On a Singular Logistic Equation with the $p$-Laplacian

Dang Dinh Hai

Abstract. We prove the existence and nonexistence of positive solutions for the boundary value problems

\[
\begin{cases}
-\Delta_p u = g(x,u) - \frac{h(x)}{u^\alpha} & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$, $\alpha \in (0, 1)$, $g : \Omega \times (0, \infty) \to \mathbb{R}$ is possibly singular at $u = 0$. An application to a singular logistic-like equation is given.

Keywords. Sup-supersolutions, singular, positive solutions

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1. Introduction

Consider the boundary value problem

\[
\begin{cases}
-\Delta_p u = g(x,u) - \frac{h(x)}{u^\alpha} & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$, $h : \Omega \to \mathbb{R}$, $g : \Omega \times (0, \infty) \to \mathbb{R}$, and $0 < \alpha < 1$.

In [4, Theorem 5.3], Drabek and Hernandez show that the logistic equation involving the $p$-Laplacian

\[
\begin{cases}
-\Delta_p u = \lambda m(x)u^{p-1} - u^{\gamma-1} & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

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where \( 1 < p < \gamma, m \in L^r(\Omega), r > \frac{N(\gamma-1)}{p(\gamma-p)}, m(x) \geq m_0 > 0 \) in \( \Omega \), has a unique positive solution \( u \) with \( u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \) for \( \lambda > \lambda_1 \). Here \( \lambda_1 \) denotes the first eigenvalue of
\[
\begin{cases}
-\Delta_p u = \lambda m(x) |u|^{p-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\] (3)

Note that the nonlinearity \( g(x,u) = \lambda m(x) u^{p-1} - u^{\gamma-1} \) is continuous in \( u \) for a.e. \( x \in \Omega \), and satisfies
\[
\lim_{u \to \infty} \frac{g(x,u)}{m(x)u^{p-1}} = -\infty, \quad \lim_{u \to 0^+} \frac{g(x,u)}{m(x)u^{p-1}} = \lambda > \lambda_1.
\] (4)

uniformly for \( x \in \Omega \).

When \( p = 2 \), Lee et al. [8] consider the singular problem
\[
\begin{cases}
-\Delta u = \lambda u - f(u) - \frac{c}{u^\alpha} & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\] (5)

where \( \lambda, c, \alpha \) are positive constants with \( \alpha < 1 \), \( f : [0, \infty) \to \mathbb{R} \) is continuous and satisfies
\[
\lambda u - M \leq f(u) \leq Au^q
\]
for all \( u \geq 0 \), where \( M, A, q \) are positive constants with \( q > 1 \). Under these assumptions, they show that (5) has a solution \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) for \( \lambda > \frac{2\lambda_1}{1+\alpha} \) and \( c \) is sufficiently small [8, Theorem 2.1]. Here \( \lambda_1 \) corresponds to \( m(x) \equiv 1 \).

Note that the nonlinearity \( g(u) = \lambda u - f(u) \) is continuous and satisfies
\[
\limsup_{u \to \infty} \frac{g(u)}{u} \leq 0, \quad \liminf_{u \to 0^+} \frac{g(u)}{u} \geq \lambda > \frac{2\lambda_1}{1+\alpha}.
\] (6)

Note that for \( f(u) = u^q \), (5) is a singular perturbation problem of (2) with \( p = 2 \) and \( m(x) \equiv 1 \), but the result in [8] is not as good as the corresponding one in [4] when \( c = 0 \). In this paper, we shall study positive solutions to the general problem (1) when \( h \) is a bounded function with small \( \sup_{\Omega} h \) and \( g(\cdot,u) \) is allowed to be singular at \( u = 0 \) and satisfies a weaker condition than (4) and (6). To be precise, we shall assume the following:

(A1) \( m \in L^\infty(\Omega) \) and there exists a constant \( m_0 > 0 \) such that \( m(x) \geq m_0 \) for a.e. \( x \in \Omega \).

(A2) \( g : \Omega \times (0, \infty) \to \mathbb{R} \) is continuous and
\[
\limsup_{u \to \infty} \frac{g(x,u)}{m(x)u^{p-1}} < \lambda_1, \quad \liminf_{u \to 0^+} \frac{g(x,u)}{m(x)u^{p-1}} > \lambda_1
\]
uniformly for \( x \in \Omega \).
(A3) There exists $\alpha \in (0, 1)$ such that
\[
\limsup_{u \to 0^+} u^\alpha g(x, u) < \infty
\]
uniformly for $x \in \Omega$.

In particular, our result can be applied to the following singular perturbation problem of (2)
\[
\begin{cases}
-\Delta_p u = \lambda m(x) u^{p-1} - u^{\gamma-1} - \frac{h(x)}{u^\alpha} & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where $1 < p < \gamma$, $\alpha \in (0, 1)$, $m$ is as above, gives the existence of a positive solution $u \in C^{1, \beta}(\bar{\Omega})$ for some $\beta \in (0, 1)$ when $\lambda > \lambda_1$ and $\sup_{\Omega} h$ is sufficiently small. Also, if $h$ is a constant, there exists a constant $h^* > 0$ such that (7) has a positive solution for $h < h^*$ and no positive solution for $h > h^*$.

Our main result complements the result in [4] and improves the corresponding result in [8] in many ways. Our approach is based on the method of sub- and supersolutions developed in [6] for singular problems. However, the type of nonlinearities $g(u)$ covered in [6] does not apply here as it requires
\[
\lim_{u \to \infty} \frac{g(u)}{u^{p-1}} = 0 \quad \text{and} \quad g(u) > 0 \quad \text{for } u \text{ large},
\]
whereas the one in this paper allows
\[
\lim_{u \to \infty} \frac{g(u)}{u^{p-1}} = -\infty \quad \text{and} \quad g(u) \to -\infty \quad \text{as } u \to \infty.
\]

Let $\lambda_1$ be the first eigenvalue of (3) with a positive, normalized corresponding eigenfunction $\phi_1$, i.e., $||\phi_1||_\infty = 1$. It is well known that $\lambda_1 > 0$, $\phi_1 \in C^1(\bar{\Omega})$, $\frac{\partial \phi_1}{\partial n} < 0$ on $\partial \Omega$, where $n$ denotes the outer unit normal vector on $\partial \Omega$ (see [1]).

By a positive solution of (1) we mean a function $u \in C^{1, \beta}(\bar{\Omega})$ for some $\beta \in (0, 1)$ with $u = 0$ and $\frac{\partial u}{\partial n} < 0$ on $\partial \Omega$ such that
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \xi \, dx = \int_{\Omega} \left( g(x, u) - \frac{h(x)}{u^\alpha} \right) \xi \, dx
\]
for all $\xi \in W_0^{1,p}(\Omega)$. Here $n$ denotes the outer unit normal vector. Our main result is

**Theorem 1.1.** Let $h \in L^\infty(\Omega)$ and suppose (A1)-(A3) hold. Then there exists a constant $\eta > 0$ such that Problem (1) has a positive solution when $\sup_{\Omega} h < \eta$. Moreover, if $h$ is a constant, then there exists a positive number $h^*$ such that (1) has a positive solution for $h < h^*$ and no positive solutions for $h > h^*$. 
2. Preliminary results

We shall denote the norms in $L^p(\Omega)$, $W^{1,p}_0(\Omega)$, $C^1(\Omega)$ and $C^{1,\alpha}(\overline{\Omega})$ by $\| \cdot \|_{p}$, $\| \cdot \|_{1,p}$, $\| \cdot \|$ and $\| \cdot \|_{1,\alpha}$, respectively.

For $x \in \Omega$, let $d(x)$ denote the distance from $x$ to $\partial \Omega$. The following regularity result in [6, Lemma 3.1] plays a key role in the proof of the main results:

**Lemma 2.1.** Let $h \in L^\infty_{\text{loc}}(\Omega)$ and suppose there exist numbers $\alpha \in (0, 1)$ and $C > 0$ such that

$$|h(x)| \leq C d^{\alpha}(x)$$

for a.e. $x \in \Omega$. Let $u \in W^{1,p}_0(\Omega)$ be the solution of

$$\begin{cases}
-\Delta_p u = h & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$

Then there exist constants $\beta \in (0, 1)$ and $M > 0$ depending only on $C, \alpha, \Omega$ such that $u \in C^{1,\beta}(\overline{\Omega})$ and $\|u\|_{1,\beta} < M$.

**Remark 2.2.** (i) Since $\frac{\partial \phi_1}{\partial n} < 0$ on $\partial \Omega$, there exists a constant $k > 0$ such that $\phi_1(x) \geq kd(x)$ for $x \in \Omega$. Hence Lemma 2.1 holds if (8) is replaced by

$$|h(x)| \leq \frac{C}{\phi_1^\delta(x)}$$

for a.e. $x \in \Omega$.

(ii) Note that under the assumptions of Lemma 2.1, (9) has a unique solution $u \in W^{1,p}_0(\Omega)$. Indeed, define $A: W^{1,p}_0(\Omega) \to W^{-1,p'}_0(\Omega)$ and $\hat{h} \in W^{-1,p'}_0(\Omega)$, where $p' = \frac{p}{p-1}$, by

$$\langle Au, \xi \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \xi \, dx, \quad \hat{h}(\xi) = \int_\Omega h \xi \, dx.$$

By Hardy’s inequality (see e.g. [2, p. 194]), we obtain

$$|\hat{h}(\xi)| \leq C \int_\Omega \left| \frac{\xi}{d^\alpha} \right| dx \leq C \|d\|_{\infty}^{1-\alpha} \int_\Omega \left| \frac{\xi}{d} \right| dx \leq \tilde{C} \|\xi\|_{1,p}$$

for all $\xi \in W^{1,p}_0(\Omega)$, where $\tilde{C}$ is a constant independent of $\xi$. Thus $\hat{h} \in W^{-1,p'}_0(\Omega)$. Since $A$ is continuous, coercive, and strictly monotone, it follows from the Minty-Browder Theorem (see [2, p. 88]) that there exists a unique $u \in W^{1,p}_0(\Omega)$ such that $Au = \hat{h}$. 

Lemma 2.3. Let $\varepsilon > 0$ and let $h, h_\varepsilon \in L^\infty_{\text{loc}}(\Omega)$ satisfy (8). Let $u, u_\varepsilon \in W^{1,p}_0(\Omega)$ be the solutions of
\[
\begin{cases}
-\Delta_p u = h & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
and
\[
\begin{cases}
-\Delta_p u_\varepsilon = h_\varepsilon & \text{in } \Omega \\
u_\varepsilon = 0 & \text{on } \partial \Omega,
\end{cases}
\]
respectively. Suppose $||h_\varepsilon - h||_1 \to 0$ as $\varepsilon \to 0$. Then
\[
|u_\varepsilon - u|_1 \to 0
\]
as $\varepsilon \to 0$.

Proof. By Lemma 2.1, there exist $\beta \in (0, 1)$, $M > 0$ such that $u, u_\varepsilon \in C^{1,\beta}(\bar{\Omega})$ and
\[
|u|_{1,\beta}, |u_\varepsilon|_{1,\beta} < M. \tag{10}
\]
Multiplying the equation $-\Delta_p u_\varepsilon - (-\Delta_p u) = h_\varepsilon - h$ in $\Omega$ by $u_\varepsilon - u$ and integrating, we obtain
\[
\int_{\Omega} (||\nabla u_\varepsilon||^{p-2}\nabla u_\varepsilon - ||\nabla u||^{p-2}\nabla u) \cdot (\nabla u_\varepsilon - \nabla u) \, dx \leq 2M||h_\varepsilon - h||_1. \tag{11}
\]
By [9, Lemma 30.1], for $x, y \in \mathbb{R}^n$,
\[
(|x| + |y|)^{2-\min(p,2)}(|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y) \geq C_0|x - y|^{\max(p,2)}, \tag{12}
\]
where $C_0 = \left(\frac{1}{2}\right)^{p-1}$ if $p \geq 2$, $C_0 = p - 1$ if $p < 2$.

Using (12) with $x = \nabla u_\varepsilon, y = \nabla u$ and the fact that $|x| + |y| < 2M$, we obtain from (11) that
\[
C_1 \int_{\Omega} |\nabla (u_\varepsilon - u)|^q \, dx \leq 2M||h_\varepsilon - h||_1,
\]
where $C_1 = C_0 \cdot (2M)^{\min(p,2) - 2}$ and $q = \max(p, 2)$.

Hence, by Poincaré’s inequality,
\[
||u_\varepsilon - u||_q \to 0 \tag{13}
\]
as $\varepsilon \to 0$. Suppose $|u_\varepsilon - u|_1 \neq 0$ as $\varepsilon \to 0$. Then there exists a sequence $(\varepsilon_n)$ which converges to 0 such that
\[
|u_{\varepsilon_n} - u|_1 \neq 0 \quad \text{as } n \to \infty. \tag{14}
\]
By (10), $(u_{\varepsilon_n})$ is bounded in $C^{1,\beta}(\bar{\Omega})$, and since $C^{1,\beta}(\bar{\Omega})$ is compactly embedded in $C^1(\bar{\Omega})$, there exist $v \in C^1(\bar{\Omega})$ and a subsequence $(u_{\varepsilon_{n_k}})$ of $(u_{\varepsilon_n})$ such that
\[
|u_{\varepsilon_{n_k}} - v|_1 \to 0 \quad \text{as } k \to \infty. \tag{15}
\]
From (13) and (15), we see that $u = v$ and so $|u_{\varepsilon_{n_k}} - u|_1 \to 0$ as $k \to \infty$, a contradiction with (14). This completes the proof of Lemma 2.3. \qed
Next, we recall some results in sub- and supersolutions method for singular boundary value problems in [6, Appendix A]. Related results can be found in [3]. Consider the problem
\begin{equation}
\begin{cases}
-\Delta_p u = h(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}
where \( h : \Omega \times (0, \infty) \to \mathbb{R} \) is continuous.

Let \( \phi, \psi \in C^1(\bar{\Omega}) \). Suppose there exist constants \( c_0, C, \alpha > 0 \) with \( \alpha < 1 \) such that \( \phi(x), \psi(x) \geq c_0d(x) \) for \( x \in \Omega \) and
\begin{equation}
|h(x, w)| \leq \frac{C}{d^\alpha(x)}
\end{equation}
for a.e. \( x \in \Omega \) and all \( w \in C(\bar{\Omega}) \) with \( \phi \leq w \leq \psi \) in \( \Omega \). Suppose \( \phi, \psi \) are sub- and supersolutions of (16) respectively, i.e., for all \( \xi \in W^{1,p}_0(\Omega) \) with \( \xi \geq 0 \),
\begin{align*}
\int_\Omega |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla \xi \, dx &\leq \int_\Omega h(x, \phi) \xi \, dx , \\
\int_\Omega |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla \xi \, dx &\geq \int_\Omega h(x, \psi) \xi \, dx ,
\end{align*}
and \( \phi \leq 0 \leq \psi \) on \( \partial \Omega \). Note that the integrals on the right-hand side are defined by virtue of Hardy’s inequality.

**Lemma 2.4.** Under the above assumptions, there exists a constant \( \beta \in (0, 1) \) such that (16) has a solution \( u \in C^{1,\beta}(\bar{\Omega}) \) with \( \phi \leq u \leq \psi \) in \( \Omega \).

### 3. Proof of the main result

Now, we are ready to give the proof of the main result.

**Proof of Theorem 1.1.** By (A2), there exists \( \lambda_0 > \lambda_1 \) and \( \delta_0 > 0 \) such that
\begin{equation}
g(x, u) \geq \lambda_0 m(x)u^{p-1}
\end{equation}
for \( x \in \Omega \) and \( u \in (0, \delta_0] \). Choose \( \delta \in (0, 1) \) so that \( \lambda_0 \delta^{p-1} > \lambda_1 \).

For \( \varepsilon > 0 \), let \( z_\varepsilon > 0 \) be the solution of
\[-\Delta_p z_\varepsilon = h_\varepsilon \equiv \begin{cases}
\lambda_1 m(x)(\delta_0 \phi_1)^{p-1} & \text{in } \{ \phi_1 > \varepsilon \} \\
-\phi_1^{-\alpha} & \text{in } \{ \phi_1 < \varepsilon \},
\end{cases} \quad z_\varepsilon = 0 \quad \text{on } \partial \Omega.
\]
Note that the existence of \( z_\varepsilon \) follows from Lemma 2.1 and Remark 2.2. Since
\[-\Delta_p (\delta_0 \phi_1) = h \equiv \lambda_1 m(x)(\delta_0 \phi_1)^{p-1} \quad \text{in } \Omega,
\]
the weak maximum principle [10, Lemma A.2] implies $z_\varepsilon \leq \delta_0 \phi_1 \leq \delta_0$ in $\Omega$.

Next,

$$||h_\varepsilon - h||_1 = \int_{\phi_1 < \varepsilon} |\lambda_1 m(x)(\delta_0 \phi_1)^{p-1} + \phi_1^{-\alpha}| \, dx \leq C_0 \int_{\phi_1 < \varepsilon} \phi_1^{-\alpha} \, dx$$

and since $\int_\Omega \phi_1^{-\alpha} \, dx < \infty$ (see [7, p. 726], it follows that $||h_\varepsilon - h||_1 \to 0$ as $\varepsilon \to 0$. By Lemma 2.3, $|z_\varepsilon - \delta_0 \phi_1|_1 \to 0$ as $\varepsilon \to 0$. Hence $|z_\varepsilon - \delta_0 \phi_1|_1 < \frac{\delta_0(1-\delta)}{k}$, if $\varepsilon$ is sufficiently small, where $k > 0$ is such that $\frac{d \phi_1}{\phi_1} \leq k$ in $\Omega$.

By the Mean Value Theorem,

$$|z_\varepsilon(x) - \delta_0 \phi_1(x)| \leq \frac{\delta_0(1-\delta)}{k} d(x) \leq \delta_0(1-\delta)\phi_1(x)$$

for $x \in \Omega$, which implies

$$z_\varepsilon \geq \delta \delta_0 \phi_1 \text{ in } \Omega \tag{19}$$

if $\varepsilon$ is sufficiently small, which we assume.

Suppose $\sup_\Omega h < \eta$, where

$$\eta = \min \left\{ \left( \lambda_0 \delta^{p-1} - \lambda_1 \right) m_0 \delta^\alpha (\delta_0 \varepsilon)^{p-1+\alpha}, (\delta \delta_0)^\alpha \right\}.$$ We shall verify that $z_\varepsilon$ is a subsolution of (1). Let $\xi \in W^{1,p}_0(\Omega)$ with $\xi \geq 0$.

Then

$$\int_\Omega |\nabla z_\varepsilon|^{p-2} \nabla z_\varepsilon \cdot \nabla \xi \, dx = -\int_\Omega (\Delta_\mu z_\varepsilon) \xi \, dx$$

$$= \lambda_1 \int_{\phi_1 > \varepsilon} m(x)(\delta_0 \phi_1)^{p-1} \xi \, dx - \int_{\phi_1 < \varepsilon} \frac{\xi}{\phi_1^p} \, dx. \tag{20}$$

In the set $\{\phi_1 > \varepsilon\}$, we have

$$(\lambda_0 \delta^{p-1} - \lambda_1) m(x)(\delta_0 \phi_1)^{p-1} \geq (\lambda_0 \delta^{p-1} - \lambda_1) m_0 (\delta_0 \varepsilon)^{p-1} \geq \frac{\eta}{(\delta \delta_0 \varepsilon)^\alpha},$$

which, together with (18), (19), implies

$$g(x, z_\varepsilon) - \frac{h(x)}{z_\varepsilon^\alpha} \geq \lambda_0 m(x) z_\varepsilon^{p-1} - \frac{1}{z_\varepsilon^\alpha} \sup_\Omega h$$

$$\geq \lambda_0 \delta^{p-1} m(x)(\delta_0 \phi_1)^{p-1} - \frac{\eta}{(\delta \delta_0 \varepsilon)^\alpha} \tag{21}$$

in $\{\phi_1 > \varepsilon\}$. On the other hand, since $\eta \leq (\delta \delta_0)^\alpha$,

$$g(x, z_\varepsilon) - \frac{h(x)}{z_\varepsilon^\alpha} \geq -\frac{1}{z_\varepsilon^\alpha} \sup_\Omega h \geq -\frac{\eta}{(\delta \delta_0 \phi_1)^\alpha} \geq -\frac{1}{\phi_1^p} \text{ in } \Omega. \tag{22}$$
Combining (20)–(22), we obtain
\[ \int_{\Omega} |\nabla z_{\varepsilon}|^{p-2} \nabla z_{\varepsilon} \cdot \nabla \xi \, dx \leq \int_{\Omega} \left( g(x, z_{\varepsilon}) - \frac{h(x)}{z_{\varepsilon}^\alpha} \right) \xi \, dx, \]
i.e., \( z_{\varepsilon} \) is a subsolution of (1).

Next, in view of (A2) and (A3), there exist constants \( b \in (0, \lambda_1) \) and \( d_0 > 0 \) such that
\[ g(x, u) \leq bm(x)u^{p-1} + \frac{d_0}{u^\alpha} \quad \text{for all } u > 0 \text{ and a.e. } x \in \Omega. \]
Choose \( \gamma \in (0, 1) \) and \( \tilde{\lambda}_1, M_0 > 0 \) so that
\[ (1 + \gamma)^{p-1} \left( b + \frac{d_0}{m_0 M_0^{p-1+\alpha}} \right) < \tilde{\lambda}_1 < \lambda_1, \]
and
\[ \frac{(1 + \gamma)^{p-1}||h||_{\infty}}{m_0 M_0^{p-1+\alpha}} < \lambda_1 - \tilde{\lambda}_1. \]

Let \( \psi_{\varepsilon} \) be the solution of
\[-\Delta_p \psi_{\varepsilon} = \begin{cases} \lambda_1 m(x) \phi_1^{p-1} & \text{in } \{ \phi_1 > \varepsilon \}, \\ \lambda_1 m(x) + \phi_1^{-\alpha} & \text{in } \{ \phi_1 < \varepsilon \}, \end{cases} \quad \psi_{\varepsilon} = 0 \text{ on } \partial \Omega.\]

Then, since \(-\Delta_p \phi_1 = \lambda_1 m(x) \phi_1^{p-1} \) in \( \Omega \), it follows from Lemma 2.3 that \(|\psi_{\varepsilon} - \phi_1|_1 \to 0 \) as \( \varepsilon \to 0 \). Hence, if \( \varepsilon \) is small enough,
\[ (1 - \gamma) \phi_1 \leq \psi_{\varepsilon} \leq (1 + \gamma) \phi_1 \quad \text{in } \Omega, \]
which we assume. We shall verify that \( Z_{\varepsilon} = M \psi_{\varepsilon} \) is a supersolution for (1) with \( Z_{\varepsilon} \geq z_{\varepsilon} \) in \( \Omega \) if \( M \) is large enough. Let \( \xi \in W_0^{1,p}(\Omega) \) with \( \xi \geq 0 \). Then we have
\[ \int_{\Omega} |\nabla Z_{\varepsilon}|^{p-2} \nabla Z_{\varepsilon} \cdot \nabla \xi \, dx = \lambda_1 \int_{\phi_1 > \varepsilon} m(x)(M \phi_1)^{p-1} \xi \, dx \\ + M^{p-1} \int_{\phi_1 < \varepsilon} (\lambda_1 m(x) + \phi_1^{-\alpha}) \xi \, dx. \]

Suppose \( M > \frac{M_0}{(1-\gamma)\varepsilon} \). Then
\[ Z_{\varepsilon} \geq M(1 - \gamma)\varepsilon > M_0 \]
in \( \{ \phi_1 > \varepsilon \} \). Since \( M \phi_1 \geq (1 + \gamma)^{-1}Z_{\varepsilon} \) in \( \Omega \), it follows from (23)-(25) that
\[ \frac{||h||_{\infty}}{m(x)(M \phi_1)^{p-1} Z_{\varepsilon}^\alpha} \leq \frac{(1 + \gamma)^{p-1} ||h||_{\infty}}{m(x) Z_{\varepsilon}^{p-1+\alpha}} \leq \frac{(1 + \gamma)^{p-1} ||h||_{\infty}}{m_0 M_0^{p-1+\alpha}} < \lambda_1 - \tilde{\lambda}_1. \]
Hence
\[ g(x, Z_\varepsilon) - \frac{h(x)}{Z_\varepsilon^\alpha} \leq g(x, Z_\varepsilon) + \frac{||h||_\infty}{Z_\varepsilon^\alpha} \leq \lambda_1 m(x)(M\phi_1)^{p-1} \]  
(29)
in \{\phi_1 > \varepsilon\}. From (23), (24), and (26), we get
\[ g(x, Z_\varepsilon) - \frac{h(x)}{Z_\varepsilon^\alpha} \leq b m(x) Z_\varepsilon^{p-1} + d_0 + ||h||_\infty \]
\[ \leq b(1 + \gamma)^{p-1} m(x)(M\phi_1)^{p-1} + d_0 + ||h||_\infty \phi_1^{-\alpha} \]
(30)
if \( M \) is large enough so that \( M^{p-1+\alpha} > (d_0 + ||h||_\infty)(1 - \gamma)^{-\alpha} \), which we assume.

Combining (27), (29), and (30), we get
\[ \int_\Omega |\nabla Z_\varepsilon|^{p-2} \nabla Z_\varepsilon \cdot \nabla \xi \ dx \geq \int_\Omega \left( g(x, Z_\varepsilon) - \frac{h(x)}{Z_\varepsilon^\alpha} \right) \xi \ dx, \]
i.e., \( Z_\varepsilon \) is a supersolution of (1) with \( Z_\varepsilon \geq z_\varepsilon \) for large \( M \).

Finally, it follows from (A3) and (19) that there exists a constant \( K > 0 \) depending on \( ||Z_\varepsilon||_\infty \) such that
\[ |g(x, w)| \leq \frac{K}{w^\alpha} \leq \frac{K}{z_\varepsilon^\alpha} \leq \frac{K}{(\delta\delta_0\phi_1)^\alpha} \]
for all \( w \in C(\bar{\Omega}) \) with \( z_\varepsilon \leq w \leq Z_\varepsilon \) in \( \Omega \). The existence of a positive solution for (1) now follows from Lemma 2.4.

Next, suppose that \( h \) is a constant. Then, as in the above, we see that there exists a constant \( h_0 > 0 \) such that (1) has a positive solution for \( h < h_0 \). We claim that (1) has no positive solutions for large \( h \). Indeed, let \( u \) be a positive solution of (1) with \( h > 0 \). Multiplying the equation \( -\Delta_p u = g(x, u) - \frac{h}{w^\alpha} \) in \( \Omega \) by \( u \) and integrating, we obtain, by (23),
\[ \int_\Omega |\nabla u|^p \ dx = \int_\Omega g(x, u) u \ dx - h \int_\Omega u^{1-\alpha} \ dx \]
\[ \leq b \int_\Omega m(x) u^p \ dx + (d_0 - h) \int_\Omega u^{1-\alpha} \ dx \]
\[ \leq b \int_\Omega m(x) u^p \ dx \]
for \( h \geq d_0 \). Since
\[ \lambda_1 = \inf_{u \in W_0^{1,p}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^p \ dx}{\int_\Omega m(x) u^p \ dx}, \]
it follows that \((1 - \frac{b}{\lambda}) \int_{\Omega} |\nabla u|^p \, dx \leq 0\), which implies \(u \equiv 0\), a contradiction. Hence the claim is proved.

Define \(h^* = \sup \{h > 0 : (1) \text{ has a positive solution} \}\). Then \(h^* \in (0, \infty)\) and (1) has no positive solutions for \(h > h^*\). Let \(h < h^*\). Then there exists \(\tilde{h} > h\) such that (1) with \(h = \tilde{h}\) has a positive solution \(u_{\tilde{h}}\). Since

\[ g(x, u_{\tilde{h}}) - \frac{\tilde{h}}{u_{\tilde{h}}^\alpha} \leq g(x, u_{\tilde{h}}) - \frac{h}{u_{\tilde{h}}^\alpha}, \]

in \(\Omega\), it follows that \(u_{\tilde{h}}\) is a subsolution for (1). As above, we obtain a supersolution \(Z_{\epsilon}\) for (1) with \(Z_{\epsilon} \geq u_{\tilde{h}}\) in \(\Omega\), and the existence of a positive solution to (1) follows. This completes the proof of Theorem 1.1.

\[\square\]

References


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