On an Evolution Equation for a Non-Hypoelliptic Linear Partial
Differential Operator from Stochastics

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Recently E. B. Dynkin [2] introduced and studied a non-hypoelliptic linear partial differential
operator of even order (with constant coefficients) which originates from the theory of multi-
parametric stochastic processes. Motivated by the considerations of Dynkin the authors have
solved a generalized Dirichlet problem for this differential operator in their work [1]. Our
aim in the present paper is to investigate the Cauchy problem for the corresponding evolution
equation (in the time variable of first order); such a Cauchy problem could have applications
to some questions from the stochastics.

1. Introduction

1.1 We consider in a bounded open set \( G \subset \mathbb{R}^n \) (which satisfies some conditions, cf.
Property P in 1.2) the non-hypoelliptic linear partial differential operator \( L(D) \) of
order \( 2k \) \( (k \in \mathbb{N}, k \leq n) \) introduced by E. B. Dynkin [2]; the exact definition of
\( L(D) \) will be given below in 1.3. In this paper we want to investigate an initial-
boundary value problem, which we call the Cauchy problem, for the “abstract”
evolution equation

\[
\frac{d}{dt} u(t) + L^\ast u(t) = f(t)
\]

where \( L^\ast \) is the closure of the differential operator \( L(D) \) in \( L^2(G) \), \( f \) a given function
defined on the positive time axis with values in \( L^2(G) \) and the solutions \( u(t) \) are
searched among functions defined on the positive time axis with values in a certain
subspace of \( L^2(G) \) which coincides with the domain \( D(L^\ast) \) of \( L^\ast \).

1.2 Let us first recall some notions, notations and results from our earlier paper [1].
Let \( G \) be a set in \( \mathbb{R}^n \) with
Property P: The bounded open set \( G \subset \mathbb{R}^n \) is the Cartesian product

\[ G = G_1 \times \cdots \times G_k \]

of bounded open sets \( G_j \subset \mathbb{R}^{m_j} \) \((1 \leq j \leq k)\), \( m_1 + \cdots + m_k = n \), \( \mathbb{R}^n = \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k} \), with sufficiently smooth boundaries \( \partial G_j \) (if the boundaries \( \partial G_j \) are of class \( C^\infty \) all our considerations surely will be valid). We write

\[ \partial_j G := G_1 \times \cdots \times G_{j-1} \times \partial G_j \times G_{j+1} \times \cdots \times G_k. \]

By \( \mathbb{N}_0^n \) we denote, as usual, the set of all ordered systems of \( n \) nonnegative integers (multi-indices). For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \) and \( \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n \), we write \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) and \( \alpha \leq \beta \) if \( \alpha_l \leq \beta_l \) for \( 1 \leq l \leq n \).

With \( n, k, m_j \in \mathbb{N} \) as in Property P we put

\[ l_1 = 0, \quad l_j := \sum_{i=1}^{j-1} m_i \quad (2 \leq j \leq k). \]

For each \( j \) \((1 \leq j \leq k)\) we define \( m_j \) multi-indices \( \epsilon_{ij} \in \mathbb{N}_0^n \) with \( |\epsilon_{ij}| = 1 \) each having its only nonvanishing coordinate in the \( t_i \)-th position, \( l_i + 1 \leq t_i \leq l_i + m_j \). We introduce the set

\[ \Gamma := \left\{ \alpha \in \mathbb{N}_0^n \mid \alpha = \sum_{j=1}^{k} \epsilon_{ij} \right\}. \]

Note that \( \Gamma \) has \( m_1 \cdots m_k \) elements. Further, we write

\[ \Gamma^j := \left\{ \gamma \in \mathbb{N}_0^n \mid \gamma = \sum_{l=1}^{k} \epsilon_{ij} \right\} \quad (1 \leq j \leq k). \]

1.3 Now we can introduce the differential operator

\[ L(D) := \sum_{\alpha \in \Gamma} D^{2\alpha} \]

where we use the abbreviation

\[ D^{\alpha} = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \quad \text{with} \quad D_j := -\sqrt{-1} \frac{\partial}{\partial x_j}. \]

This operator has also the expression

\[ L(D) = \Lambda_1 \cdots \Lambda_k \]

where

\[ \Lambda_j := \sum_{s=1}^{m_k} D_{l_j+s} \quad (1 \leq j \leq k) \]

denotes the Laplace operator, which acts on functions defined on subsets of \( \mathbb{R}^{m_j} (\subset \mathbb{R}^n) \). As we have shown in [1] the operator \( L(D) \) is not hypoelliptic.

1.4 Let \( C^k(G) \), \( k \in \mathbb{N}_0 \), be the linear space of all complex valued functions \( u \) which are \( k \) times continuously differentiable in \( G \). By \( C_0^k(G) \) we denote the space of all functions \( u \in C^k(G) \) each having a compact support in \( G \). We write also \( C_0^\infty(G) = \bigcap_{k \in \mathbb{N}_1} C_0^k(G) \). Further let be

\[ C_\star^k(G) := \left\{ u \in C^k(G) \mid D^{\alpha} u \in L^2(G), \alpha \in \mathbb{N}_0^n, |\alpha| \leq k \right\}. \]
In $C^{k}(G)$ (with $K$ from 1.2) we associate with the operator $L(D)$ the sesquilinear form

$$B(u, v) := \sum_{\alpha \in \Gamma} \int_{G} D^\alpha u(x) D^\alpha v(x) \, dx.$$  \hfill (1.1)

We put

$$(u, v)_r := B(u, v) + (u, v)_0$$  \hfill (1.2)

where $(., .)_0$ denotes the scalar product in $L^2(G)$, $$(u, v)_0 := \int_{G} u(x) v(x) \, dx.$$ Thus, we have a scalar product $(., .)_r$ on $C^k(G)$ with the corresponding norm $\| \cdot \|_r$. We denote the completion of $C^k(G)$ with respect to the scalar product (1.2) by $H^r(G)$. For the closure of $C^\infty(G)$ in $H^r(G)$ we write $H_0^r(G)$.

On the space $H_0^r(G)$ the sesquilinear form (1.1) defines a scalar product equivalent to (1.2), in particular one has the inequalities

$$B(u, u) \leq \| u \|_r^2 \leq c^* B(u, u) \quad \text{for all } u \in H_0^r(G)$$  \hfill (1.3)

with a constant $c^*$ (cf. [1: Lemma 2]). The norm induced by $B(., .)_r$ on $H_0^r(G)$ will be denoted by $\| \cdot \|_r$.

The elements of $H_0^r(G)$ can be interpreted as functions with generalized homogeneous boundary data: Namely, for each $u \in H_0^r(G) \cap C^{k-1}(\overline{G})$ the relation

$$D^\beta u|_{\partial G} = 0$$  \hfill (1.4)

holds for all $\beta \in \mathbb{N}_0^n$ with $\beta \leq \gamma$ for some $\gamma \in \Gamma^j$ ($1 \leq j \leq k$) (we proved this assertion in [1: Theorem 4], cf. also [4: p. 28]).

Further for every $u \in H_0^r(G)$ the strong $L^2$-derivative $D^\tau u$ exists for all $\tau \in \mathbb{N}_0^*$ with $\tau \leq \sigma$ for some $\sigma \in \Gamma$ (the corresponding assertion for $u \in H^r(G)$ is not valid; cf. [1: Lemma 1], and also [5: Theorem 1]).

Because the relation (1.4) is valid for functions $u \in H_0^r(G) \cap C^{k-1}(\overline{G})$ one can apply partial integration for functions from the set $X := H_0^r(G) \cap C^k(\overline{G})$ and gets

$$\int_{G} D^\alpha u(x) D^\alpha v(x) \, dx = \int_{G} D^\alpha u(x) v(x) \, dx$$  \hfill (1.5)

for all $u, v \in X$ and all $\alpha \in \Gamma$ (cf. [1: Lemma 5]).

1.5 In the Hilbert space $L^2(G)$ we associate to the differential operator $L(D)$ a linear operator $L$ by

$$D(L) := X(\subset L^2(G)),$$

$$Lu := L(D) u \quad \text{for all } u \in X.$$  \hfill (1.6)

This operator is densely defined and has an adjoint operator $L^*$, and by partial integration one sees that $D(L^*) \supset X$ and that $L^*$ is also densely defined. The operator $L$ is therefore closable with the closure (smallest closed extension) $L^* := L^{**}$.

On the other hand, we have on $X$ a scalar product defined by

$$(u, \phi)_X := (L(D) u, L(D) \phi)_0 + (u, \phi)_0.$$  \hfill (1.7)
for all $u \in X$ and $\phi \in C_0^\infty(G)$ and since (1.7) depends continuously on $u$ in the topology defined by (1.6) we have further

$$B(u, \phi) = (L^* w, \phi)_0$$

for all $w \in H^{(k)}(G)$ and $\phi \in C_0^\infty(G)$.

In [1] we proved that the elements of $H^{(k)}(G)$ have the same boundary behaviour as the elements of $H_0^r(G)$, i.e.,

$$H^{(k)}(G) \cap H_0^r(G) = H^{(k)}(G).$$

1.6 In [1] we proved the existence and uniqueness of the solution of a generalized Dirichlet problem. We formulate the results for homogeneous boundary data:

For a given $f \in L^2(G)$ there exists a unique element $u \in H_0^r(G)$ such that

$$B(u, \phi) = (f, \phi)_0$$

holds for all $\phi \in C_0^\infty(G)$.

From the regularity result of [1: Theorem 9] it follows that the solution $u$ of (1.10) lies in $H^{(k)}(G)$.

2. The resolvent of $L$

2.1 We first derive a priori estimates for the operator $L^*$. Take a finite open interval $J := \{ t \in \mathbb{R} | -T < t < T \}$ and let $G$ be a bounded open set in $\mathbb{R}^n$ with Property P. The set of all functions which obey the condition

$$\omega(x, \cdot) \in C_0^\infty(J) \quad \text{for each } x \in G,$$

$$\omega(\cdot, t) \in X \quad \text{for each } t \in J$$

will be denoted by $\mathscr{C}(G \times J)$. On this set we introduce the norm

$$\|\omega\|_{k,2},G \times J := \int_T^{T} \int_G \{|L(D) \omega(x, t)|^2 + |\partial_t \omega(x, t)|^2 + |\omega(x, t)|^2\} \, dt \, dx$$

where we use the notation $\partial_t$ for the differential operator $\partial/\partial t$.

**Lemma 1:** The estimate

$$\|\{L(D) + e^{it} \partial_t^2\} \omega\|_{k,2},G \times J \geq \|\omega\|_{k,2},G \times J - \|\omega\|_{0,0},G \times J$$

holds for each $\theta \in \mathbb{R}$ with $\pi/2 \leq \theta \leq 3\pi/2$ and for all $\omega \in \mathscr{C}(G \times J)$.

**Proof:** For $\omega \in \mathscr{C}(G \times J)$ we have

$$\|\{L(D) + e^{it} \partial_t^2\} \omega\|_{k,2},G \times J = \int_T^{T} \int_G \{|L(D) + e^{it} \partial_t^2\} \omega(x, t)|^2 \, dt \, dx$$

$$= \int_T^{T} \int_G \{|L(D) \omega(x, t)|^2 + |\partial_t \omega(x, t)|^2\} \, dt \, dx + e^{it}x + e^{-it}x$$

$$= \|\omega\|_{k,2},G \times J - \|\omega\|_{0,0},G \times J + 2 \Re(e^{it}x)$$

with

$$x := \int_T^{T} \int_G L(D) \omega(x, t) \, dt \, dx.$$
By partial integration one gets (note that the boundary terms vanish)

\[ x = \int_{-T}^{T} \sum_{\gamma \in \Gamma} \int_{\Gamma} D_{\gamma}^2 \omega(x, t) \partial_{\gamma}^2 \omega(x, t) \, dt \, dx \]

\[ = \int_{-T}^{T} \sum_{\gamma \in \Gamma} D_{\gamma}^2 \omega(x, t) \, dt \, dx \]

\[ = -\int_{-T}^{T} \sum_{\gamma \in \Gamma} |D_{\gamma}^2 \partial_{\gamma} \omega(x, t)|^2 \, dt \, dx. \]

Thus one has \( x \in \mathbb{R}, \ x \leq 0 \). We get therefore by the assumption \( 2 \Re (e^{i\theta} x) = 2x \cos \theta \geq 0 \), and (2.2) gives the desired result.

**Lemma 2:** Let \( G \) be a bounded open set with Property \( P \). To the operator \( L^* \) there exist two real constants \( c^* > 0 \) and \( \lambda_0 > 0 \) such that

\[ c^* \| (L^* - \lambda I) w \|_{0,G} \geq (1 + |\lambda|) \| w \|_{0,G} \]

for all \( w \in H^{(k)}(G) \) and for all \( \lambda \in \mathbb{C} \) with \( |\lambda| \geq \lambda_0 \) and \( \Re \lambda \leq 0 \).

**Proof:** We choose a real valued function \( \rho \in C_0^\infty(\mathbb{R}) \) satisfying the conditions

\[ \rho(t) = \begin{cases} 1 & \text{for } |t| \leq 1 \\ 0 & \text{for } |t| \geq 2 \end{cases} \]

and \( 0 \leq \rho(t) \leq 1 \) elsewhere. For an arbitrary function \( u \in X \) and an arbitrary \( \mu \in \mathbb{R} \) we define the function \( \omega \) by

\[ \omega(x, t) := u(x) \rho(t) e^{i\mu t}. \]

Now let us take \( T > 2 \). Then one has \( \omega \in \mathcal{C}(G \times J) \), and we get by Lemma 1 for \( \theta \in \mathbb{R}, \pi/2 \leq \theta \leq 3\pi/2, \)

\[ \| (L(D) + e^{i\theta} \partial_\gamma^2) \omega \|_{0,G \times J} \leq \| (L(D) + e^{i\theta} \partial_\gamma^2) \omega \|_{0,G} + \| \omega \|_{0,G \times J}. \]  

(2.3)

We estimate now the first term on the right hand side of (2.3):

\[ \| (L(D) + e^{i\theta} \partial_\gamma^2) \omega \|_{0,G \times J} \]

\[ = \int_{-T}^{T} \int_{\Gamma} |L(D) \omega(x, t) + e^{i\theta} u(x) e^{i\mu \rho(t)} + 2i\mu \rho(t) - \mu^2 \rho(t)| \, dt \, dx \]

\[ \leq 2 \int_{-T}^{T} \int_{\Gamma} |\rho(t)| \, |L(D) u(x) - \mu^2 e^{i\theta} u(x)|^2 + |u(x)|^2 |\rho''(t) + 2i\mu \rho'(t)|^2 \, dt \, dx \]

\[ \leq C_1 \int_{\Gamma} |L(D) u(x) - \mu^2 e^{i\theta} u(x)|^2 \, dx + C_2 \int_{\Gamma} |u(x)|^2 \, dx \]

\[ = C_1 \| (L(D) - \mu^2 e^{i\theta}) u \|_{0,G}^2 + C_2 \| u \|_{0,G}^2. \]  

(2.4)

with two positive constants \( C_1 \) and \( C_2 \) (independent of \( u \) and \( T \)).

1) By \( I \) we denote the identity operator on \( L^2(G) \).
The second term on the right hand side of (2.3) satisfies the estimate
\[ \|\omega\|_{L^2(G)}^2 \leq \int_0^T \int_{\partial G} |u(x, t)| e^{it} dx dt \leq C_3 \|u\|_{L^2(G)}^2 \] (2.5)
with a third positive constant \( C_3 \).

By (2.1) we receive for the left hand term of (2.3)
\[ \|\omega\|_{L^2(G)}^2 \leq \int_0^T \int_{\partial G} |L(D) \omega(x, t)| + |\partial_t \omega(x, t)| + |\omega(x, t)| \] (2.6)
dt dx,
\[ \leq 2 \|L(D) u\|_{L^2(G)}^2 + 2 \mu^4 \|u\|_{L^2(G)}^2 + 2 \|u\|_{L^2(G)}^2 \]
\[ \geq (2\mu^4 + 2) \|u\|_{L^2(G)}^2. \]

From (2.3) we get by the estimates (2.4)-(2.6)
\[ (2\mu^4 + 2 - C_2 - C_3) \|u\|_{L^2(G)}^2 \leq C_1 \|\omega\|_{L^2(G)}^2 \]
We choose \( \lambda_0 = 1 + (C_2 + C_3)^{1/2} \). Then for all \( \lambda = \mu^2 e^{i\theta} \) with \( \mu \in \mathbb{R}, \mu^2 \geq \lambda_0 \) and \( \pi/2 \leq \theta \leq 3\pi/2 \) the inequality
\[ c^* \|\omega - \lambda I u\|_{L^2(G)}^2 \leq (1 + |\lambda|) \|u\|_{L^2(G)}^2 \]
with \( c^* := C_4^{1/2} \) is valid for all \( u \in X \). By continuous extension to \( H^{(k)}(G) \) we get the assertion of the lemma.

2.2 Analogously to the elliptic case we prove now

**Lemma 3:** Let \( G \) be a bounded open set with Property P. Then the range of the operator \( L^- - \lambda I : H^{(k)}(G) \to L^2(G) \) coincides with the whole space \( L^2(G) \) for all \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda \leq 0 \), and the operator \( L^- - \lambda I \) is bijective.

**Proof:** Let be \( \lambda \in \mathbb{C}, \text{Re} \lambda \leq 0; \) and take an arbitrary element \( f \in L^2(G) \). We have to prove that there exists a uniquely determined element \( u \in H^{(k)}(G) \) with \( (L^- - \lambda I) u = f \).

We define
\[ B_1(u, \psi) := B(u, \psi) - \lambda(u, \psi)_{L^2(G)} \text{ for all } u, \psi \in X. \]
Now one has by (1.3) (as \( \text{Re} \lambda \leq 0 \) the estimate
\[ \text{Re} B_1(u, u) = B(u, u) - \lambda(u, u)_{L^2(G)} \leq \|u\|_{L^2(G)}^2. \]
(2.7)

On the other hand, the sesquilinear form \( B_1(\cdot, \cdot) \) is bounded on \( X, \|B_1(u, v)\| \leq (1 + |\lambda|) \|u\|_{L^2(G)} \|v\|_{L^2(G)} \). As \( B_1(\cdot, \cdot) \) can be continuously extended to a bounded sesquilinear form with property (2.7) to \( H_0^{(k)}(G) \), we get by the theorem of LAX-MILGRAM (cf. [3: pp. 41-46]) the existence of a unique element \( u \in H_0^{(k)}(G) \) such that \( (f, \phi)_{L^2(G)} = B_1(u, \phi) \) is valid for all \( \phi \in C_0^\infty(G) \). By the regularity result for the solution \( u \in H_0^{(k)}(G) \) of the homogeneous Dirichlet problem mentioned in 1.6 we have \( u \in H^{(k)}(G) \).

By partial integration (see (1.5)) we get further
\[ (f, \psi)_{L^2(G)} = B_1(u, \psi) = \langle (L^- - \lambda I) u, \psi \rangle \]
for all \( \phi \in C_0^\infty(G) \). This proves that the operator \( L^- - \lambda I : H^{(k)}(G) \to L^2(G) \) is bijective. \[ \Box \]
2.3 By Lemma 3 the domain of the resolvent \((\lambda I - L^\ast)^{-1}\) of the operator \(L^\ast\) coincides with the whole space \(L^2(G)\) for all \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda \leq 0\), hence for every such \(\lambda\) the operator \((\lambda I - L^\ast)^{-1}\) is bounded. Furthermore we prove the following estimate for the resolvent.

**Lemma 4:** Let \(G\) be a bounded open set with Property \(P\). Then the resolvent \((\lambda I - L^\ast)^{-1}\) of the operator \(L^\ast\) satisfies the estimate

\[
\|(\lambda I - L^\ast)^{-1}\| \leq \frac{c}{1 + |\lambda|}
\]  

for all \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda \leq 0\) where \(c\) is a positive constant.

**Proof:** The operator \((\lambda I - L^\ast)^{-1}\) is bounded for every \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda \leq 0\), i.e., for every such \(\lambda\) there exists a positive constant \(c_1\) such that \(\|(\lambda I - L^\ast)^{-1}\| \leq c_1\). Then every \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda \leq 0\) has a neighborhood \(U_2\) in \(\mathbb{C}\) for which

\[
\|(\lambda' I - L^\ast)^{-1}\| \leq 2c_1 \quad \text{for all } \lambda' \in U_2.
\]

Thus, the resolvent \((\lambda I - L^\ast)^{-1}\) is uniformly bounded on each compact subset of the half plane \(\text{Re} \lambda \leq 0\), and we have with a constant \(c_0 > 0\) (which is independent of \(\lambda\))

\[
\|(\lambda I - L^\ast)^{-1}\| \leq c_0
\]  

for all \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda \leq 0\) and \(|\lambda| \leq \lambda_0\) (for \(\lambda_0\) see Lemma 2). On the other hand, by Lemma 2 we have

\[
\|(\lambda I - L^\ast)^{-1}\| \leq \frac{c^*}{1 + |\lambda|}
\]

for all \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda \leq 0\) and \(|\lambda| \geq \lambda_0\). Hence with \(c := \max\{c^*, c_0(1 + \lambda_0)\}\) it follows from (2.9) and (2.10) that the relation

\[
\|(\lambda I - L^\ast)^{-1}\| \leq \frac{c}{1 + |\lambda|}
\]

holds for all \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda \leq 0\).

3. The Cauchy Problem

3.1 We will now investigate the generalized Cauchy problem mentioned in the Introduction. Because of (1.9) this problem is an initial-boundary value problem with generalized homogeneous boundary values on \(\partial G\) and with non-homogeneous initial values on \(G\).

**Problem C:** Let \(G\) be a bounded open set in \(\mathbb{R}^n\) with Property \(P\). Further let\(^\)\( f : \mathbb{R}^+_0 \to L^2(G)\) be a given uniformly Hölder continuous (with an exponent \(\beta\), \(0 < \beta \leq 1\)) function on \(\mathbb{R}^+_0\) with values in \(L^2(G)\), \(f \in C^{0,\beta}(\mathbb{R}^+_0, L^2(G))\), and \(u_0\) a given element of \(H^{(\beta)}(G)\). We want to find all functions \(u : \mathbb{R}^+_0 \to H^{(\beta)}(G)\) from the class \(C^{0,\beta}(\mathbb{R}^+_0, L^2(G)) \cap C^1([0, \infty), L^2(G))\) which solve the generalized evolution equation

\[
\frac{d}{dt} u(t) + L^\ast u(t) = f(t) \quad \text{for } t > 0
\]

and satisfy the initial condition \(u(0) = u_0\).

\(^\)\( We use the notations \(\mathbb{R}^+ := \{r \in \mathbb{R} \mid r > 0\}\) and \(\mathbb{R}^+_0 := \mathbb{R}^+ \cup \{0\}\).
3.2 We are now able to use for this problem the theory presented by A. Friedman in [3: Part 2, especially 2.1—2.13, pp. 101—158] (cf. also [6: pp. 85—109]). For this theory it is not necessary that \( L^* \) is the closure of an elliptic differential operator but that

\[
D(L^*) = H^{(k)}(G)
\]

is dense in \( L^2(G) \) and that with a constant \( c > 0 \) the estimate

\[
\|(\lambda I - L^*)^{-1}\| \leq \frac{c}{1 + |\lambda|}
\]

is valid for all \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda \leq 0 \). These conditions guarantee that the operator \(-L^*\) is an infinitesimal generator of an analytic semigroup of bounded linear operators in \( L^2(G) \), with the help of this fact: one proves the existence and uniqueness of the fundamental solution \( V(\cdot, \tau) \) for the operator \( \frac{d}{dt} + L^* \).

By a fundamental solution we mean a function \(^3\)

\[
V(\cdot, \tau) : \{ t \in \mathbb{R} \mid t \leq t < \infty \} \times \{ \tau \in \mathbb{R} \mid \tau \geq 0 \} \rightarrow \mathcal{B}(L^2(G))
\]

with the properties:

I. The operator \( V(t, \tau) \) (\( \in \mathcal{B}(L^2(G)) \)) is strongly continuous in \( t, \tau \) for \( 0 \leq t \leq t < \infty \).

II. The derivative \( \frac{\partial}{\partial t} V(t, \tau) \) exists in the strong topology of \( \mathcal{B}(L^2(G)) \) and belongs to \( \mathcal{B}(L^2(G)) \) for \( 0 \leq \tau < t < \infty \) and is also strongly continuous in \( t \) for \( \tau < t < \infty \).

III. The range of \( V(t, \tau) \) lies in \( D(L^*) \) (\( = H^{(k)}(G) \)) for all \( t, \tau \) with \( 0 \leq \tau < t < \infty \).

IV. The function \( V(\cdot, \tau) \) is the solution of the Cauchy problem

\[
\frac{\partial}{\partial t} V(t, \tau) + L^* V(t, \tau) = 0 \quad \text{for} \quad \tau < t < \infty
\]

and \( V(\tau, \tau) = I \).

Finally we get from the considerations of A. Friedman [3: p. 109].

Theorem 5: Problem C has a unique solution \( u \). This solution has the expression

\[
u(t) := V(t, 0) u_0 + \int_0^t V(t, s) f(s) \, ds.
\]

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