

# 2-Modified Characteristic Fredholm Determinants, Hill's Method, and the Periodic Evans Function of Gardner

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**Abstract.** Using the relation established by Johnson-Zumbrun between Hill's method of approximating spectra of periodic-coefficient ordinary differential operators and a generalized periodic Evans function given by the 2-modified characteristic Fredholm determinant of an associated Birman-Schwinger system, together with a Volterra integral computation introduced by Gesztesy-Makarov, we give an explicit connection between the generalized Birman-Schwinger-type periodic Evans function and the standard Jost function-type periodic Evans function defined by Gardner in terms of the fundamental solution of the eigenvalue equation written as a first-order system. This extends to a large family of operators the results of Gesztesy-Makarov for scalar Schrödinger operators and of Gardner for vector-valued second-order elliptic operators, in particular recovering by independent argument the fundamental result of Gardner that the zeros of the Evans function agree in location and (algebraic) multiplicity with the periodic eigenvalues of the associated operator.

**Keywords.** Hill's method, Fredholm determinant, Evans function

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## 1. Introduction

The purpose of this note, generalizing results of [5–7] for asymptotically constant-coefficient operators and of Gesztesy-Makarov [7] for of periodic-coefficient Schrödinger operators, is to give an explicit connection between two objects related to the spectra of periodic-coefficient operators on the line, namely, a Birman-Schwinger-type *characteristic Fredholm determinant*  $D(\lambda)$  introduced in [3, 12] and the Jost function-type *periodic Evans function*  $E(\lambda)$  of Gardner [4].

Both of these objects are analytic functions whose zeros have been shown in various contexts to agree in location and multiplicity with the eigenvalues of the associated operator. Thus, on the mutual domain for which these properties

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have been established, it follows that they must agree up to a nonvanishing analytic factor. Here, we determine explicitly this nonvanishing factor, thus illuminating the relation between the two functions while extending the key property of agreement with eigenvalues to the union of the domains on which it has been established for each function separately. Knowledge of this scaling factor is useful also for numerical approximation of spectra as in [1–3], allowing convenient comparison of different methods.

Our method of proof proceeds by comparison with an interpolating first-order Birman-Schwinger determinant, using a simple version of the Volterra integral computation of [7] together with the relation established in [12] between Birman-Schwinger determinants and Hill's method, a spectral Galerkin algorithm for finite approximation of spectra. We carry out the analysis here for the simplest interesting case of a second-order elliptic operator; however, our arguments extend readily to the full class of operators discussed in [12].

Consider the eigenvalue problem for a general second-order periodic coefficient ordinary differential operator  $L = B_0(x)^{-1}(\partial_x^2 + \partial_x A_1(x) + A_0(x))$ , written in form

$$(\partial_x^2 + \partial_x A_1(x) + A_0(x) - \lambda B_0(x))U = 0, \quad (1)$$

where  $L$  is defined on complex vector-valued functions  $U \in L^2[0, X]$  with periodic boundary conditions,  $A_j, B_j \in L^2$  are matrix-valued and periodic on  $x \in [0, X]$ , and  $B_0$  is positive or negative definite in the sense that its symmetric part  $\Re B_0 := \frac{1}{2}(B_0 + B_0^*)$  is positive or negative definite.<sup>1</sup> (Note that, by Lyapunov's Lemma, definiteness can always be achieved by a variable-coefficient change of coordinates, provided that the eigenvalues of  $B_0$  are of purely positive or purely negative real part.) Taking the Fourier transform, we may express (1), equivalently, as an infinite-dimensional matrix system

$$(\mathcal{D}^2 + \mathcal{D}\mathcal{A} + \mathcal{B})\mathcal{U} = 0, \quad (2)$$

$\mathcal{B}_{jk} = \hat{A}_{0,j-k} - \lambda \hat{B}_{0,j-k}$ ,  $\mathcal{A}_{jk} = \hat{A}_{1,j-k}$ ,  $\mathcal{D}_{jk} = \frac{2\pi}{X} \delta_j^k i j$ ,  $\mathcal{U}_j = \hat{U}(j)$ , where  $\hat{f}$  denotes the discrete Fourier transform of  $f$ . (Here and elsewhere  $i = \sqrt{-1}$ .) We shall alternate between these two representations as is convenient.

## 2. The Birman-Schwinger Evans function

By a Birman-Schwinger-type procedure, applying  $(\partial_x^2 - 1)^{-1}$  to (1) on the left, we obtain an equivalent problem

$$(I + K(\lambda))U = 0,$$

where  $K = K_1 + K_0$ , with  $K_1 = \partial_x(\partial_x^2 - 1)^{-1}A_1$ ,  $K_0 = (\partial_x^2 - 1)^{-1}(A_0 + 1 - \lambda B_0)$ .

<sup>1</sup>The assumption of symmetry of  $B_0$  made in [12] is not necessary for the arguments there.

**Lemma 2.1** ([12]). *For  $A_j, B_j \in L^2$ , the operator  $K$  is Hilbert-Schmidt.*

*Proof.* Expressing  $K$  in matrix form  $\mathcal{K}$  with respect to the infinite-dimensional Fourier basis  $\left\{ \frac{\exp(\frac{2\pi i j x}{X})}{\sqrt{X}} \right\}$ , we find that  $\mathcal{K}_{1,jk} = \frac{\frac{2\pi i j}{X}}{1+(\frac{2\pi}{X})^2 j^2} \hat{A}_{1,j-k}$ , and similarly for  $\mathcal{K}_2$ , where  $\hat{A}_j$  denotes Fourier transform and  $i := \sqrt{-1}$ . Taking without loss of generality  $X = 2\pi$ , we find that

$$\|K_1\|_{\mathcal{B}_2} = \|\mathcal{K}_1\|_{\mathcal{B}_2} = \sum_j \frac{j^2}{(1+j^2)^2} \sum_k |A_1(j-k)|^2 = \frac{j^2}{(1+j^2)^2} \|A_1\|_{L^2(x)} < +\infty,$$

and similarly for  $\mathcal{K}_2$ . □

**Definition 2.2.** We define the *Birman-Schwinger Evans function* as

$$D(\lambda) := \det_2(I - K(\lambda)),$$

where  $\det_2$  denotes the 2-modified Fredholm determinant, defined for Hilbert-Schmidt perturbations of the identity; see Appendix A for a review of the relevant theory.

**Proposition 2.3** ([3, 12]). *For  $A_j, B_j \in L^2$ ,  $D$  is analytic in  $\lambda$ ; moreover, its zeros correspond in location and multiplicity with the eigenvalues of  $L := B_0^{-1}(\partial_x^2 + \partial_x A_1 + A_0)$ .*

*Proof.* Analyticity follows by analyticity of the finite-dimensional Galerkin approximations by which the Fredholm determinant is defined, plus uniform convergence of the Galerkin approximations, a consequence of (15). Correspondence in location is immediate from the fact that  $\det_2(I - K) = 0$  if and only if  $(I - K)$  has a kernel, while correspondence in multiplicity may be deduced by consideration of a special sequence of Galerkin approximations on successive eigenspaces of  $L$ , for which the Galerkin approximant can be seen to be a non-vanishing multiple of the characteristic polynomial for the restriction of  $L$  to these finite-dimensional invariant subspaces. See [12] for further details. □

### 3. Hill’s method

Hill’s method consists of truncating (2) at wave number  $J$ , i.e., considering the  $(2j + 1)$ -dimensional minor  $|j| \leq J$ , and solving the resulting finite-dimensional system to obtain approximate eigenvalues for  $L$ : that is, the eigenvalues of the  $(2J + 1) \times (2J + 1)$  matrix

$$L_J = (\mathcal{B}_{0,J})^{-1}(\mathcal{D}_J^2 + \mathcal{D}_J \mathcal{A}_J + \mathcal{B}_J),$$

$\mathcal{B}_{0,J,jk} = \hat{B}_{0,j-k}$ , where subscripts  $J$  indicate truncation at wave number  $J$ , or, equivalently, restriction to the  $J$ th centered minor as described above.

Left-multiplying by  $(\mathcal{D}_J^2 - 1)^{-1}$ , similarly as above, we may rewrite the truncated eigenvalue equation  $(L_J - \lambda)U = (\mathcal{D}_J^2 + \mathcal{D}_J \mathcal{A}_J + \mathcal{B}_J)U = 0$  equivalently as  $(I + \mathcal{K}_J)U = 0$ , where  $\mathcal{K}_J = \mathcal{K}_{1,J} + \mathcal{K}_{2,J}$  is the truncation of the Fourier representation  $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$  of operator  $K$ , that is,  $\mathcal{K}_1 = \mathcal{D}_J(\mathcal{D}_J^2 - I)^{-1}\mathcal{A}_{1,J}$ ,  $\mathcal{K}_2 = (\mathcal{D}_J^2 - I)^{-1}(\mathcal{A}_{0,J} + 1 - \lambda\mathcal{B}_{0,J})$ . We define the *truncated Birman-Schwinger Evans function*, accordingly, as

$$D_J(\lambda) := \det_2(I - \mathcal{K}_J) = \det(\mathcal{D}_J^2 - 1) \det \mathcal{B}_{0,J} e^{\text{tr} \mathcal{K}_J(\lambda)} \det(L_J - \lambda). \quad (3)$$

**Proposition 3.1** ([12]). *Each  $D_J$  is analytic in  $\lambda$ ; moreover, the zeros of  $D_J$  correspond in location and multiplicity with those of  $L_J$ .*

*Proof.* This follows immediately by (3), the assumed uniform positive definiteness of  $B_0$ , hence also of  $\mathcal{B}_0$  and  $\mathcal{B}_{0,J}$ , and properties of the characteristic polynomial. □

**Proposition 3.2** ([12]). *For  $A_j, B_j \in L^2$ ,  $D_J(\lambda) \rightarrow D(\lambda)$  as  $J \rightarrow \infty$ , uniformly on  $|\lambda| \leq R$ .*

*Proof.* By diagonality of  $\mathcal{D}$ ,  $\mathcal{D}_J \mathcal{A}_J = (\mathcal{D} \mathcal{A})_J$ , hence  $D_J$  is a sequence of Galerkin approximations of the Fredholm determinant  $D = \det_2(I - \mathcal{K})$  on a complete set of successively larger subspaces. By definition of the Fredholm determinant, therefore,  $D_J \rightarrow D$  as  $J \rightarrow \infty$ . □

**Corollary 3.3** (Convergence of Hill’s method [12]). *For  $A_j, B_j \in L^2$ , the eigenvalues of  $L_J$  approach the eigenvalues of  $L$  in location and multiplicity as  $J \rightarrow \infty$ , uniformly on  $|\lambda| \leq R$ .*

*Proof.* This follows immediately by Lemma 2.3, Propositions 3.1 and 3.2, and properties of uniformly convergent analytic functions. □

### 4. The Evans function of Gardner

Rewriting (1) as a first-order system

$$\partial_x W = A(\lambda)W, \quad W := \begin{pmatrix} U \\ U_x \end{pmatrix}, \quad \mathbb{A}(\lambda) := \begin{pmatrix} 0 & I \\ (\lambda B_0 - A_0 - \partial_x A_1) & -A_1 \end{pmatrix}, \quad (4)$$

we may alternatively define an Evans function by shooting, as done by Gardner.

**Definition 4.1.** We define the *Jost-type Evans function* of Gardner, following [4], as

$$E(\lambda) := \det(\Psi(X) - I),$$

where  $\Psi$  denotes the fundamental solution of (4), satisfying  $\Psi(0) = I$ , and  $X$  is the period.

**Proposition 4.2** ([4]). *For  $A_j, B_j \in C^1$ ,  $D$  is analytic in  $\lambda$ ; moreover, its zeros correspond in location and multiplicity with the (periodic) eigenvalues of  $L := B_0^{-1}(\partial_x^2 + \partial_x A_1 + A_0)$ .*

*Proof.* Analyticity and agreement in location follow immediately by analytic dependence of solutions of ODE and the fact that  $\lambda$  is an eigenvalue precisely if there exists a solution  $W(x) = \Psi(x)W_0$  of (4) for which  $W(X) = \Psi(X)W_0 = W_0$ . Agreement in multiplicity may be shown by a more detailed argument based on choice of a Jordan basis as in [4]. □

### 5. Connections

Define now the intermediate *first-order Birman-Schwinger Evans function*

$$F(\lambda) := \det_2((\partial_x - 1)^{-1}(\partial_x - \mathbb{A})) = \det_2(I - \mathbb{K}),$$

where  $\mathbb{K} := (\partial_x - 1)^{-1}(\mathbb{A} - 1)$ .

**Theorem 5.1.** *For  $A_j, B_j \in L^2$ ,*

$$F(\lambda) = \gamma(e^X - 1)^{-2n} E(\lambda), \tag{5}$$

where

$$\begin{aligned} \gamma &:= \exp\left(\frac{e^X}{1 - e^X} \int_0^X (\text{tr}(\mathbb{A}) - 2n)(y) dy\right) \\ &= \exp\left(\frac{e^X}{e^X - 1} (-\text{tr}(A_{1,ave}) + 2n)X\right), \end{aligned} \tag{6}$$

and  $A_{1,ave}$  denotes the mean over one period of  $A_1$ .

**Remark 5.2.** From (5), (6), we obtain  $F(\lambda) \sim E(\lambda)e^{-\int_0^X \text{tr}(\mathbb{A})(y)dy} = \det(I - \Psi(X)^{-1})$  as  $X \rightarrow \infty$ , so that  $F$  is asymptotic to a “backward” version of Gardner’s Evans function.

*Proof.* By a straightforward computation taking into account periodicity together with the jump condition  $G(y^+, y) - G(y_-, y) = 1$  at  $x = y$ , we find that the Green kernel associated with  $(\partial_x - 1)^{-1}$  is semiseparable, of form

$$G(x, y) = \begin{cases} \frac{e^{x-y}}{1 - e^X}, & x > y, \\ \frac{e^{X+x-y}}{1 - e^X}, & x < y. \end{cases}$$

Following [7], we may thus decompose  $G$  as the sum  $G(x, y) = f(x)g(y) + J(x, y)$  of a separable kernel

$$f(x)g(y) : \quad f(x) = e^x, \quad g(y) = \frac{e^X e^{-y}}{1 - e^X}, \tag{7}$$

and a Volterra kernel

$$J(x, y) = \begin{cases} e^{x-y}, & x > y, \\ 0, & x < y. \end{cases}$$

Likewise, the kernel  $\kappa(x, y)$  of  $\mathbb{K}$  decomposes as

$$\kappa(x, y) = f(x)g(y)(\mathbb{A}(y) - 1) + H(x, y),$$

where  $H(x, y) = J(x, y)(\mathbb{A}(y) - 1)$  is again a Volterra kernel, vanishing for  $x < y$ .

Using the Fredholm determinant facts (16), (17) compiled in Appendix A, together with the fact that  $\det_2(I - H) = 1$  for a Volterra operator  $H$ ,<sup>2</sup> we find (here, with slight abuse of notation, denoting operators by their associated kernels) that

$$\begin{aligned} & \det_2(I - (\partial_x - 1)^{-1}(\mathbb{A} - 1)) \\ &= \det_2(I - H - fg(\mathbb{A} - 1)) \\ &= \det_2(I - H)\det_2(I - (I - H)^{-1}fg(\mathbb{A} - 1))e^{-\text{tr}(H(I - H)^{-1}fg(\mathbb{A} - 1))} \quad (8) \\ &= \det_2(I - (I - H)^{-1}fg(\mathbb{A} - 1))e^{\text{tr}(fg(\mathbb{A} - 1))}e^{-\text{tr}((I - H)^{-1}fg(\mathbb{A} - 1))} \\ &= \gamma \det_{\mathbb{C}^{2n}}(I - \langle g(\mathbb{A} - 1), (I - H)^{-1}f \rangle), \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  denotes  $L^2[0, X]$  inner product and

$$\gamma := \exp(\text{tr}(\langle g(\mathbb{A} - 1), f \rangle)) = \exp\left(\frac{e^X}{e^X - 1}(\text{tr}(A_{1,ave}) + 2n)X\right).$$

Computing  $(I - H)\Psi(x) = \Psi(x) - \int_0^x e^{x-y}(\mathbb{A} - 1)\Psi(y)dy$ , and recalling that  $(\partial_x - 1)\Psi = (\mathbb{A} - 1)\Psi$ , where  $\Psi$  denotes the fundamental solution of  $(\partial_x - \mathbb{A})\Psi = 0$ ,  $\Psi(0) = I$ , so that, by Duhamel's principle/variation of constants,  $\Psi(x) = e^x + \int_0^x e^{x-y}(\mathbb{A} - 1)\Psi(y)dy$ , we find that  $(I - H)\Psi = e^x = f$ , or  $(I - H)^{-1}f = \Psi$ . Substituting into (8), we obtain, noting that  $\Theta := e^{-y}\Psi(y)$  satisfies  $\partial_y\Theta = (\mathbb{A} - 1)\Theta$ ,

$$\begin{aligned} \det_2(I - (\partial_x - 1)^{-1}(\mathbb{A} - 1)) &= \gamma \det_{\mathbb{C}^{2n}}\left(I - \left(\frac{e^X}{1 - e^X}\right) \int_0^X (\mathbb{A} - 1)e^{-y}\Psi(y)dy\right) \\ &= \gamma \det_{\mathbb{C}^{2n}}\left(I - \left(\frac{e^X}{1 - e^X}\right) e^{-y}\Psi(y)|_0^X\right) \\ &= \gamma \det_{\mathbb{C}^{2n}}\left(I - \left(\frac{e^X}{1 - e^X}\right) (e^{-X}\Psi(X) - I)\right) \\ &= \gamma(e^X - 1)^{-2n} \det_{\mathbb{C}^{2n}}(\Psi(X) - I), \end{aligned}$$

yielding the result. □

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<sup>2</sup>Heuristically a strictly lower triangular perturbation of the identity;  $H$  has no nonzero eigenvalues.

Now, define the *truncated first-order Birman-Schwinger Evans function* as

$$F_J(\lambda) := \det_2(I - \hat{\mathcal{K}}_J), \tag{9}$$

where  $\hat{\mathcal{K}}_J$  denotes the  $J$ th Fourier truncation of the Fourier transform  $\hat{\mathcal{K}}$  of  $\mathbb{K}$ .

**Lemma 5.3.** *For  $A_j, B_j \in L^2$ ,  $F_J(\lambda) \rightarrow F(\lambda)$  as  $J \rightarrow \infty$ , uniformly on  $|\lambda| \leq R$ .*

*Proof.* As in the argument of Proposition 3.2, this follows essentially by the definition of the Fredholm determinant.  $\square$

**Lemma 5.4.** *Similarly as in (3), we have*

$$F_J(\lambda) = \det(\mathcal{D}_J - 1)^2 \det \mathcal{B}_{0,J} e^{\text{tr} \hat{\mathcal{K}}_J(\lambda)} \det(L_J - \lambda). \tag{10}$$

*Proof.* Using  $(I - \mathbb{K}) = (\partial_x - 1)^{-1}(\partial_x - \mathbb{A})$ ,  $\mathbb{A}(\lambda) = \begin{pmatrix} 0 & I \\ (\lambda B_0 - A_0 - \partial_x A_1) & -A_1 \end{pmatrix}$  to obtain

$$(I - \hat{\mathcal{K}}_J) = \begin{pmatrix} \mathcal{D}_J - I_J & 0 \\ 0 & \mathcal{D}_J - I_J \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{D}_J & -I_J \\ \mathcal{B}_J + \widehat{(\partial_x A_1)}_J & \mathcal{D}_J + \mathcal{A}_J \end{pmatrix},$$

and, using the block determinant formula  $\det \begin{pmatrix} a & -I \\ b & c \end{pmatrix} = \det(ca + b)$ ,

$$\begin{aligned} \det \begin{pmatrix} \mathcal{D}_J & -I_J \\ \mathcal{B}_J + \widehat{(\partial_x A_1)}_J & \mathcal{D}_J + \mathcal{A}_J \end{pmatrix} &= \det ((\mathcal{D}_J + \mathcal{A})\mathcal{D}_J + \widehat{(\partial_x A_1)}_J + \mathcal{B}_J) \\ &= \det (\mathcal{D}_J^2 + \mathcal{D}_J \mathcal{A}_J + \mathcal{B}_J) \\ &= \det \mathcal{B}_{0,J} \det(L_J - \lambda). \end{aligned} \tag{11}$$

**Corollary 5.5.** *For  $A_j, B_j \in L^2$ ,  $F(\lambda) = \epsilon e^{\hat{\delta} - \delta} D(\lambda)$ , where, for  $X = 2\pi$ ,*

$$\begin{aligned} \hat{\delta} &:= \text{tr}(\hat{A}_{1,0} + 2I) \left( 1 + \sum_{1 \leq |j|} \frac{1}{j^2 + ij} \right), \\ \delta &:= -\text{tr}(\hat{A}_{1,0} + I - \lambda \hat{B}_{0,0}) \sum_j \frac{1}{j^2 + 1}, \\ \epsilon &:= \lim_{J \rightarrow \infty} \prod_{|j| \leq J} \left( 1 + \frac{2}{ij + 1} \right). \end{aligned} \tag{11}$$

*Proof.* We may readily calculate that

$$\begin{aligned} \text{tr } \mathcal{K} &= \text{tr}(\hat{A}_{1,0}) \sum_{|j| \leq J} \frac{j}{-j^2 - 1} + \text{tr}(\hat{A}_{1,0} + I - \lambda \hat{B}_{0,0}) \sum_{|j| \leq J} \frac{1}{-j^2 - 1} \\ &= -\text{tr}(\hat{A}_{1,0} + I - \lambda \hat{B}_{0,0}) \sum_{|j| \leq J} \frac{1}{j^2 + 1} \end{aligned}$$

and

$$\begin{aligned} \operatorname{tr} \hat{\mathcal{K}} &= \begin{pmatrix} -(\mathcal{D}_J - I_J)^{-1} & I \\ * & (\mathcal{D}_J - I_J)^{-1}(-A_1 - I) \end{pmatrix} \\ &= -\operatorname{tr}(\hat{A}_{1,0} + 2I) \sum_{|j| \leq J} \frac{1}{ij - 1} \\ &= -\operatorname{tr}(\hat{A}_{1,0} + 2I) \left( -1 + \sum_{1 \leq |j| \leq J} \left( \frac{1}{ij - 1} - \frac{1}{ij} \right) \right) \\ &= \operatorname{tr}(\hat{A}_{1,0} + 2I) \left( 1 + \sum_{1 \leq |j| \leq J} \frac{1}{j^2 + ij} \right), \end{aligned}$$

both uniformly convergent by absolute convergence of  $\sum \frac{1}{j^2}$ . Similarly,

$$\prod_{|j| \leq J} \frac{(ij - 1)^2}{-j^2 - 1} = \prod_{|j| \leq J} \left( 1 + \frac{2}{ij + 1} \right) \sim e^{\sum_{|j| \leq J} \frac{2}{ij+1} + O(j^{-2})} \sim e^{\sum_{|j| \leq J} O(j^{-2})}$$

may be seen to be uniformly convergent. Comparing (3) and (10) and taking the limit as  $J \rightarrow \infty$ , we obtain the result.  $\square$

Collecting information, we have our final result, relating  $D$ ,  $E$ , and  $F$ .

**Corollary 5.6.** *For  $A_j, B_j \in L^2$ , and  $X = 2\pi$ ,*

$$D(\lambda) = \frac{e^{\delta - \hat{\delta}}}{\epsilon} F(\lambda) = \frac{e^{\delta - \hat{\delta}}}{\epsilon} \gamma(e^X - 1)^{-2n} E(\lambda), \tag{12}$$

where  $\gamma$  is as in (6) and  $\delta, \hat{\delta}$ , and  $\epsilon$  are as in (11).

**Remark 5.7.** Since  $\frac{e^{\delta - \hat{\delta}}}{\epsilon} \gamma(e^X - 1)^{-2n}$  is analytic in  $\lambda$  and nonvanishing, Corollary 5.6 together with Proposition 2.3 gives an alternative proof of Proposition 4.2.

### A. 2-modified Fredholm determinants

In this appendix, we recall for completeness the basic properties of 2-modified Fredholm determinants of Hilbert-Schmidt perturbations of the identity; see [9–11, 13, 14]. For a Hilbert space  $\mathcal{H}$ , the Hilbert-Schmidt class  $\mathcal{B}_2(\mathcal{H})$  is defined as the set of linear operators on  $\mathcal{H}$  for which

$$\|A\|_{\mathcal{B}_2(\mathcal{H})} := \sum_{j,k} |\langle Ae_j, e_k \rangle|^2 = \operatorname{tr}(A^*A) < +\infty,$$



where  $\{e_j\}$  is any orthonormal basis. Evidently,  $\|\cdot\|_{\mathcal{B}_2(\mathcal{H})}$  is independent of the basis.

For a finite-rank operator  $A$ , the 2-modified Fredholm determinant is defined as

$$\det_{2,\mathcal{H}}(I_{\mathcal{H}} - A) := \det_{\mathcal{H}}((I_{\mathcal{H}} - A)e^A) = \det_{\mathcal{H}}(I_{\mathcal{H}} - A) e^{\text{tr}_{\mathcal{H}}(A)}, \quad (13)$$

where  $\det_{\mathcal{H}}$  denotes the usual determinant restricted to  $\text{Range}(A)$ . There holds

$$e^{-C\|A\|_{\mathcal{B}_2(\mathcal{H})}^2} \leq |\det_{2,\mathcal{H}}(I_{\mathcal{H}} - A)| \leq e^{C\|A\|_{\mathcal{B}_2(\mathcal{H})}^2} \quad (14)$$

and

$$|\det_{2,\mathcal{H}}(I_{\mathcal{H}} - A) - \det_{2,\mathcal{H}}(I_{\mathcal{H}} - B)| \leq \|A - B\|_{\mathcal{B}_2(\mathcal{H})} e^{C[\|A\|_{\mathcal{B}_2(\mathcal{H})} + \|B\|_{\mathcal{B}_2(\mathcal{H})} + 1]^2}, \quad (15)$$

where  $C > 0$  is a constant independent of the dimension of the space.

For  $A \in \mathcal{B}_2(\mathcal{H})$ , the 2-modified Fredholm determinant is defined as the limit

$$\det_{2,\mathcal{H}}(I_{\mathcal{H}} - A) := \lim_{J \rightarrow \infty} \det_{2,\mathcal{H}_J}(I_{\mathcal{H}_J} - A_J),$$

where  $\mathcal{H}_J$  is any increasing sequence of finite-dimensional subspaces filling up  $\mathcal{H}$ , and  $A_J$  denotes the Galerkin approximation  $P_{\mathcal{H}_J}A|_{\mathcal{H}_J}$ , where  $P_J$  is the orthogonal projection onto  $\mathcal{H}_J$ . Equivalently,  $\det_{2,\mathcal{H}}(I_{\mathcal{H}} - A) := \prod_j (1 - \alpha_j) e^{\alpha_j}$ , where  $\alpha_j$  are the eigenvalues of  $A$ . For  $A \in \mathcal{B}_2$ ,  $(I - A)$  is invertible if and only if  $\det_2(I - A) \neq 0$ .

Properties (13), (14) are evidently inherited by continuity. Likewise, from the corresponding properties of finite-dimensional determinants, we obtain in the limit

$$\det_{2,\mathcal{H}}((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - B)) = \det_{2,\mathcal{H}}(I_{\mathcal{H}} - A) \det_{2,\mathcal{H}}(I_{\mathcal{H}} - B) e^{-\text{tr}_{\mathcal{H}}(AB)} \quad (16)$$

for all  $A, B \in \mathcal{B}_2$  (here we are implicitly using the fact that  $AB$  is in trace class, with  $\|AB\|_{\mathcal{B}_1} \leq \|A\|_{\mathcal{B}_2} \|B\|_{\mathcal{B}_2}$ , so that  $\text{tr}(AB)$  is well-defined) and

$$\det_{2,\mathcal{H}'}(I_{\mathcal{H}'} - AB) = \det_{2,\mathcal{H}}(I_{\mathcal{H}} - BA) \quad (17)$$

for all  $A \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ ,  $B \in \mathcal{B}(\mathcal{H}', \mathcal{H})$  such that  $BA \in \mathcal{B}_2(\mathcal{H})$ ,  $AB \in \mathcal{B}_2(\mathcal{H}')$ .

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