Pointwise Limits of Sequences of Right-Continuous Functions and Measurability of Functions of Two Variables

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Abstract. In this article I prove that the pointwise limit $f: \mathbb{R} \to \mathbb{R}$ of a sequence of right-continuous functions has some special property (G) and that bounded functions of two variables $g: \mathbb{R}^2 \to \mathbb{R}$ whose vertical sections $g_x$, $x \in \mathbb{R}$, are derivatives and horizontal sections $g^y$, $y \in \mathbb{R}$, are pointwise limits of sequences of right-continuous functions, are measurable and sup-measurable in the sense of Lebesgue.

Keywords. Pointwise convergence, right-continuity, Baire 1 class, derivative, approximate continuity, measurability of functions of two variables

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1. Introduction

Denote by $c_0$ be the class of all continuous functions $f: \mathbb{R} \to \mathbb{R}$, by $r_0$ the class of all right-continuous functions $f: \mathbb{R} \to \mathbb{R}$ , and by $j_0$ the class of all regulated functions $f: \mathbb{R} \to \mathbb{R}$ (a function $f: \mathbb{R} \to \mathbb{R}$ is regulated if for each point $x \in \mathbb{R}$ the both unilateral limits $f(x+)$ and $f(x−)$ exist and are finite). Moreover, let $c_1$ (resp. $r_1$ or $j_1$) be the class of all pointwise limits of sequences of functions from $c_0$ (resp. from $r_0$ or $j_0$). Similarly, if we take pointwise limits of sequences of functions from $c_1$, $r_1$ or $j_1$ we define the classes $c_2$, $r_2$ and $j_2$.

In [10], Reed obtained very interesting characterizations of $c_1$, $r_1$ and $j_1$. He proved that $c_1 \subset r_1 \subset j_1$, $c_1 \neq r_1 \neq j_1$ and $c_2 = r_2 = j_2$. Note that Reed’s considerations in [10] concern functions from $[0, 1]$ to $\mathbb{R}$, but his theorems are true for functions from $\mathbb{R}$ to $\mathbb{R}$.

In [6, 7] it is proved the Lebesgue measurability of bounded functions $g: \mathbb{R}^2 \to \mathbb{R}$ whose vertical sections $g_x(t) = g(x, t)$, $x \in \mathbb{R}$, are derivatives and horizontal sections $g^y(t) = g(t, y)$, $y \in \mathbb{R}$, belong to $c_1$.

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In [3,4] it is shown that the Continuum Hypothesis (CH) implies that there is a Lebesgue nonmeasurable function $h: \mathbb{R}^2 \to [0,1]$ with approximately continuous sections $h_x$, $x \in \mathbb{R}$, and such that for each $y \in \mathbb{R}$ the set $\mathbb{R} \setminus (h^y)^{-1}(0)$ is countable. Since every function $f: \mathbb{R} \to \mathbb{R}$ with the countable set $\mathbb{R} \setminus f^{-1}(0)$ belong to $j_1$ and every approximately bounded function is a derivative [2], we obtain that CH implies that there is a Lebesgue nonmeasurable function $h: \mathbb{R}^2 \to [0,1]$ with vertical sections $h_x$, $x \in \mathbb{R}$, being derivatives and horizontal sections $h^y$, $y \in \mathbb{R}$, belonging to $j_1$.

Hence the following natural question arises.

**Problem.** Let a bounded function $h: \mathbb{R}^2 \to \mathbb{R}$ be such that the vertical sections $h_x$, $x \in \mathbb{R}$, are derivatives and the vertical sections $h^y$, $y \in \mathbb{R}$, belong to $r_1$. Is the function $h$ Lebesgue measurable?

In this article I prove that the answer is affirmative.

2. Main results

In [7] I introduce the following property (G) for the investigation of the Lebesgue measurability of functions of two variables. This definition bases on the notion of the density topology $T_d$ [2].

For a point $x \in \mathbb{R}$ and for a Lebesgue measurable set $A \subset \mathbb{R}$ we define the lower density $D_l(A,x)$ of $A$ at $x$ as

$$\liminf_{h \to 0^+} \frac{\mu(A \cap [x-h,x+h])}{2h},$$

where $\mu$ denotes the Lebesgue measure on $\mathbb{R}$. If $D_l(A,x) = 1$ then $x$ is called a density point of $A$. If $B$ is arbitrary subset of $\mathbb{R}$ then $x$ is said a density point of $B$ if there is a Lebesgue measurable subset $A \subset B$ with $D_l(A,x) = 1$. A nonempty set $B \subset \mathbb{R}$ belongs to the density topology $T_d$ if every point $x \in B$ is a density point of $B$. All sets belonging to $T_d$ are Lebesgue measurable [2].

**Definition.** A function $f: \mathbb{R} \to \mathbb{R}$ has the property (G) if for each nonempty set $A \in T_d$ and each real $\eta > 0$ there is an open interval $I$ with $I \cap A \neq \emptyset$ such that the diameter $d(f(I \cap A))$ of the image of $f(I \cap A)$ is less than $\eta$.

**Theorem 2.1.** If a function $f: \mathbb{R} \to \mathbb{R}$ belongs to $r_1$ then it has the property (G).

**Proof.** Let $A \in T_d$ be a nonempty set and let $\eta$ be a positive real. Since $f \in r_1$, there is a sequence of continuous on the right functions $f_n: \mathbb{R} \to \mathbb{R}$ which pointwise converges to $f$. For each point $x \in \mathbb{R}$ we find a positive integer $n(x)$ such that

$$|f_k(x) - f(x)| < \frac{\eta}{30} \quad \text{for } k \geq n(x).$$
Denote by $B$ the closure of the set $A$ and for $k \geq 1$ let $A_k = \{ x \in B : n(x) = k \}$. Since $B$ is of the second category in itself and since $B = \bigcup_{k \geq 1} A_k$, there is a positive integer $m$ such that $A_m$ is of the second category in $B$. So there is an open interval $I_1$ such that $I_1 \cap B \neq \emptyset$ and the intersection $I_1 \cap A_m$ is dense in $I_1 \cap B$. Since $B = \text{cl}(A)$, there is a point $u \in A \cap I_1$. Let $i = \max(m, n(u))$. From the continuity on the right of the function $f_i$ it follows that there is an open interval $I \subset I_1$ for which $u$ is the left endpoint, and such that $|f_i(t) - f_i(u)| \leq \frac{\eta}{30}$ for $t \in I$. Since $u$ is a density point of $A$, the intersection $I \cap A \neq \emptyset$. For $w \in I \cap A_m$ we have

$$|f(w) - f(u)| \leq |f(w) - f_i(w)| + |f_i(w) - f_i(u)| + |f_i(u) - f(u)|$$

$$(*)$$

We will prove that

$$f(t) \in \left[ f(u) - \frac{\eta}{3}, f(u) + \frac{\eta}{3} \right] \quad \text{for} \quad t \in I \cap A.$$  

Suppose, contrary to our claim, that $|f(s) - f(u)| > \frac{2}{3}$ for some point $s \in A \cap I$. Let $j > i$ be a positive integer such that $|f_k(s) - f(s)| < \frac{\eta}{30}$ for $k \geq j$. From the continuity on the right of the function $f_j$ it follows that there is an open interval $K \subset I$ with the left endpoint $s$ such that $|f_j(t) - f_j(s)| < \frac{\eta}{30}$ for $t \in K$. Since $s$ is a density point of $A$, the set $K \cap A \neq \emptyset$. But $K \subset I$, so the intersection $K \cap A_m$ is dense in $K \cap A$. Consequently, there is a point $w_1 \in A_m \cap I$. We have

$$|f(w_1) - f(s)| \leq |f(w_1) - f_j(w_1)| + |f_j(w_1) - f_j(s)| + |f_j(s) - f(s)|$$

$$< \frac{\eta}{30} + \frac{\eta}{30} + \frac{\eta}{30} = \frac{\eta}{10}.$$  

Thus

$$|f(w_1) - f(u)| = |(f(s) - f(u)) + (f(w_1) - f(s))|$$

$$\geq |f(s) - f(u)| - |f(w_1) - f(s)|$$

$$> \frac{\eta}{3} - \frac{\eta}{10}$$

$$> \frac{\eta}{10},$$

contradicting $(*)$. So the oscillation of $f$ on $I \cap A$ is $\leq \frac{2\eta}{3} < \eta$ and $f$ has the property (G).  

From the above Theorem 2.1 and from [7, Theorem 4] we obtain the following theorem.
Theorem 2.2. Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be a bounded function such that the vertical sections \( g_x, x \in \mathbb{R} \), are derivatives and the horizontal sections \( g^y, y \in \mathbb{R} \), belong to \( r_1 \). Then the function \( g \) is measurable in the sense of Lebesgue.

The continuity of functions \( f : \mathbb{R} \to \mathbb{R} \) considered as some applications from \((\mathbb{R}, T_d)\) to \((\mathbb{R}, T_e)\), where \( T_e \) denotes the natural topology in \( \mathbb{R} \), is said approximate continuity [2]. Since bounded approximately continuous functions are derivatives [2], from the above Theorem 2.2 we obtain the following.

Theorem 2.3. Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be a bounded function. If the vertical sections \( g_x, x \in \mathbb{R} \), are approximately continuous and the horizontal sections \( g^y, y \in \mathbb{R} \), belong to \( r_1 \), then \( g \) is measurable in the sense of Lebesgue.

Observe that in Theorem 2.3 the condition of boundedness of the function \( g \) can be omitted, since the class of approximately continuous functions and \( r_1 \) are both invariant under outer homeomorphisms.

3. Final observations

3.1. Property (K). Earlier in [5] I introduce the property (K) which is more special than the property (G). A function \( f : \mathbb{R} \to \mathbb{R} \) has the property (K) if for each nonempty closed set \( A \subset \mathbb{R} \) such that for each open interval \( I \) with \( I \cap A \neq \emptyset \), the intersection \( I \cap A \) is of positive Lebesgue measure, the restricted function \( f|_A \) is continuous at a point \( x \in A \). Evidently all functions from \( c_1 \) have the property (K) and if a function \( f \) has the property (K) then it has also the property (G).

However, there are functions \( f \in r_1 \) without the property (K).

Example 3.1. Let \( A \subset (0,1) \) be a nonempty nowhere dense closed set such that for each open interval \( I \) with \( I \cap A \neq \emptyset \), the intersection \( I \cap A \) is of positive Lebesgue measure. If \( x \in A \) is isolated in \( A \) from the right then we put \( f(x) = 1 \). For other points \( x \in \mathbb{R} \) we put \( f(x) = 0 \). Then evidently \( f \in r_1 \), but \( f \) does not have the property (K).

3.2. Lebesgue sup-measurability. Recall that a function \( g : \mathbb{R}^2 \to \mathbb{R} \) is said to be Lebesgue sup-measurable if for each Lebesgue measurable function \( f : \mathbb{R} \to \mathbb{R} \) the Carathéodory superposition \( x \mapsto g(x, f(x)) \) is Lebesgue measurable [7]. It is known that the Lebesgue measurability of a bounded function \( g : \mathbb{R}^2 \to \mathbb{R} \) with the vertical sections \( g_x, x \in \mathbb{R} \), being derivatives implies its Lebesgue sup-measurability [8]. So from Theorem 2.2 we obtain the following.

Theorem 3.2. Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be a bounded function such that the vertical sections \( g_x, x \in \mathbb{R} \), are derivatives and the horizontal sections \( g^y, y \in \mathbb{R} \), belong to \( r_1 \). Then the function \( g \) is Lebesgue sup-measurable.
In [7] it is shown an example of Lebesgue measurable bounded function \( g: \mathbb{R}^2 \to \mathbb{R} \) with constant horizontal sections \( g^y, y \in \mathbb{R} \), and almost everywhere continuous (so having the property (K)) vertical sections \( g_x, x \in \mathbb{R} \), which is not Lebesgue sup-measurable.

On the other hand Borel functions are Lebesgue sup-measurable and bounded functions \( g: \mathbb{R}^2 \to \mathbb{R} \), whose the vertical sections \( g_x, x \in \mathbb{R} \), belong to \( c_1 \) and whose the horizontal sections \( g^y, y \in \mathbb{R} \), are approximately continuous, are Borel functions of Baire class 2 [9]. On applying the same argument as in the proof of Theorem 1 from [9] we obtain that bounded functions \( g: \mathbb{R}^2 \to \mathbb{R} \), whose the vertical sections \( g_x, x \in \mathbb{R} \), belong to \( c_1 \) and whose the horizontal sections \( g^y, y \in \mathbb{R} \), are derivatives, are Borel functions of Baire class 2.

In this situation the following natural problems are open.

**Problems.** Let \( g: \mathbb{R}^2 \to \mathbb{R} \) be a bounded function whose the vertical sections \( g_x \in c_1 \) for \( x \in \mathbb{R} \) and the horizontal sections \( g^y, y \in \mathbb{R} \), are derivative. Is \( g \)

(1) a Borel function?

(2) a Lebesgue sup-measurable function?

In the investigation of the sup-measurability very important role play the numerous contributions of Isaak V. Shragin. In particular, Shragin obtained many closely related results on sup-measurable functions which should be compared with Theorem 3.2 (see for example [11]). Moreover, the book [1] contains a whole chapter dedicated to this topic.

**References**


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