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On the Convergence to Stationary Solutions for a Semilinear Wave Equation with an Acoustic Boundary Condition

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Abstract. We consider a semilinear wave equation equipped with an acoustic boundary condition. More precisely, we study a system consisting of the wave equation for the evolution of an unknown function in a three-dimensional domain Ω, i.e., the velocity potential u, coupled with an ordinary differential equation for the evolution of an unknown function on $\partial\Omega$, i.e., the normal displacement δ. The system is completed with a third condition expressing the impenetrability of the boundary. This problem, inspired on a model for acoustic wave motion of a fluid in a domain with locally reacting boundary surface, originally proposed by J. T. Beale and S. I. Rosencrans in [Bull. Amer. Math. Soc. 80 (1974), 1276 – 1278], has been studied by S. Frigeri in [J. Evol. Equ. 10 (2010), 29 – 58] from the point of view of the global asymptotic analysis. The goal of this paper is to analyze the asymptotic behavior of single trajectories, proving that, when the nonlinearity $f(u)$ is analytic, every weak solution converges to a stationary state. The result is obtained by suitably using an argument due to Haraux-Jendoubi and based on the Simon-Łojasiewicz inequality. Furthermore, we provide an estimate for the decay rate to equilibrium.

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1. Introduction

Consider a bounded domain $\Omega \subset \mathbb{R}^3$ filled with a fluid which is at rest except for perturbations due to acoustic waves. The physical state of the fluid is described by the velocity potential $u = u(x,t)$, $x \in \Omega$ which is connected to the particle velocity $\mathbf{v} = \mathbf{v}(x,t)$, $x \in \Omega$, by $\mathbf{v} = -\nabla u(x,t)$.

From theoretical acoustics (see, e.g., [12]) it is well known that $u(x,t)$ satisfies the wave equation

$$u_{tt} = c^2 \Delta u \quad \text{in } \Omega, \tag{1}$$

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where c is the speed of sound in the medium. Now, suppose that the boundary $\Gamma = \partial\Omega$ of the domain is not rigid, but subject to small perturbations. Namely, we describe the boundary as a distributed system of independent springs which interact with the fluid by reacting to the excess pressure of the acoustic wave like a resistive armonic oscillator. If we introduce the normal displacement $\delta = \delta(x, t)$, $x \in \Gamma$, of the boundary into the domain, then, according with this model, δ satisfies the equation for motion of a damped and forced harmonic oscillator at each point $x \in \Gamma$, i.e.,

$$m(x)\delta_{tt}(x,t) + d(x)\delta_t(x,t) + k(x)\delta(x,t) = -\rho u_t(x,t) \quad \text{on } \Gamma, \tag{2}$$

where ρ is the density of the fluid, m, d and k are the mass per unit area, the resistivity and the spring constant of the boundary, respectively.

The model is finally completed with a third condition expressing the impenetrability of the boundary, i.e., the fact that on Γ we have $\mathbf{v} \cdot \mathbf{n} = -\delta$ ($\mathbf{n}$ is the outward normal). This compatibility condition is therefore

$$\frac{\partial u}{\partial \mathbf{n}} = \delta_t \quad \text{on } \Gamma. \tag{3}$$

System (1)–(3) has been proposed by J. T. Beale and S. I. Rosencrans in their pioneering paper [1].

We point out that system (1)–(3) can be considered as a partial differential equation (i.e., the wave equation (1)) describing the evolution of the velocity potential in the domain Ω, coupled with an ordinary differential equation (i.e., the harmonic type oscillator equation (2)) which governs the evolution of the displacement δ of the boundary Γ, with a compatibility condition (cf. (3)) added for physical reasons.

The model we consider in this paper is closely related with the original model by Beale and Rosencrans. More precisely, in our model we introduce a nonlinear term $f(u)$ in the wave equation (1) which accounts for nonlinear effects in the small wave motion of the fluid. Furthermore, we also consider dissipation effects inside the domain Ω by adding the term ωu_t to the wave equation. Notice that the dissipation on the boundary is given by the term $d\delta_t$ in (2). For the sake of simplicity we set all the coefficients, with the only exception of the damping parameters, equal to 1. The problem we consider is therefore the following

$$\begin{cases} u_{tt} + \omega u_t - \Delta u + u + f(u) = 0 & \text{in } \Omega \times (0, +\infty) \\ \delta_{tt} + \nu\delta_t + \delta = -u_t & \text{on } \Gamma \times (0, +\infty) \\ \delta_t = \dfrac{\partial u}{\partial \mathbf{n}} & \text{on } \Gamma \times (0, +\infty), \end{cases} \tag{4}$$

where ω and $\nu > 0$ are the interior and surface damping parameter, respectively.

Problem (1)–(3) has been considered in [1–3] for the exterior domain case. In these papers well-posedness (cf. [2]) and some spectral properties of the evolution semigroup generator (cf. [2, 3]) have been studied.

More recently wave equations with acoustic boundary conditions have been considered by many authors. In particular we mention [4, 5, 8, 9, 13, 18] as far as well-posedness and spectral results are concerned, and [7] which, to the best of our knowledge, is the first paper where the asymptotic behavior of solutions has been analyzed. Indeed, in [7], problem (4) has been considered from the point of view of the theory of infinite-dimensional dissipative dynamical systems and the existence of a bounded absorbing set, of the global attractor and of an exponential attractor of optimal regularity have been established.

In this paper our aim is to study the convergence towards equilibria of solutions to system (4), under the assumption of analytic nonlinearity, by means of the Simon-Łojasiewicz technique. We point out that, in general, this is not a trivial issue since the set of stationary solutions can be a continuum. In particular, there might be initial conditions whose ω-limit sets do not reduce to a singleton even though the nonlinear term is C^∞ (see [15,16], where a parabolic semilinear equation with a nonanalytic nonlinearity is considered). However, if the nonlinearity is real analytic, in many cases it is possible to show that any (sufficiently smooth) trajectory converges to a single equilibrium. A well-known technique is based on Simon-Łojasiewicz type inequalities.

The key step in the application of the classical Haraux-Jendoubi argument (see [11]) is the introduction of a suitable auxiliary functional (cf. (21) below).

Furthermore, another goal of the paper is to provide an estimate for the convergence rate to equilibrium. This will be established by means of a decomposition method of the solution inspired by [14]. This method requires the additional assumption that f' has to be bounded from below.

We observe that there are only few works in literature on the asymptotic behavior (and, in particular, on the convergence to stationary solutions) for evolution equations with *dynamic* boundary conditions (see, e.g., [10, 17] and references therein). The present paper, along with the previous one [7], aim to be a contribution in this direction.

2. Preliminary results

Let us introduce some notation. We denote by $(\cdot, \cdot)$ and $\|\cdot\|$ the inner product and the norm on $L^2(\Omega)$, respectively. For every $s \in \mathbb{R}$, the inner product and the norm in the Sobolev space $H^s(\Omega)$ will be denoted by $(\cdot, \cdot)_s$ and $\|\cdot\|_s$, respectively. The inner product on $L^2(\Gamma)$ is $\langle \cdot, \cdot \rangle$ and the corresponding norm is simply $\|\cdot\|_{L^2(\Gamma)}$.

We now state the assumptions we need on the nonlinear term $f \in C^1(\mathbb{R})$.

$$|f'(u)| \leq c_1(1+u^2), \quad c_1 \geq 0 \tag{5}$$

$$\liminf_{|u|\to+\infty} \frac{f(u)}{u} > -1. \tag{6}$$

For some results, in place of (5) we shall assume that $f \in C^2(\mathbb{R})$ satisfies

$$|f''(u)| \leq c_2(1+|u|), \quad c_2 \geq 0 \tag{7}$$

$$f'(u) \geq -l, \quad l \geq 0. \tag{8}$$

Existence and uniqueness of a weak solution to (4) has been obtained for the linear case ($f = 0$) without interior dissipation ($\omega = 0$) in [2]. In [7], by exploiting the results in [2], the existence of the solution semigroup $S(t)$ is proved for our system with f and ω different from zero.

The finite energy phase space for $S(t)$ is the Hilbert space

$$\mathcal{H} := H^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma) \times L^2(\Gamma),$$

endowed with the norm

$$\|w\|^2_{\mathcal{H}} = \|w_1\|^2_1 + \|w_2\|^2 + \|w_3\|^2_{L^2(\Gamma)} + \|w_4\|^2_{L^2(\Gamma)},$$

for every $w := (w_1, w_2, w_3, w_4) \in \mathcal{H}$. Indeed, it is easy to write an energy identity, which can be shown to be satisfied by all weak solutions $w = (u, u_t, \delta, \delta_t)$ of the problem, and which has the form

$$\frac{d\mathcal{E}_w}{dt} = -\omega\|u_t\|^2 - \nu\|\delta_t\|^2_{L^2(\Gamma)}, \tag{9}$$

where $\mathcal{E}_w(t) = E_w(t) + \int_\Omega F(u)$ and $E_w(t) = \frac{1}{2}\|w(t)\|^2_{\mathcal{H}}$. We have set $F(s) = \int_0^s f(\sigma)d\sigma$. In order to consider strong solutions we also introduce the following (second order) phase space

$$\mathcal{H}_1 = \left\{w = (w_1, w_2, w_3, w_4) \in H^2(\Omega) \times H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) : \partial_{\mathbf{n}} w_1 = w_4\right\}$$

which is Hilbert with respect to the norm

$$\|w\|^2_{\mathcal{H}_1} = \|w_1\|^2_2 + \|w_2\|^2_1 + \|w_3\|^2_{H^{\frac{1}{2}}(\Gamma)} + \|w_4\|^2_{H^{\frac{1}{2}}(\Gamma)}.$$

We now recall the well-posedness result (see [7]).

Theorem 2.1. *Let* (5) *and* (6) *hold and assume that* $w_0 \in \mathcal{H}$. *Then, there exists a unique weak solution* $w \in C^0([0, +\infty); \mathcal{H})$ *to* (4). *For each weak solution* $\mathcal{E}_w(\cdot) \in C^1([0, +\infty))$ *and the energy equation* (9) *holds. Furthermore, if* w_{01}

and w_{02} *are two sets of data in* $\mathcal{H}$ *and* w_1*,* w_2 *the corresponding solutions on* $[0,+\infty)$*, there exists* $\theta > 0$*, depending only on the* $\mathcal{H}$*-norms of the data and independent of* ω*,* ν*, such that*

$$\|w_2(t) - w_1(t)\|_{\mathcal{H}} \le e^{\theta t}\|w_{02} - w_{01}\|_{\mathcal{H}}, \quad \forall t \ge 0.$$

Assuming, in addition, that f *fulfills* (7) *and that* $w_0 \in \mathcal{H}_1$*, the corresponding weak solution satisfies the regularity property*

$$w \in C^1([0,+\infty);\mathcal{H}) \cap C^0([0,+\infty);\mathcal{H}_1).$$

Let us now introduce the Riesz map $\widehat{B} : H^1(\Omega) \to H^1(\Omega)'$, i.e., the linear bounded isomorphism between H^1 and its dual associated to $(\cdot,\cdot)_1$. Furthermore, set

$$\mathcal{M}(u) := \widehat{B}u + f(u), \quad \forall u \in H^1(\Omega)$$

which is the Frechét derivative of the functional $\mathcal{G} \in C^1(H^1)$ given by

$$\mathcal{G}(u) := \frac{1}{2}\|u\|_1^2 + \int_\Omega F(u), \quad \forall u \in H^1(\Omega).$$

We denote by $\mathcal{S}$ the set $\mathcal{S} := \{u_\infty \in D(B) : \mathcal{M}(u) = 0\}$, where $D(B) = \{v \in H^2(\Omega) : \partial_{\mathbf{n}} v = 0\}$ is the domain of the operator $\widehat{B}$ in $L^2(\Omega)$. With this notation, the set of equilibria for problem (4) is $\mathcal{E} = \{(u_\infty, 0, 0, 0) \in \mathcal{H} : u_\infty \in \mathcal{S}\}$. We first have the following.

Lemma 2.2. *Let* (5)–(8) *hold. Consider* $w_0 = (u_0, u_1, \delta_0, \delta_1) \in \mathcal{H}$ *and the corresponding trajectory* $w(t) = (u(t), u_t(t), \delta(t), \delta_t(t))$*. Then,* $\cup_{t\ge 0}\{w(t)\}$ *is precompact in* $\mathcal{H}$*, and we have*

$$u_t(t) \to 0 \quad \text{in } L^2(\Omega), \quad \text{as } t \to +\infty \tag{10}$$

$$\delta_t(t) \to 0 \quad \text{in } L^2(\Gamma), \quad \text{as } t \to +\infty. \tag{11}$$

Furthermore,

$$\omega(w_0) \subset \mathcal{E}, \tag{12}$$

where $\omega(w_0)$ *is the* ω*-limit set of the trajectory.*

Proof. The precompactness of the trajectory is an immediate consequence of the existence of the global attractor, which is ensured under assumptions (5)–(8) (see [7, Theorem 3]).

The proof of (10) follows the same argument used in [6, Lemma 1]. In order to prove (11), we decompose the solution w as (cf. [7]) $w = w^d + w^c$, where $w^d := (u^d, u^d_t, \delta^d, \delta^d_t)$ and $w^c := (u^c, u^c_t, \delta^c, \delta^c_t)$ are solutions to

$$\begin{cases} u^d_{tt} + \omega u^d_t - \Delta u^d + u^d + \psi(u) - \psi(u^c) = 0 & \text{in } \Omega \times (0,\infty) \\ \delta^d_{tt} + \nu \delta^d_t + \delta^d = -u^d_t & \text{on } \Gamma \times (0,\infty) \\ \delta^d_t = \dfrac{\partial u^d}{\partial \mathbf{n}} & \text{on } \Gamma \times (0,\infty) \\ w^d(0) = w_0 & \text{in } \Omega \end{cases} \tag{13}$$

and

$$\begin{cases} u^c_{tt} + \omega u^c_t - \Delta u^c + u^c + \psi(u^c) = \theta u & \text{in } \Omega \times (0,\infty) \\ \delta^c_{tt} + \nu \delta^c_t + \delta^c = -u^c_t & \text{on } \Gamma \times (0,\infty) \\ \delta^c_t = \dfrac{\partial u^c}{\partial \mathbf{n}} & \text{on } \Gamma \times (0,\infty) \\ w^c(0) = 0 & \text{in } \Omega, \end{cases} \tag{14}$$

respectively. This decomposition technique is inspired by the one used in [14]. Here we have set $\psi(s) := f(s) + \theta s$, with $\theta \geq l$ fixed large enough. In [7] it is shown that there exist three constants M, N and β (depending on ω and ν) such that

$$\|w^d(t)\|_{\mathcal{H}} \leq M e^{-\beta t}, \quad \forall t \geq 0, \tag{15}$$

whereas

$$\|w^c(t)\|_{\mathcal{H}_1} \leq N, \quad \forall t \geq 0. \tag{16}$$

Then, we have $\delta^d_t(t) \to 0$ in $L^2(\Gamma)$. Furthermore, there holds

$$\frac{d}{dt}\|\delta^c_t\|^2_{L^2(\Gamma)} = 2\langle \delta^c_t, \delta^c_{tt}\rangle = -2\nu\|\delta^c_t\|^2_{L^2(\Gamma)} - 2\langle \delta^c_t, \delta^c\rangle - 2\langle \delta^c_t, u^c_t\rangle.$$

Hence, setting $h(t) = \|\delta^c_t\|^2_{L^2(\Gamma)}$, by (16) and [7, Lemma 5], we have $|h'(t)| \leq c$, for every $t \in (0, +\infty)$. This fact, along with the dissipation integral

$$\int_0^\infty \Big(\omega\|u_t(\tau)\|^2 + \nu\|\delta_t(\tau)\|^2_{L^2(\Gamma)}\Big) d\tau \leq \Lambda(\|w_0\|_{\mathcal{H}})$$

(see [7, Corollary 2]) and (15), which imply $h \in L^1(0, +\infty)$, yields $\delta^c_t(t) \to 0$ in $L^2(\Gamma)$ as $t \to +\infty$. Hence we get (11). Finally, (12) can be obtained by arguing, e.g., as in [6, Lemma 1]. □

3. The main result

The key ingredient for the proof of the convergence to equilibria result is the well known Simon-Łojasiewicz inequality, which we shall use in the form deduced in [11, Theorem 2.2 and Proposition 5.3.1]. Thus, the assumption we need is the following:

$$f \text{ is real analytic.} \tag{17}$$

Lemma 3.1. *Let* (5) *and* (17) *hold. If* $u_\infty \in \mathcal{S}$*, then, there exist* $\theta \in (0, \frac{1}{2})$*,* $\sigma > 0$ *and* $c_0 > 0$ *such that*

$$|\mathcal{G}(u) - \mathcal{G}(u_\infty)|^{1-\theta} \leq c_0 \|\mathcal{M}(u)\|_{(H^1)'} \tag{18}$$

for every $u \in H^1(\Omega)$ *such that* $\|u - u_\infty\|_1 < \sigma$*.*

We now can state the convergence to equilibria result.

Theorem 3.2. *Assume* (5)–(8) *and* (17)*. Consider* $w_0 \in \mathcal{H}$ *and the corresponding trajectory* $w(t) = S(t)w_0$*. Then, there exists* $u_\infty \in \mathcal{S}$ *such that*

$$w(t) \to w_\infty \quad \text{in } \mathcal{H}, \quad \text{as } t \to +\infty, \tag{19}$$

where $w_\infty = (u_\infty, 0, 0, 0) \in \mathcal{E}$*. Furthermore, there exist a time* $t_0 > 0$*, two positive constants* a_1*,* a_2 *and* $\theta \in (0, \frac{1}{2})$ *such that*

$$\|w(t) - w_\infty\|_{\mathcal{H}} \leq a_1(1 + a_2 t)^{-\frac{\theta}{1-2\theta}} \quad \forall t \geq t_0. \tag{20}$$

Proof. Consider an element $w_\infty = (u_\infty, 0, 0, 0)$, $u_\infty \in \mathcal{S}$, in the ω-limit set $\omega(w_0)$ of the trajectory $w(t) = (u(t), u_t(t), \delta(t), \delta_t(t))$, and introduce the functional

$$\begin{aligned}\Phi(t) = \frac{1}{2}\|u_t\|^2 + \mathcal{G}(u) - \mathcal{G}(u_\infty) + \epsilon\big(\widehat{B}^{-1}\mathcal{M}(u), u_t\big) + \frac{1+\epsilon\nu}{2}\|\delta\|^2_{L^2(\Gamma)} \\ + \frac{1}{2}\|\delta_t\|^2_{L^2(\Gamma)} + \epsilon\big\langle \delta, \delta_t\big\rangle + \epsilon\big\langle \delta, \widehat{B}^{-1}\mathcal{M}(u)\big\rangle,\end{aligned} \tag{21}$$

where $\epsilon > 0$ will be fixed later. After some calculations we obtain

$$\begin{aligned}\Phi'(t) = &-(\omega - \epsilon)\|u_t\|^2 - \epsilon\|\delta\|^2_{L^2(\Gamma)} - (\nu - \epsilon)\|\delta_t\|^2_{L^2(\Gamma)} - \epsilon\omega\big(u_t, \widehat{B}^{-1}\mathcal{M}(u)\big) \\ &- \epsilon\big\langle \mathcal{M}(u), \widehat{B}^{-1}\mathcal{M}(u)\big\rangle_{(H^1)', H^1} + 2\epsilon\big\langle \delta_t, \widehat{B}^{-1}\mathcal{M}(u)\big\rangle \\ &+ \epsilon\big(u_t, \widehat{B}^{-1}f'(u)u_t\big) + \epsilon\big\langle \delta, \widehat{B}^{-1}f'(u)u_t\big\rangle.\end{aligned} \tag{22}$$

Now observe that

$$\big\langle \mathcal{M}(u), \widehat{B}^{-1}\mathcal{M}(u)\big\rangle_{(H^1)', H^1} = \|\mathcal{M}(u)\|^2_{(H^1)'}. \tag{23}$$

Furthermore we have

$$-\epsilon\omega\big(u_t, \widehat{B}^{-1}\mathcal{M}(u)\big) \leq \frac{\epsilon}{4}\|\mathcal{M}(u)\|^2_{(H^1)'} + \epsilon\omega^2\|u_t\|^2 \tag{24}$$

$$2\epsilon\big\langle \delta_t, \widehat{B}^{-1}\mathcal{M}(u)\big\rangle \leq \frac{\epsilon}{4}\|\mathcal{M}(u)\|^2_{(H^1)'} + \epsilon c\|\delta_t\|^2_{L^2(\Gamma)}, \tag{25}$$

and, by $\|f'(u)u_t\|_{(H^1)'} \leq c\|f'(u)u_t\|_{L^{\frac{6}{5}}(\Omega)} \leq c\|f'(u)\|_{L^3(\Omega)}\|u_t\|$, also

$$\epsilon\big(u_t, \widehat{B}^{-1}f'(u)u_t\big) \leq \epsilon\|u_t\|\|f'(u)u_t\|_{(H^1)'} \leq \epsilon c\|u_t\|^2 \tag{26}$$

$$\epsilon\big\langle \delta, \widehat{B}^{-1}f'(u)u_t\big\rangle \leq \frac{\epsilon}{2}\|\delta\|^2_{L^2(\Gamma)} + \epsilon c\|u_t\|^2. \tag{27}$$

Therefore, by means of (23)–(27), from (22) we get

$$\Phi'(t) \le -\big(\omega - c\epsilon(1+\omega^2)\big)\|u_t\|^2 - \frac{\epsilon}{2}\|\mathcal{M}(u)\|^2_{(H^1)'} - (\nu - \epsilon c)\|\delta_t\|^2_{L^2(\Gamma)} - \frac{\epsilon}{2}\|\delta\|^2_{L^2(\Gamma)},$$

and by choosing $\epsilon \in (0, \overline{\epsilon}]$, with $\overline{\epsilon}$ small enough (depending on ω and ν), we are led to

$$\Phi'(t) \le -\frac{\epsilon}{2} c_1 \big\{\|u_t\| + \|\mathcal{M}(u)\|_{(H^1)'} + \|\delta_t\|_{L^2(\Gamma)} + \|\delta\|_{L^2(\Gamma)}\big\}^2. \tag{28}$$

We now take a sequence $t_n \to +\infty$ such that

$$u(t_n) \to u_\infty \quad \text{in } H^1(\Omega); \quad \delta(t_n) \to 0 \quad \text{in } L^2(\Gamma). \tag{29}$$

Since Φ is non-increasing on $[0, +\infty)$, by (10), (11) and (29), we have that $\Phi(t) \to 0$, as $t \to +\infty$, and hence $\Phi \ge 0$ on $[0, +\infty)$. We can write

$$\begin{aligned}\Phi^{1-\theta} \le c\Big\{&\|u_t\|^{2(1-\theta)} + |\mathcal{G}(u) - \mathcal{G}(u_\infty)|^{1-\theta} + \|\mathcal{M}(u)\|_{(H^1)'} + \|u_t\|^{\frac{1-\theta}{\theta}} \\ &+ \|\delta\|^{2(1-\theta)}_{L^2(\Gamma)} + \|\delta_t\|^{2(1-\theta)}_{L^2(\Gamma)} + \|\delta\|^{\frac{1-\theta}{\theta}}_{L^2(\Gamma)} + \|\delta\|_{L^2(\Gamma)} + \|\delta_t\|^{\frac{1-\theta}{\theta}}_{L^2(\Gamma)}\Big\},\end{aligned} \tag{30}$$

where θ is the same as in (18). Hence, when $\|u_t\| < 1$, $\|\delta_t\|_{L^2(\Gamma)} < 1$ and $\|u - u_\infty\|_1 < \sigma$, by (18) and on account of the fact that $2(1-\theta) > 1$, $\frac{1-\theta}{\theta} > 1$, inequality (30) entails

$$\Phi^{1-\theta} \le c_2 \big\{\|u_t\| + \|\mathcal{M}(u)\|_{(H^1)'} + \|\delta_t\|_{L^2(\Gamma)} + \|\delta\|_{L^2(\Gamma)}\big\}. \tag{31}$$

Now, by virtue of (28), (31) and of

$$-\frac{d}{dt}\Phi^\theta = -\theta\Phi^{\theta-1}\frac{d\Phi}{dt} \ge \frac{c_1\theta\epsilon}{2c_2}\big\{\|u_t\| + \|\mathcal{M}(u)\|_{(H^1)'} + \|\delta_t\|_{L^2(\Gamma)} + \|\delta\|_{L^2(\Gamma)}\big\},$$

(the inequality holds when $\|u_t\| < 1$, $\|\delta_t\|_{L^2(\Gamma)} < 1$ and $\|u - u_\infty\|_1 < \sigma$), one can exploit the classical argument (see, e.g., [6, Theorem 2]) which relies on the integration of the last relation and on the precompactness of the trajectory in $\mathcal{H}$, in order to conclude that there exists $\tau \ge 0$ such that

$$u_t \in L^1(\tau, +\infty; L^2(\Omega)), \quad \delta_t \in L^1(\tau, +\infty; L^2(\Gamma)).$$

Hence $u(t) \to u_\infty$ in $L^2(\Omega)$ and $\delta(t) \to 0$ in $L^2(\Gamma)$, as $t \to +\infty$. By precompactness and (10), (11) we finally get (19).

As far as the convergence rate is concerned we first observe that there exist nonnegative constants b_1, b_2 and a time t^* such that

$$\|u(t) - u_\infty\| + \|\delta(t)\|_{L^2(\Gamma)} \le b_1(1 + b_2 t)^{-\frac{\theta}{1-2\theta}}, \quad \forall t \ge t^*. \tag{32}$$

Furthermore, we decompose the solution w according to $w = w^d + w^c$, where w^d and w^c are solutions to systems (13) and (14), respectively, and introduce $\overline{u}^c := u^c - u_\infty$, $\overline{\delta}^c := \delta^c$. By multiplying $(14)_1$ in $L^2(\Omega)$ by $\overline{u}^c_t + \epsilon\overline{u}^c$ and $(14)_2$ in $L^2(\Gamma)$ by $\overline{\delta}^c_t + \epsilon\overline{\delta}^c$, by taking into account $(14)_3$ and adding the resulting equations, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{\|\overline{u}^c\|_1^2 + \|\overline{u}^c_t\|^2 + 2\epsilon(\overline{u}^c_t, \overline{u}^c) + \big\|\overline{\delta}^c\big\|^2_{L^2(\Gamma)} + \big\|\overline{\delta}^c_t\big\|^2_{L^2(\Gamma)} \\
&\quad + 2\epsilon\big\langle\overline{\delta}^c_t, \overline{\delta}^c\big\rangle + 2\epsilon\big\langle\overline{u}^c, \overline{\delta}^c\big\rangle\Big\} + \epsilon\|\overline{u}^c\|_1^2 + (\omega - \epsilon)\|\overline{u}^c_t\|^2 \\
&\quad + \epsilon\omega(\overline{u}^c_t, \overline{u}^c) + \epsilon\big\|\overline{\delta}^c\big\|^2_{L^2(\Gamma)} + (\nu - \epsilon)\big\|\overline{\delta}^c_t\big\|^2_{L^2(\Gamma)} + \epsilon\nu\big\langle\overline{\delta}^c_t, \overline{\delta}^c\big\rangle \\
&= 2\epsilon\big\langle\overline{u}^c, \overline{\delta}^c_t\big\rangle + \theta(u, \overline{u}^c_t) + \epsilon\theta(u, \overline{u}^c) \\
&\quad - \big(\psi(u^c) - f(u_\infty), \overline{u}^c_t\big) - \epsilon\big(\psi(u^c) - f(u_\infty), \overline{u}^c\big).
\end{aligned} \tag{33}$$

Now, the terms on the right hand side of (33) can be rewritten and estimated as follows

$$\begin{aligned}
&\theta(u, \overline{u}^c_t) + \epsilon\theta(u, \overline{u}^c) - \big(\psi(u^c) - f(u_\infty), \overline{u}^c_t\big) - \epsilon\big(\psi(u^c) - f(u_\infty), \overline{u}^c\big) \\
&= -\big(f(u^c) - f(u_\infty), \overline{u}^c_t\big) - \epsilon\big(f(u^c) - f(u_\infty), \overline{u}^c\big) + \theta(u^d, \overline{u}^c_t) + \epsilon\theta(u^d, \overline{u}^c) \\
&\le \frac{\omega}{2}\|\overline{u}^c_t\|^2 + c\epsilon^2\|\overline{u}^c\|_1^2 + c\|f(u^c) - f(u_\infty)\|^2 + c\|u^d\|^2
\end{aligned} \tag{34}$$

$$2\epsilon\big\langle\overline{u}^c, \overline{\delta}^c_t\big\rangle \le \frac{\nu}{2}\big\|\overline{\delta}^c_t\big\|^2_{L^2(\Gamma)} + c\epsilon^2\|\overline{u}^c\|_1^2. \tag{35}$$

If we now define

$$\Lambda := \|\overline{u}^c\|_1^2 + \|\overline{u}^c_t\|^2 + 2\epsilon(\overline{u}^c_t, \overline{u}^c) + \big\|\overline{\delta}^c\big\|^2_{L^2(\Gamma)} + \big\|\overline{\delta}^c_t\big\|^2_{L^2(\Gamma)} + 2\epsilon\big\langle\overline{\delta}^c_t, \overline{\delta}^c\big\rangle + 2\epsilon\big\langle\overline{u}^c, \overline{\delta}^c\big\rangle,$$

then, by taking $\epsilon > 0$ small enough and introducing $\overline{w}^c = w^c - w_\infty$, we have

$$k_1\|\overline{w}^c(t)\|^2_{\mathcal{H}} \le \Lambda(t) \le k_2\|\overline{w}^c(t)\|^2_{\mathcal{H}}, \tag{36}$$

with k_1 and k_2 two positive constants. Hence, plugging (34), (35) into (33) and using also (36), for ϵ small enough we deduce the following differential inequality

$$\frac{d\Lambda}{dt} + c\Lambda \le c\|f(u^c) - f(u_\infty)\|^2 + c\|u^d\|^2. \tag{37}$$

Notice that, by (5), (15) and (32) we have

$$\begin{aligned}
\|f(u^c) - f(u_\infty)\|^2 &\le c(1 + \|u^c\|^2_\infty + \|u_\infty\|^2_\infty)\|\overline{u}^c\|^2 \\
&\le c\|\overline{u}^c\|^2 \\
&\le c\|\overline{u}\|^2 + c\|\overline{u}^d\|^2 \\
&\le c(1 + b_2 t)^{-\frac{2\theta}{1-2\theta}}.
\end{aligned}$$

Therefore, from (37) we are led to $\frac{d\Lambda}{dt} + c\Lambda \leq c(1+b_2t)^{-\frac{2\theta}{1-2\theta}}$ for all $t \geq t_1$. From this inequality, by means of the standard Gronwall lemma we easily deduce (20) (see, e.g., the end of the proof of [6, Theorem 2]). □

Remark 3.3. For $\Omega \subset \mathbb{R}^2$ another technique is available to get a convergence-rate estimate. Such technique, which does not make use of the decomposition of the solution, but only relies on the Gagliardo-Nirenberg interpolation inequality in 2D (cf. [6]), apparently does not need the stronger assumptions (7) and (8), though it provides a less sharp convergence rate to equilibrium [6]. Nevertheless, the possibility of getting rid of assumptions (7) and (8) in order to obtain the convergence (11), and hence for the convergence result as well, seems highly problematic. For this reason the above mentioned technique for the two-dimensional case has not been considered here.

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