}^{\alpha''} \partial_y^\gamma (\zeta R_\sigma)(\cdot, x'', y)\|_{V^l_{p,\beta}(K)} \le \frac{c\,|y'|^{\lambda_1^+ - |\gamma'| - \varepsilon}}{(|y'|^2 + |x''-y''|^2)^{\frac12(\sigma+\lambda_1^+ + n - 2 + |\alpha''| + |\gamma''| - \varepsilon)}}.$$

If $|\alpha'| < l - \frac{m}{p}$, then

$$\|r^{\beta - l + |\alpha'| + \frac{m}{p}} \partial_x^\alpha \partial_y^\gamma (\zeta R_\sigma)(\cdot, x'', y)\|_{L_\infty(K)} \le c\, \|\partial_{x''}^{\alpha''} \partial_y^\gamma (\zeta R_\sigma)(\cdot, x'', y)\|_{V^l_{p,\beta}(K)}$$

with a constant c independent of x'' and y (see e.g. [12, Lemma 1.2.3]). The last two inequalities directly imply (5). Theorem 2.2 is proved. □

2.4. An alternative method for the proof of Theorem 2.2. It is also possible to prove Theorem 2.2 using the asymptotics of the Green function $\mathcal{G}(x, y, t)$ for the heat equation in the dihedron $\mathcal{D}$. Here, we shortly discuss this method which was proposed by one of the reviewers of the present paper.

The asymptotics of the Green function for a parabolic equation in a cone was studied by V. A. Kozlov in [4, Theorem 4.1]. Using Kozlov's result together with [13, Theorem 4.1], one obtains the decomposition (see also [5, Theorem 2.1])

$$\mathcal{G}(x, y, t) = \left(\sum_{\lambda_j^+ < \sigma} \sum_{k=0}^{m_j} \frac{\partial_t^k C_j(y', t)\, |x'|^{\lambda_j^+ + 2k}\, \phi_j(\omega_x)}{4^k\, k!\, (\sigma_j + k)_{(k)}} + r_\sigma(x', y', t) \right) \Phi(x'' - y'', t),$$

where $\Phi(x'',t)$ is the fundamental solution of the heat equation in $\mathbb{R}^{n-m}$, i.e.

$$\Phi(x'',t) = (4\pi t)^{\frac{m-n}{2}} \exp\Big(-\frac{|x''|^2}{4t}\Big) \quad \text{for } t>0, \quad \Phi(x'',t)=0 \quad \text{for } t<0,$$

and

$$C_j(y',t) = \frac{2}{\Gamma(1+\sigma_j)}\,(4t)^{-1-\sigma_j}\,|y'|^{\lambda_j^+}\,\phi_j(\omega_y).$$

The remainder r_σ satisfies the estimate

$$\begin{aligned}\big|\partial_{x'}^\alpha \partial_{y'}^\gamma \partial_t^k r_\sigma(x',y',t)\big| \le c\, t^{-k-\frac{m+|\alpha|+|\gamma|}{2}} \Big(\frac{|x'|}{|y'|+\sqrt{t}}\Big)^{\sigma-|\alpha|} \\ \times \Big(\frac{|y'|}{|y'|+\sqrt{t}}\Big)^{\lambda_1^+-|\gamma|-\varepsilon} \exp\Big(\frac{-\kappa|x'-y'|^2}{t}\Big).\end{aligned} \tag{33}$$

for $|x'|\le 2|y'|$ and an analogous estimate for $|y'|<2|x'|$. Integrating with respect to t, we get

$$G(x,y) = -\sum_{\lambda_j^+<\sigma}\sum_{k=0}^{m_j}\frac{|x'|^{\lambda_j^++2k}\phi_j(\omega_x)}{4^k\,k!(\sigma_j+k)_{(k)}}\int_0^\infty \Phi(x''-y'',t)\partial_t^k C_j(y',t)\,dt + R_\sigma(x,y), \tag{34}$$

where

$$R_\sigma(x,y) = -\int_0^\infty r_\sigma(x',y',t)\,\Phi(x''-y'',t)\,dt.$$

Integrating by parts and using the equality $\partial_t\Phi(x''-y'',t) = \Delta_{x''}\Phi(x''-y'',t)$ for $x''\ne y''$, we obtain

$$\begin{aligned}&\int_0^\infty \Phi(x''-y'',t)\,\partial_t^k C_j(y',t)\,dt \\ &= (-\Delta_{x''})^k \int_0^\infty \Phi(x''-y'',t)\partial_t^k C_j(y',t)\,dt \\ &= \frac{2\,|y'|^{\lambda_j^+}\phi_j(\omega_y)}{\pi^{\frac{n-m}{2}}\Gamma(1+\sigma_j)}(-\Delta_{x''})^k\int_0^\infty \exp\Big(-\frac{|y'|^2+|x''-y''|^2}{4t}\Big)\frac{dt}{(4t)^{\lambda_j^++\frac{n}{2}}} \\ &= \frac{\Gamma(\lambda_j^+-1+\frac{n}{2})\,|y'|^{\lambda_j^+}\phi_j(\omega_y)}{2\pi^{\frac{n-m}{2}}\Gamma(1+\sigma_j)}(-\Delta_{x''})^k\big(|y'|^2+|x''-y''|^2\big)^{-\lambda_j^++1-\frac{n}{2}}.\end{aligned}$$

Thus, (34) implies

$$G(x,y) = \sum_{\lambda_j^+<\sigma}\sum_{k=0}^{m_j}\frac{(-\Delta_{x''})^k c_j(y',x''-y'')}{4^k\,k!\,(\sigma_j+k)_{(k)}}\,|x'|^{\lambda_j^++2k}\,\phi_j(\omega_x) + R_\sigma(x,y)$$

with the functions c_j defined by (14). It remains to deduce the estimate (5) for the function R_σ from (33).

3. Asymptotics of solutions of the Poisson equation

Now, we consider the variational solution $u \in \overset{\circ}{W}{}_2^1(\mathcal{D})$ of the problem (1), (2) with the right-hand side $f \in W^{-1}(\mathcal{D}) \cap V_{p,\beta}^{l-2}(\mathcal{D})$. Here and in the sequel, we assume that $1 < p < \infty$, l is an integer, $l \geq 2$, and that the number

$$\sigma = l - \beta - \frac{m}{p}$$

satisfies the condition (4). Using the asymptotics of the Green function $G(x, y)$, we are able to describe the singularities of the solution

$$u(x) = \int_{\mathcal{D}} G(x, y)\, f(y)\, dy$$

of the problem (1), (2). Let ψ be a smooth function on $\mathbb{R}_+ = (0, \infty)$, $\psi(r) = 1$ for $r < \frac{1}{2}$, $\psi(r) = 0$ for $r > \frac{3}{4}$, and let the function ζ be defined by (30). Furthermore, let G_σ be the same function (16) as in Section 2. Then by Theorem 2.2, the function u admits the decomposition

$$u(x) = S(x) + v_1(x) + v_2(x),$$

where

$$\begin{aligned} S(x) &= \int_{\mathcal{D}} \zeta(x, y)\, G_\sigma(x, y)\, f(y)\, dy, \\ v_1(x) &= \int_{\mathcal{D}} \zeta(x, y)\, R_\sigma(x, y)\, f(y)\, dy, \\ v_2(x) &= \int_{\mathcal{D}} \big(1 - \zeta(x, y)\big)\, G(x, y)\, f(y)\, dy, \end{aligned}$$

and R_σ satisfies the estimate (5) for $|x'| < \frac{3}{4}|x - y|$. Obviously,

$$S(x) = \sum_{\lambda_j^+ < \sigma} \sum_{k=0}^{m_j} \frac{H_{j,k}(x)}{4^k\, k!\, (\sigma_j + k)_{(k)}}\; r^{\lambda_j^+ + 2k} \phi_j(\omega_x), \tag{35}$$

where

$$H_{j,k}(x) = \int_{\mathcal{D}} \zeta(x, y)\, (-\Delta_{x''})^k\, c_j(y', x'' - y'')\, f(y)\, dy.$$

3.1. Estimation of the remainder v_1+v_2. We will show that $v_1+v_2 \in V^l_{p,\beta}(\mathcal{D})$ if $f \in V^{l-2}_{p,\beta}(\mathcal{D})$. Let $L_{p,\beta}(\mathcal{D})$ denote the weighted L_p space $V^0_{p,\beta}(\mathcal{D})$. For the estimation of the $L_{p,\beta-l}(\mathcal{D})$-norm of v_1, we will employ the following lemma.

Lemma 3.1. *Let w be defined as*

$$w(x) = \int_{\mathcal{D}} K(x,y)\, f(y)\, dy,$$

where $f \in L_{p,\alpha+\beta-\gamma+n}(\mathcal{D})$, $K(x,y) = 0$ for $|x-y| < \delta|x'|$ (δ is a given positive number) and

$$\big|K(x,y)\big| \le c\, \frac{|x'|^\alpha\, |y'|^\beta}{|x-y|^\gamma}.$$

If $\alpha > -\frac{m}{p}$ and $\alpha - \gamma < m - n - \frac{m}{p}$, then

$$\|w\|_{L_p(\mathcal{D})} \le c\, \|f\|_{L_{p,\alpha+\beta-\gamma+n}(\mathcal{D})}$$

with a constant c independent of f.

Proof. One can easily show that

$$\int\limits_{\substack{\mathcal{D}\\ |x-y|>\delta|x'|}} \frac{|y'|^\beta}{|x-y|^\gamma}\, dy \le c\, |x'|^{n+\beta-\gamma} \quad \text{if } \beta > -m,\ \beta-\gamma < -n \tag{36}$$

(cf. [12, Lemma 2.6.1] in the case $m=2$, $n=3$). Under our conditions on α and γ, there exist numbers β_1 and γ_1 such that

$$-\frac{m}{p'} < \beta_1 < \gamma - \alpha - n \quad \text{and} \quad -\alpha - \frac{m}{p} < \beta_1 - \gamma_1 + \frac{n}{p'} < 0,$$

where $p' = \frac{p}{p-1}$. Let $\beta_2 = \beta - \beta_1$ and $\gamma_2 = \gamma - \gamma_1$. Then, by Hölder's inequality and (36),

$$\begin{aligned} |w(x)|^p &\le c\, |x'|^{p\alpha} \Bigg(\int\limits_{\substack{\mathcal{D}\\ |x-y|>\delta|x'|}} \frac{|y'|^{p'\beta_1}\, dy}{|x-y|^{p'\gamma_1}} \Bigg)^{p-1} \int\limits_{\substack{\mathcal{D}\\ |x-y|>\delta|x'|}} \frac{|y'|^{p\beta_2}\, |f(y)|^p\, dy}{|x-y|^{p\gamma_2}} \\ &\le c\, |x'|^{p(\alpha+\beta_1-\gamma_1+n)-n} \int\limits_{\substack{\mathcal{D}\\ |x-y|>\delta|x'|}} \frac{|y'|^{p\beta_2}}{|x-y|^{p\gamma_2}}\, |f(y)|^p\, dy. \end{aligned}$$

This together with (36) implies

$$\begin{aligned} \int_{\mathcal{D}} |w(x)|^p\, dx &\le c \int_{\mathcal{D}} |y'|^{p\beta_2}\, |f(y)|^p \Bigg(\int\limits_{\substack{\mathcal{D}\\ (\delta+1)|x-y|>\delta|y'|}} \frac{|x'|^{p(\alpha+\beta_1-\gamma_1+n)-n}}{|x-y|^{p\gamma_2}}\, dx \Bigg)\, dy \\ &\le c \int_{\mathcal{D}} |y'|^{p(\alpha+\beta-\gamma+n)}\, |f(y)|^p\, dy. \end{aligned}$$

The lemma is proved. $\square$

Using the last lemma, we can easily estimate the $L_{p,\beta-l}$-norm of v_1.

Lemma 3.2. *Let l,p,β satisfy the condition* (6). *Then the function v_1 satisfies the estimate*

$$\|v_1\|_{L_{p,\beta-l}(\mathcal{D})} \le c\,\|f\|_{L_{p,\beta-l+2}(\mathcal{D})} \tag{37}$$

with a constant c independent of f.

Proof. Let ε be a sufficiently small positive number. Then $G_\sigma = G_{\sigma+\varepsilon}$ and $R_\sigma = R_{\sigma+\varepsilon}$. Consequently, the estimate (5) yields

$$|x'|^{\beta-l}\,\big|v_1(x)\big| \le c \int \frac{|x'|^{\beta-l+\sigma+\varepsilon}\,|y'|^{\lambda_1^+-\varepsilon}}{|x-y|^{\sigma+\lambda_1^++n-2}}\,\big|f(y)\big|\,dy,$$

where the integration is extended over the set $\{y\in\mathcal{D}:\ 3|x-y|>4|x'|\}$. Since $\beta-l+\sigma+\varepsilon > -\frac{m}{p}$ and $\lambda_1^- = 2-m-\lambda_1^+ < l-\beta-\frac{m}{p}$, we can apply Lemma 3.1 and obtain (37). □

We prove the same estimate for the function v_2.

Lemma 3.3. *Suppose that l,p,β satisfy the condition* (6). *Then*

$$\|v_2\|_{L_{p,\beta-l}(\mathcal{D})} \le c\,\|f\|_{L_{p,\beta-l+2}(\mathcal{D})}$$

with a constant c independent of f.

Proof. By (3), the Green function satisfies the estimate

$$\big|G(x,y)\big| \le c\,|x-y|^{2-n}\left(\frac{|y'|}{|y'|+|x-y|}\right)^{\lambda_1^+-\varepsilon} \quad \text{for } |x-y|<2|x'|.$$

Consequently,

$$\big|v_2(x)\big| \le A+B,$$

where

$$A = \int\limits_{\substack{\mathcal{D}\\ |x-y|<2\min(|x'|,|y'|)}} \frac{|f(y)|\,dy}{|x-y|^{n-2}}, \quad B = \int\limits_{\substack{\mathcal{D}\\ 2|y'|<|x-y|<2|x'|}} \frac{|y'|^{\lambda_1^+-\varepsilon}\,|f(y)|\,dy}{|x-y|^{n-2+\lambda_1^+-\varepsilon}}.$$

By Hölder's inequality,

$$\begin{aligned} A^p &\le c \int\limits_{\substack{\mathcal{D}\\ |x-y|<2\min(|x'|,|y'|)}} \frac{\big|f(y)\big|\,dy}{|x-y|^{n-2}}\left(\int_{|x-y|<2|x'|}\frac{dy}{|x-y|^{n-2}}\right)^{p-1} \\ &\le c\,|x'|^{2(p-1)} \int\limits_{\substack{\mathcal{D}\\ |x-y|<2\min(|x'|,|y'|)}} |x-y|^{2-n}\,\big|f(y)\big|\,dy. \end{aligned}$$

Since $\frac{|y'|}{3} < |x'| < 3|y'|$ for $|x-y| < 2\min(|x'|, y'|)$, it follows that

$$\int_{\mathcal{D}} |x'|^{p(\beta-l)} \, |A|^p \, dx \le c \int_{\mathcal{D}} |y'|^{p(\beta-l+2)-2} \, |f(y)|^p \left(\int_{|x-y|<2|y'|} \frac{dx}{|x-y|^{n-2}} \right) dy$$
$$\le c \int_{\mathcal{D}} |y'|^{p(\beta-l+2)} \, |f(y)|^p \, dy.$$

We consider the term B. Obviously $2|x'| < 3|x-y| < 6|x'|$ for $2|y'| < |x-y| < 2|x'|$. Hence,

$$B \le c \, |x'|^{2-n-\lambda_1^+ + \varepsilon} \int\limits_{\substack{\mathcal{D} \\ 2|y'|<|x-y|<2|x'|}} |y'|^{\lambda_1^+ - \varepsilon} \, |f(y)| \, dy.$$

Using Hölder's inequality, we get

$$B^p \le c \, |x'|^{2-n-\lambda_1^+ + \varepsilon} \int |y'|^{p(\lambda_1^+ - \varepsilon)+(p-1)(m-\varepsilon)} \, \big| f(y) \big|^p \, dy \left(\int \frac{dy}{|y'|^{m-\varepsilon}} \right)^{p-1},$$

where the integration is extended over the set of all $y \in \mathcal{D}$ such that $2|y'| < |x-y| < 2|x'|$. Here

$$\int_{2|y'|<|x-y|<2|x'|} \frac{dy}{|y'|^{m-\varepsilon}} \le \int_{|y'|<|x'|} \frac{dy'}{|y'|^{m-\varepsilon}} \int_{|y''-x''|<2|x'|} dy'' \le c \, |x'|^{n-m+\varepsilon}.$$

Therefore,

$$B^p \le c \, |x'|^{p(2-m-\lambda_1^+ + 2\varepsilon)-n+m-\varepsilon} \int\limits_{\substack{\mathcal{D} \\ 2|y'|<|x-y|<2|x'|}} |y'|^{p(\lambda_1^+ + m - 2\varepsilon)-m+\varepsilon} \, \big| f(y) \big|^p \, dy$$

and

$$\int_{\mathcal{D}} |x'|^{p(\beta-l)} |B|^p \, dx$$
$$\le c \int_{\mathcal{D}} |y'|^{p(\lambda_1^+ + m - 2\varepsilon)-m+\varepsilon} |f(y)|^p \left(\int_{2|y'|<|x-y|<2|x'|} |x'|^{p(\beta-l+2-m-\lambda_1^+ + 2\varepsilon)-n+m-\varepsilon} dx \right) dy,$$

where

$$\int_{2|y'|<|x-y|<2|x'|} |x'|^{p(\beta-l+2-m-\lambda_1^+ + 2\varepsilon)-n+m-\varepsilon} \, dx$$
$$\le \int_{|x'|\ge|y'|} |x'|^{p(\beta-l+2-m-\lambda_1^+ + 2\varepsilon)-n+m-\varepsilon} \left(\int_{|x''-y''|<2|x'|} dx'' \right) dx'$$
$$= c \int_{|x'|\ge|y'|} |x'|^{p(\beta-l+2-m-\lambda_1^+ + 2\varepsilon)-\varepsilon} \, dx'$$
$$= C \, |y'|^{p(\beta-l+2-m-\lambda_1^+ + 2\varepsilon)+m-\varepsilon}$$

since $p(\beta - l + 2 - m - \lambda_1^+) + m < 0$. Consequently,

$$\int_{\mathcal{D}} |x'|^{p(\beta-l)}\, |B(x)|^p\, dx \le c \int_{\mathcal{D}} |y'|^{p(\beta-l+2)}\, |f(y)|^p\, dy.$$

This proves the lemma. □

Next, we consider the function S defined by (35). We prove that $\Delta S \in V_{p,\beta}^{l-2}(\mathcal{D})$ if $f \in V_{p,\beta}^{l-2}(\mathcal{D})$.

Lemma 3.4. *Let $f \in V_{p,\beta}^{l-2}(\mathcal{D})$, where l, p, β satisfy the condition* (6)*. Then the function* (35) *satisfies the estimate*

$$\|\Delta S\|_{V_{p,\beta}^{l-2}(\mathcal{D})} \le c\, \|f\|_{V_{p,\beta}^{l-2}(\mathcal{D})}$$

with a constant c independent of f.

Proof. Using the formula (20), we obtain

$$\Delta_x G_\sigma(x,y) = -\sum_{\lambda_j^+ < \sigma} \frac{(-\Delta_{x''})^{m_j+1} c_j(y', x''-y'')}{4^{m_j}\, m_j!\, (\sigma_j + m_j)_{(m_j)}}\, r^{\lambda_j^+ + 2m_j}\, \phi_j(\omega_x).$$

We consider the term

$$T(x) = \int_{\mathcal{D}} \zeta(x,y)\, \Delta_x G_\sigma(x,y)\, f(y)\, dy = \sum_{\lambda_j^+ < \sigma} T_j(x),$$

where

$$T_j(x) = -r^{\lambda_j^+ + 2m_j}\, \phi_j(\omega_x) \int_{\mathcal{D}} \zeta(x,y)\, \frac{(-\Delta_{x''})^{m_j+1} c_j(y', x''-y'')}{4^{m_j}\, m_j!\, (\sigma_j + m_j)_{(m_j)}}\, f(y)\, dy.$$

Obviously,

$$\big|\partial_x^\alpha \zeta(x,y)\big| \le c\, r^{-|\alpha|} \tag{38}$$

for $|\alpha| \le l-2$. Since (11) is satisfied for $4|x'| < 3|x-y|$, we furthermore get

$$\begin{aligned} \big|\partial_{x''}^{\alpha''} (-\Delta_{x''})^{m_j+1} c_j(y', x''-y'')\big| &\le c\, |y'|^{\lambda_j^+}\, |x-y|^{-(2\lambda_j^+ + n + 2m_j + |\alpha''|)} \\ &\le c'\, |x'|^{-|\alpha''|}\, |y'|^{\lambda_j^+}\, |x-y|^{-(2\lambda_j^+ + n + 2m_j)} \end{aligned} \tag{39}$$

for $4|x'| < 3|x-y|$. Thus,

$$r^{\beta-l+2+|\alpha|}\, \big|\partial_x^\alpha T_j(x)\big| \le c\, r^{\beta-l+2+\lambda_j^+ + 2m_j} \int\limits_{\substack{\mathcal{D} \\ 3|x-y|>4|x'|}} \frac{|y'|^{\lambda_j^+}\, |f(y)|\, dy}{|x-y|^{2\lambda_j^+ + n + 2m_j}}.$$

Here $\beta - l + 2 + \lambda_j^+ + 2m_j > -\frac{m}{p}$. Since moreover $l - \beta - \frac{m}{p} > \lambda_j^- = 2 - m - \lambda_j^+$, Lemma 3.1 implies $\|r^{\beta-l+2+|\alpha|}\,\partial_x^\alpha T\|_{L_p(\mathcal{D})} \le c\,\|f\|_{L_{p,\beta-l+2}(\mathcal{D})}$ for $|\alpha| \le l-2$. Analogously, this estimate holds for the term

$$\Delta_x S(x) - T(x) = \int_{\mathcal{D}} \Big(\big(\Delta_x \zeta(x,y)\big)\, G_\sigma(x,y) + 2\nabla_x \zeta(x,y) \cdot \nabla_x G_\sigma(x,y) \Big)\, f(y)\, dy.$$

Here one can use the fact that $\frac{|x-y|}{2} \le |x'| \le \frac{3|x-y|}{4}$ on the support of $\nabla_x \zeta$. Hence,

$$\|\Delta S\|_{V^{l-2}_{p,\beta}(\mathcal{D})} \le c\,\|f\|_{L_{p,\beta-l+2}(\mathcal{D})}.$$

The lemma is proved. □

The last three lemmas allow us to deduce the following result.

Theorem 3.5. *Suppose that* $u \in \overset{\circ}{W}{}^1_2(\mathcal{D})$ *is a solution of the problem* (1), (2) *with the right-hand side* $f \in W^{-1}(\mathcal{D}) \cap V^{l-2}_{p,\beta}(\mathcal{D})$, *where* l, p, β *satisfy the condition* (6). *Then*

$$u(x) = S(x) + v(x),$$

where S *is defined by* (35) *and* $v \in V^l_{p,\beta}(\mathcal{D})$. *Furthermore,*

$$\|v\|_{V^l_{p,\beta}(\mathcal{D})} \le c\,\|f\|_{V^{l-2}_{p,\beta}(\mathcal{D})}$$

with a constant c *independent of* f.

Proof. By Lemmas 3.2 and 3.3, we have $u = S + v$, where $v \in V^0_{p,\beta-l}(\mathcal{D})$ and

$$\|v\|_{V^0_{p,\beta-l}(\mathcal{D})} \le c\,\|f\|_{V^{l-2}_{p,\beta}(\mathcal{D})}.$$

Using [12, Theorem 2.2.9], Lemma 3.4 and the equality $\Delta v = f - \Delta S$, we conclude that $v \in V^l_{p,\beta}(\mathcal{D})$ and

$$\|v\|_{V^l_{p,\beta}(\mathcal{D})} \le c\left(\|\Delta v\|_{V^{l-2}_{p,\beta}(\mathcal{D})} + \|v\|_{V^0_{p,\beta-l}(\mathcal{D})}\right) \le c'\,\|f\|_{V^{l-2}_{p,\beta}(\mathcal{D})}.$$

The theorem is proved. □

3.2. On the coefficients in the asymptotics. The coefficients $H_{j,k}$ in the representation (35) of S are extensions of the functions

$$h_{j,k}(x'') = \int_{\mathcal{D}} (-\Delta_{x''})^k\, c_j(y', x'' - y'')\, f(y)\, dy$$

to the domain $\mathcal{D}$. We show in the last subsection that these coefficients can be replaced by other suitable extensions of the functions $h_{j,k}$.

First, we show that the assertion of Theorem 3.5 remains true if we replace $S(x)$ by the expression

$$\tilde{S}(x) = \sum_{\lambda_j^+<\sigma} \sum_{k=0}^{m_j} \frac{r^{\lambda_j^+ + 2k}\phi_j(\omega_x)}{4^k\, k!\, (\sigma_j+k)_{(k)}} (-\Delta_{x''})^k \int_{\mathcal{D}} \zeta(x,y)\, c_j(y',x''-y'')\, f(y)\, dy.$$

Obviously, this follows directly from the next lemma.

Lemma 3.6. *Suppose that the conditions of Theorem* 3.5 *on* f, l, p, β *are satisfied. Then* $S-\tilde{S} \in V^l_{p,\beta}(\mathcal{D})$ *and*

$$\|S-\tilde{S}\|_{V^l_{p,\beta}(\mathcal{D})} \le c\, \|f\|_{V^{l-2}_{p,\beta}(\mathcal{D})} \tag{40}$$

with a constant c *independent of* f.

Proof. We denote the expression

$$\frac{(-1)^k r^{\lambda_j^+ + 2k}\phi_j(\omega_x)}{4^k\, k!\, (\sigma_j+k)_{(k)}} \Big(\zeta(x,y)\Delta^k_{x''} c_j(y',x''-y'') - \Delta^k_{x''}\zeta(x,y)\, c_j(y',x''-y'')\Big)$$

by $K_{j,k}(x,y)$. Then

$$S(x)-\tilde{S}(x) = \sum_{\lambda_j^+<\sigma} \sum_{k=0}^{m_j} \int_{\mathcal{D}} K_{j,k}(x,y)\, f(y)\, dy.$$

Using the estimates (38), (39) and the fact that $\frac{|x-y|}{2} \le |x'| \le \frac{3|x-y|}{4}$ on the support of the function $\nabla_{x''}\zeta$, we obtain

$$r^{\beta-l+|\alpha|} \left|\partial_x^\alpha K_{j,k}(x,y)\right| \le c\, \frac{|y'|^{\lambda_j^+}}{|x-y|^{l-\beta+\lambda_j^+ + n-2}}$$

for $|\alpha| \le l$. Thus, we can conclude from Lemma 3.1 that the operator with the kernel $r^{\beta-l+|\alpha|}\, \partial_x^\alpha K_{j,k}(x,y)$ is bounded from $L_{p,\beta-l+2}(\mathcal{D})$ into $L_p(\mathcal{D})$. This implies (40). □

We introduce the function

$$H_j(x) = \int_{\mathcal{D}} \zeta(x,y)\, c_j(y',x''-y'')\, f(y)\, dy. \tag{41}$$

Then

$$\tilde{S}(x) = \sum_{\lambda_j^+<\sigma} \sum_{k=0}^{m_j} \frac{(-\Delta_{x''})^k\, H_j(x)}{4^k\, k!\, (\sigma_j+k)_{(k)}}\, r^{\lambda_j^+ + 2k}\, \phi_j(\omega_x).$$

It is evident that the functions $H_j(x)$ depend only on $r=|x'|$ and x''. Furthermore, these functions have the following properties.

Lemma 3.7. *Suppose that* $f \in V^{l-2}_{p,\beta}(\mathcal{D})$ *and* l, p, β *satisfy the condition* (6). *Then*

(i) $\partial_x^\alpha H_j = \partial_{x'}^{\alpha'} \partial_{x''}^{\alpha''} H_j \in L_{p,\beta-l+\lambda_j^+ + |\alpha|}(\mathcal{D})$ *if* $|\alpha'| \ge 1$.

(ii) $\partial_{x''}^{\alpha''} H_j \in L_{p,\beta-l+\lambda_j^+ + |\alpha''|}(\mathcal{D})$ *if* $|\alpha''| > \sigma - \lambda_j^+$.

(iii) *The trace of* H_j *on* M *coincides with the function* (8) *and belongs to the Besov space* $B_p^{\sigma-\lambda_j^+}(\mathbb{R}^{n-m})$.

Proof. Using (11), we get

$$\left|\partial_x^\alpha \zeta(x,y)\, c_j(y', x''-y'')\right| \le c_\alpha\, |y'|^{\lambda_j^+}\, |x-y|^{2-2\lambda_j^+ - n - |\alpha|}$$

for every multi-index α.

If $\alpha = (\alpha', \alpha'')$ and $|\alpha'| \ge 1$, then the function $\partial_x^\alpha \zeta(x,y)\, c_j(y', x''-y'')$ vanishes outside the region $\frac{|x-y|}{2} < |x'| < \frac{3|x-y|}{4}$. Therefore,

$$r^{\beta-l+\lambda_j^+ + |\alpha|} \left|\partial_x^\alpha \zeta(x,y)\, c_j(y', x''-y'')\right| \le c_\alpha\, \frac{|y'|^{\lambda_j^+}}{|x-y|^{\lambda_j^+ - \beta + l - 2 + n}}$$

Consequently, it follows from Lemma 3.1 that

$$\|r^{\beta-l+\lambda_j^+ + |\alpha|}\, \partial_x^\alpha H_j\|_{L_p(\mathcal{D})} \le c_\alpha\, \|f\|_{L_{p,\beta-l+2}(\mathcal{D})}$$

with a constant c_α independent of f.

Now, let $\alpha' = 0$ and $|\alpha''| > l - \beta - \frac{m}{p} - \lambda_j^+$. Then

$$r^{\beta-l+\lambda_j^+ + |\alpha''|} \left|\partial_{x''}^{\alpha''} \zeta(x,y)\, c_j(y', x''-y'')\right| \le c_{\alpha''}\, \frac{|x'|^{\beta-l+\lambda_j^+ + |\alpha''|}\, |y'|^{\lambda_j^+}}{|x-y|^{2\lambda_j^+ - 2 + n + |\alpha''|}}.$$

Applying again Lemma 3.1, we obtain

$$\|r^{\beta-l+\lambda_j^+ + |\alpha''|}\, \partial_{x''}^{\alpha''} H_j\|_{L_p(\mathcal{D})} \le c_{\alpha''}\, \|f\|_{L_{p,\beta-l+2}(\mathcal{D})}.$$

If we consider H_j as a function on $\mathbb{R}_+ \times \mathbb{R}^{n-m}$ (with the variables r and x''), then it follows from (i) and (ii) in particular that

$$r^{\beta-l+\lambda_j^+ + k + |\alpha''| + \frac{m-1}{p}}\, \partial_r^k \partial_{x''}^{\alpha''} H_j \in L_p(\mathbb{R}_+ \times \mathbb{R}^{n-m}) \quad \text{for } k + |\alpha''| > \sigma - \lambda_j^+.$$

Hence by [15, Section 2.9.2, Theorem 1]), the trace $h_j = H_j|_{x'=0}$ of H_j on the edge M exists and belongs to the Besov space $B_p^{\sigma-\lambda_j^+}(M)$. Since $\zeta(x,y) = 1$ for $x' = 0$, the trace h_j is given by the formula (8). This proves the lemma. □

Finally, we show that the coefficients (41) of $\tilde{S}(x)$ can be replaced by other extensions of h_j satisfying the conditions (i) and (ii) of Lemma 3.7. In particular, these extensions can be defined by means of extension operator

$$(\mathcal{E}h)(x',x'') = \int_{\mathbb{R}^{n-m}} T(y'')\, h(x''-ry'')\, dy'', \tag{42}$$

where T is a smooth function on $\mathbb{R}^{n-m}$ with support in the unit ball $|y''| \le 1$ satisfying the following condition with $N_0 = [\sigma - \lambda_1^+]$:

$$\int_{\mathbb{R}^{n-m}} T(y'')\, dy'' = 1, \quad \int_{\mathbb{R}^{n-m}} (y'')^{\alpha''}\, T(y'')\, dy'' = 0 \text{ for } 1 \le |\alpha''| \le N_0. \tag{43}$$

We show that the functions $\mathcal{E}h_j$ have the properties (i) and (ii) of Lemma 3.7.

Lemma 3.8. *If $f \in V^{l-2}_{p,\beta}(\mathcal{D})$ and l,p,β satisfy the condition* (6), *then*

(i) $\partial_x^\alpha \mathcal{E}h_j = \partial_{x'}^{\alpha'} \partial_{x''}^{\alpha''} \mathcal{E}h_j \in L_{p,\beta-l+\lambda_j^++|\alpha|}(\mathcal{D})$ *if* $|\alpha'| \ge 1$.

(ii) $\partial_{x''}^{\alpha''} \mathcal{E}h_j \in L_{p,\beta-l+\lambda_j^++|\alpha''|}(\mathcal{D})$ *if* $|\alpha''| > \sigma - \lambda_j^+$.

Proof. It can be easily seen that

$$(\mathcal{E}h_j)(x) = \int_{\mathcal{D}} K_j(x,y)\, f(y)\, dy,$$

where

$$K_j(x,y) = r^{m-n} \int_{\mathbb{R}^{n-m}} T\Big(\frac{x''-z''}{r}\Big)\, c_j(y', z''-y'')\, dz''.$$

Since $(\mathcal{E}h_j)(x)$ depends only on r and x'', we consider the derivatives

$$\partial_r^k \partial_{x''}^{\alpha''} (\mathcal{E}h_j)(x) = \int_{\mathcal{D}} \partial_r^k \partial_{x''}^{\alpha''} K_j(x,y)\, f(y)\, dy.$$

We introduce the following subsets

$$\begin{aligned} \mathcal{D}_1 &= \{y \in \mathcal{D} :\ 3|x-y| < r\}, \\ \mathcal{D}_2 &= \{y \in \mathcal{D} :\ \frac{r}{3} < |x-y| < 3r\}, \\ \mathcal{D}_3 &= \{y \in \mathcal{D} :\ |x-y| > 3r\} \end{aligned}$$

of $\mathcal{D}$ and show that the functions

$$U_{\nu,k,\alpha''}(x) = \int_{\mathcal{D}_\nu} \partial_r^k \partial_{x''}^{\alpha''} K_j(x,y)\, f(y)\, dy, \quad \nu = 1,2,3,$$

satisfy the estimate

$$\|U_{\nu,k,\alpha''}\|_{L_{p,\beta-l+\lambda_j^++k+|\alpha''|}(\mathcal{D})} \le c\, \|f\|_{L_{p,\beta-l+2}(\mathcal{D})} \tag{44}$$

if $k \geq 1$ or $|\alpha''| > \sigma - \lambda_j^+$.

We start with the cases $\nu = 1$ and $\nu = 2$. Obviously,

$$\partial_r^k \partial_{x''}^{\alpha''} K_j(x,y) = \int_{\mathbb{R}^{n-m}} \partial_r^k \, r^{m-n-|\alpha''|} \, T^{(\alpha'')}\Big(\frac{x''-z''}{r}\Big) \, c_j(y', z''-y'') \, dz'',$$

where $T^{(\alpha'')}(x'') = \partial_{x''}^{\alpha''}(x'')$. Since

$$\Big|\partial_r^k \, r^{m-n-|\alpha''|} \, T^{(\alpha'')}\Big(\frac{x''-z''}{r}\Big)\Big| \leq c \, r^{m-n-k-|\alpha''|},$$

we obtain

$$\begin{aligned}\big|\partial_r^k \partial_{x''}^{\alpha''} K_j(x,y)\big| &\leq c \, r^{m-n-k-|\alpha''|} \int_{\mathbb{R}^{n-m}} \frac{\rho^{\lambda_j^+} \, dz''}{(\rho^2 + |y''-z''|^2)^{\lambda_j^+ - 1 + \frac{n}{2}}} \\ &= C \, r^{m-n-k-|\alpha''|} \, \rho^{2-m-\lambda_j^+}.\end{aligned}$$

If $3|x-y| < r$, then $2|x-y| < \rho$ and $2r < 3\rho < 4r$. Consequently by Hölder's inequality,

$$\begin{aligned}|U_{1,k,\alpha}(x)|^p &\leq c \, r^{p(2-n-\lambda_j^+ - k - |\alpha''|)} \int_{\mathcal{D}_1} |f(y)|^p \, dy \Big(\int_{3|x-y|<r} dy\Big)^{p-1} \\ &= C \, r^{p(2-\lambda_j^+ - k - |\alpha''|) - n} \int_{\mathcal{D}_1} |f(y)|^p \, dy.\end{aligned}$$

This implies

$$\int_{\mathcal{D}} r^{p(\beta - l + \lambda_j^+ + k + |\alpha''|)} \, |U_{1,k,\alpha}|^p \, dx \leq c \int_{\mathcal{D}} \rho^{p(\beta-l+2)-n} |f(y)|^p \, \Big(\int_{2|x-y|<\rho} dx\Big) \, dy.$$

Thus, (44) holds for $\nu = 1$. If $\frac{r}{3} < |x-y| < 3r$, then

$$r^{\beta-l+\lambda_j^+ + k + |\alpha''|} \, \big|\partial_r^k \partial_{x''}^{\alpha''} K_j(x,y)\big| \leq c \, \frac{r^{\beta-l+\lambda_j^+ + \kappa} \, \rho^{2-m-\lambda_j^+}}{|x-y|^{n-m+\kappa}}$$

with arbitrary κ. Hence Lemma 3.1 yields (44) for $\nu = 2$.

We consider the case $\nu = 3$. First let $k = 0$ and $|\alpha''| > \sigma - \lambda_j^+$. Integrating by parts, we get

$$\partial_{x''}^{\alpha''} K_j(x,y) = r^{m-n} \int_{\mathbb{R}^{n-m}} T\Big(\frac{x''-z''}{r}\Big) \, \partial_{z''}^{\alpha''} c_j(y', z''-y'') \, dz''. \tag{45}$$

If $|x-y| > 3r$ and $|x''-z''| \leq r$, then $\frac{5}{18}\,|x-y|^2 \leq |y'|^2 + |y''-z''|^2 \leq \frac{22}{9}\,|x-y|^2$. Therefore,

$$\big|\partial_{z''}^{\alpha''} c_j(y', z''-y'')\big| \leq \frac{c \, \rho^{\lambda_j^+}}{(\rho^2 + |y''-z''|)^2)^{\lambda_j^+ - 1 + \frac{n+|\alpha''|}{2}}} \leq c \, \frac{\rho^{2-n-\lambda_j^+ - |\alpha''| + \kappa}}{|x-y|^{\kappa}}$$

for $|x-y| > 3r$ and $|x''-z''| \le r$ provided that $\kappa \le 2\lambda_j^+ - 2 + n + |\alpha''|$. This means that

$$r^{\beta-l+\lambda_j^+ +|\alpha''|} \left|\partial_{x''}^{\alpha''} K_j(x,y)\right| \le c\, \frac{r^{\beta-l+\lambda_j^+ +|\alpha''|}\, \rho^{2-n-\lambda_j^+ -|\alpha''|+\kappa}}{|x-y|^\kappa}$$

for $|x-y| > 3r$. If $|\alpha''| > l-\beta-\lambda_j^+ - \frac{m}{p}$, then we can apply Lemma 3.1 and obtain $\|r^{\beta-l+\lambda_j^+ +|\alpha''|} U_{3,0,\alpha''}\|_{L_p(\mathcal{D})} \le c\,\|f\|_{L_{p,\beta-l+2}(\mathcal{D})}$. Finally, let α'' be an arbitrary multi-index and $k \ge 1$. By (45),

$$\partial_r^k \partial_{x''}^{\alpha''} K_j(x,y) = \int_{\mathbb{R}^{n-m}} T(z'')\, \partial_r^k\, (-r)^{-|\alpha''|}\, \partial_{z''}^{\alpha''} c_j(y', x''-y''-rz'')\, dz''.$$

Here, $\partial_r^k\, r^{-|\alpha''|}\, \partial_{z''}^{\alpha''} c_j(y', x''-y''-rz'')$ is a finite sum of terms of the form

$$\Psi(\rho, z'', x''-y''-rz'') = c\, \frac{\rho^{\lambda_j^+}\, (z'')^\gamma\, (x''-y''-rz'')^\delta}{\big(\rho^2 + |x''-y''-rz''|^2\big)^{\lambda_j^+ + k + |\alpha''| - \nu - 1 + \frac{n}{2}}}\,,$$

where $|\gamma| = k$ and $2\nu + |\delta| = k + |\alpha''|$. Using the Taylor expansion of the function Ψ with respect to the variable r and the condition (43), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^{n-m}} T(z'')\, \Psi(\rho, z'', x''-y''-rz'')\, dz'' \\ &\le c\, r^{N_0+1-k} \sup_{0<\theta<r} \left|\partial_\theta^{N_0+1-k} \Psi(\rho, z'', x''-y''-\theta z'')\right|.\end{aligned}$$

If $0 < \theta < r$, $|z''| \le 1$ and $y \in \mathcal{D}_3$ (i.e. $|x-y| > 3r$), then $3|x''-y''-\theta z''| < 4|x-y|$ and $\frac{5}{18}\, |x-y|^2 \le \rho^2 + |x''-y''-\theta z''|^2 \le \frac{22}{9}\, |x-y|^2$. This implies

$$\left|\partial_\theta^{N_0+1-k} \Psi(\rho, z'', x''-y''-\theta z'')\right| \le c\, \frac{\rho^{\lambda_j^+}}{|x-y|^{2\lambda_j^+ + n - 1 + |\alpha''| + N_0}}.$$

Consequently, we get

$$r^{\beta-l+\lambda_j^+ +k+|\alpha''|} \left|\partial_r^k \partial_{x''}^{\alpha''} K_j(x,y)\right| \le c\, \frac{r^{\beta-l+1+\lambda_j^+ +|\alpha''|+N_0}\, \rho^{\lambda_j^+}}{|x-y|^{2\lambda_j^+ + n - 1 + |\alpha''| + N_0}}$$

for $y \in \mathcal{D}_3$ and $k \ge 1$. Since $N_0 = [\sigma - \lambda_1^+] > l - \beta - 1 - \lambda_j^+ - |\alpha''| - \frac{m}{p}$, we can apply Lemma 3.1 and get the estimate

$$\|r^{\beta-l+\lambda_j^+ +k+|\alpha''|} U_{3,k,\alpha''}\|_{L_p(\mathcal{D})} \le c\, \|f\|_{L_{p,\beta-l+2}(\mathcal{D})}$$

for $k \ge 1$. The proof of the lemma is complete. $\square$

Now it is easy to prove the main result of this section.

Theorem 3.9. *Suppose that* $f \in W_2^{-1}(\mathcal{D}) \cap V_{p,\beta}^{l-2}(\mathcal{D})$ *and* l, p, β *satisfy the condition* (6). *Then the solution* $u \in \overset{\circ}{W}{}_2^1(\mathcal{D})$ *of the problem* (1), (2) *admits the decomposition* (7) *with the remainder* $v \in V_{p,\beta}^l(\mathcal{D})$.

Proof. It follows from Theorem 3.5 and Lemma 3.6 that

$$u(x) = \sum_{\lambda_j^+ < \sigma} \sum_{k=0}^{m_j} \frac{(-\Delta_{x''})^k H_j(x)}{4^k\, k!\, (\sigma_j + k)_{(k)}}\, r^{\lambda_j^+ + 2k} \phi_j(\omega_x) + w(x), \tag{46}$$

where $w \in V_{p,\beta}^l(\mathcal{D})$. We show that

$$\partial_x^\alpha \big(H_j - \mathcal{E} h_j\big) \in L_{p,\beta-l+\lambda_j^+ + |\alpha|}(\mathcal{D}) \tag{47}$$

for every multi-index α. If $\partial_x^\alpha = \partial_{x'}^{\alpha'} \partial_{x''}^{\alpha''}$ and $|\alpha'| \ge 1$, this follows immediately from Lemmas 3.7 and 3.8. This is also the case if $\alpha' = 0$ and $|\alpha''| > \sigma - \lambda_j^+$. Suppose that $|\alpha''| < \sigma - \lambda_j^+$. Then $\partial_{x''}^{\alpha''} H_j|_M = \partial_{x''}^{\alpha''} \mathcal{E} h_j|_M = \partial_{x''}^{\alpha''} h_j \in B_p^{\sigma - \lambda_j^+ - |\alpha''|}(M)$ and Hardy's inequality implies

$$\begin{aligned} &\int_{\mathcal{D}} r^{p(\beta - l + \lambda_j^+ + |\alpha''|)} \left|\partial_{x''}^{\alpha''}(H_j - \mathcal{E} h_j)\right|^p dx \\ &\le c \int_{\mathcal{D}} r^{p(\beta - l + \lambda_j^+ + |\alpha''| + 1)} \left|\partial_r \partial_{x''}^{\alpha''}(H_j - \mathcal{E} h_j)\right|^p dx \\ &< \infty, \end{aligned}$$

since $\beta - l + \lambda_j^+ + |\alpha''| = \lambda_j^+ + |\alpha''| - \sigma - \frac{m}{p} < -\frac{m}{p}$.

Finally, let $|\alpha''| = \sigma - \lambda_j^+$, i.e. $\beta - l + \lambda_j^+ + |\alpha''| = -\frac{m}{p}$. Since $\partial_{x'}^{\alpha'}(H_j - \mathcal{E} h_j) \in L_{p,\beta-l+\lambda_j^+ + |\alpha'|}(\mathcal{D})$, we have

$$\partial_{x'}^{\alpha'}(H_j - \mathcal{E} h_j)|_M = 0 \quad \text{for } |\alpha'| < \sigma - \lambda_j^+$$

and

$$\int_{\mathcal{D}} r^{-m} \left|\partial_{x'}^{\alpha'}(H_j - \mathcal{E} h_j)\right|^p dx < \infty \quad \text{for } |\alpha'| = \sigma - \lambda_j^+.$$

Moreover, $\partial_x^\alpha (H_j - \mathcal{E} h_j) \in L_{p,1-\frac{m}{p}}(\mathcal{D})$ for $|\alpha| = \sigma - \lambda_j^+ + 1$. This implies (cf. [12, Lemma 6.2.6] for $m = 2$, in the case $m > 2$ this result holds analogously) that $H_j - \mathcal{E} h_j \in V_{p,1-\frac{m}{p}}^{\sigma - \lambda_j^+ + 1}(\mathcal{D})$ and, in particular, that

$$\int_{\mathcal{D}} r^{-m} \left|\partial_{x'}^{\alpha''}(H_j - \mathcal{E} h_j)\right|^p dx < \infty \quad \text{for } |\alpha''| = \sigma - \lambda_j^+.$$

Thus, (47) is proved for all multi-indices α. As an immediate consequence, we get

$$\Delta_{x''}^k (H_j - \mathcal{E} h_j)\, r^{\lambda_j^+ + 2k} \phi_j(\omega_x) \in V_{p,\beta}^l(\mathcal{D}).$$

Consequently, we can replace H_j by $\mathcal{E} h_j$ in (46). The theorem is proved. □

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