Fredholmness and finite section method for Toeplitz operators in \( L^p(\mathbb{Z}_+ \times \mathbb{Z}_+) \) with piecewise continuous symbols II

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In this paper we prove sufficient conditions for Fredholmness of discrete Toeplitz operators with piecewise continuous symbols on the space \( L^p \) over the quarter-plane and for the applicability, of the finite section method to such operators. The methods used here are based on a bilocalization technique and the local principle of Douglas and Krupnik. Part I of this work contained the proofs of the necessity of the corresponding conditions, the necessary definitions, and the formulation of the main results.

This paper continues the paper [1] and it is devoted to the proof of the sufficiency part of the Theorems 1 and 2 of [1]. All definitions and notations used here and not being explicitly explained were introduced in [1].

§ 5. Further auxiliary propositions on one-dimensional Toeplitz operators

With regard to a theory of the finite section method for two-dimensional Toeplitz operators some one-dimensional results have to be precised. This is the purpose of the present section.

Set
\[
F_p = \{ \{A_n\}_{n=0}^{\infty} : A_n : \text{Im} P_n \to \text{Im} P_n, \quad \|A_n\| = \sup_{n} \|A_n P_n\|_{L^p(\mathbb{Z})} < \infty \},
\]
\[
G_p = \{ \{A_n\}_{n=0}^{\infty} \in F_p : \|A_n P_n\|_{L^p(\mathbb{Z})} \to 0 \quad (n \to \infty) \}.
\]

By \( A_p \) we denote the closure in \( F_p \) of the collection of all sequences of the form
\[
\{A_n\} = \left\{ \sum_{j=1}^{r} \prod_{k=1}^{s} T_n(a_{jk}) \right\}.
\]


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where $r, s \in \mathbb{Z}_+$, $a_{jk} \in PC_0$. Finally, let $I_p$ be the set of all sequences $\{A_n\} \in F_p$ which are of the form

$$\{A_n\} = \{P_n KP_n + W_n K_1 W_n + C_n\},$$

where $K, K_1 \in \mathcal{S}_p$ and $\{C_n\} \in \mathcal{G}_p$ (the operators $P_n$ and $W_n$ were defined in Section 2).

As in [2] the inclusion $I_p \subseteq \mathcal{A}_p$, can be proved and then in the same fashion as in [8] it can be verified that $I_p$ forms a closed two-sided ideal in the Banach algebra $\mathcal{A}_p$ and that $\mathcal{A}_p/I_p$ is commutative. Let $\tau_p$ denote the canonical projection of $\mathcal{A}_p$ onto $\mathcal{A}_p/I_p$, $N_p$ the maximal ideal space of $\mathcal{A}_p/I_p$ and $T_{N_p}$ the Gelfand map of $\mathcal{A}_p/I_p$ into $C(N_p)$. Note that $\{T_n(a)\} \in \mathcal{A}_p$ if $a \in PC_p(T)$.

**Proposition 3:** Let $a \in PC_p(T)$. If $T(a) \in G(\Omega(l^r))$ for every $r \in [p, q]$, $1/p + 1/q = 1$, then $\tau_p(T_n(a)) \in G(\mathcal{A}_p/I_p)$.

Recall that $a \in PC_p(T)$ implies $T(a) \in \Omega(l^r) \forall r \in [p, q]$.

**Proof:** Applying the local principle of I. C. Gohberg and N. Yu. Krupnik [4] with the method of [8] we find that $\tau_p(T_n(a))$ is locally equivalent at $t_0 \in T$ to $\tau_p(T_n(a_{t_0}))$, where $a_{t_0}$ is defined by

$$a_{t_0}(t) = \begin{cases} a(t_0 + 0), & \text{arg } t_0 < \text{arg } t < \text{arg } t_0 + \pi \\ a(t_0 - 0), & \text{arg } t_0 - \pi < \text{arg } t < \text{arg } t_0, \end{cases}$$

t \in T. Let $\mathcal{Q}_p$ denote the closed "lentiform" domain in $C$ which has as its boundary the two circular arcs

$$\{(1 - s_p(\mu)) a(t_0 - 0) + s_p(\mu) a(t_0 + 0) : \mu \in [0, 1]\},$$

$$\{(1 - s_p(\mu)) a(t_0 + 0) + s_p(\mu) a(t_0 - 0) : \mu \in [0, 1]\}$$

(recall the notation (2.2) of Section 2). According to Theorem II of Section 2 we have $T(a_{t_0}) - \lambda I \in \Pi_p(P_n)$ for $\lambda \notin \mathcal{Q}_p$.

Now, let $\mathfrak{M}_p$ be the Banach algebra playing the dominant part in [8], i.e. $\mathfrak{M}_p$ consists of all sequences $\{A_n\}_{n=0}^\infty$, $A_n : \text{Im } P_n \rightarrow \text{Im } P_n$ for which there exist operators $A, A_1 \in \mathfrak{M}(l^p)$ such that $A_n P_n \rightarrow A, A_n^* P_n \rightarrow A^*, W_n A_n W_n \rightarrow A_1, W_n A_n^* W_n \rightarrow A_1^*$ (the convergence in the strong sense, the asterisk denoting the Hermitian conjugate). $I_p$ again forms a closed two-sided ideal in $\mathfrak{M}_p$. Let us by $\tau_p$ denote the canonical projection of $\mathfrak{M}_p$ onto $\mathfrak{M}_p/I_p$.

The results of [8] now imply that $\tau_p(T_n(a_{t_0})) - \lambda I \in \Pi_p(P_n)$ if only $T(a_{t_0}) - \lambda I \in \Pi_p(P_n)$. Then $\lambda$ does not belong to the spectrum of $\tau_p(T_n(a_{t_0}))$ in $\mathfrak{M}_p/I_p$, i.e.

$$\lambda \notin \text{spec}_{\mathfrak{M}_p/I_p}(\tau_p(T_n(a_{t_0}))).$$

Hence

$$\text{spec}_{\mathfrak{M}_p/I_p}(\tau_p(T_n(a_{t_0}))) \subseteq \mathcal{Q}_p.$$

By [7: 10.18] we conclude

$$\text{spec}_{\mathfrak{M}_p/I_p}(\tau_p(T_n(a_{t_0}))) \subseteq \mathcal{Q}_p.$$  \tag{1}

On condition that $T(a) \in G(\Omega(l^r))$ for every $r \in [p, q]$, $1/p + 1/q = 1$, from Theorem G of Section 2 follows that $0 \notin \mathcal{Q}_p$ for every $t_0 \in T$. Thus (1) shows that $\tau_p(T_n(a_{t_0})) \in G(\mathcal{A}_p/I_p)$ and application of the local principle of [4] gives $\tau_p(T_n(a)) \in G(\mathcal{A}_p/I_p)$.

**Proposition 4:** Let $N \in N_p$. Then there exist $r \in [p, q]$ $(1/p + 1/q = 1)$, $\xi \in T$, and $\mu \in [0, 1]$ such that

$$J_{N_p, \tau_p}(T_n(a)) (N) = (1 - s_r(\mu)) a(\xi - 0) + s_r(\mu) a(\xi + 0)$$

for every $a \in PC_p(T)$. 


Proof: Let $W$ be the Wiener algebra of all functions on $T$ with absolutely convergent Fourier series. Define the map $\omega$ by

$$\omega: W \to \mathbb{A}_p/I_p, a \mapsto \tau_p[T_n(a)].$$

Obviously, $\omega$ is a continuous algebraic homomorphism. If $\varphi_N$ is the complex homomorphism (continuous multiplicative linear functional) associated with $N \in N_p$, i.e. $N = \text{Ker} \varphi_N$, then $\varphi_N \circ \omega$ is a complex homomorphism on $W$. Consequently, there is a $\zeta \in T$ with

$$\varphi_N(\tau_p[T_n(a)]) = a(\zeta) \tag{2}$$

for every $a \in W$. For $\vartheta \in T$ define the function $a_\vartheta \in PC_0$ by

$$a_\vartheta(t) = \begin{cases} 1, & \text{arg } \vartheta < \text{arg } t < \text{arg } \vartheta + \pi \\ 0, & \text{arg } \vartheta - \pi < \text{arg } t < \text{arg } \vartheta \end{cases}$$

and the complex number $c$ by

$$c = \varphi_N(\tau_p[T_n(a_\zeta)]) \tag{3}$$

($\zeta$ given by (2)). From the inclusion (1) we get the existence of an $r \in [p, q]$ and of a $\mu \in [0, 1]$ such that

$$c = (1 - s_r(\mu)) \cdot 0 + s_r(\mu) \cdot 1 = s_r(\mu). \tag{4}$$

Now, the identity $a_\zeta + a_{-\zeta} = 1$ gives

$$\varphi_N(\tau_p[T_n(a_{-\zeta})]) = 1 - s_r(\mu). \tag{5}$$

Thus we have evaluated $\varphi_N$ at $\tau_p[T_n(a_{\zeta})]$ and at $\tau_p[T_n(a_{-\zeta})]$. Let $c(\vartheta)$ be the value of $\varphi_N$ at $\tau_p[T_n(a_{\vartheta})]$ for $\vartheta \in T$, $\vartheta \pm \zeta$, i.e. define $c(\vartheta)$ by

$$c(\vartheta) = \varphi_N(\tau_p[T_n(a_{\vartheta})]). \tag{6}$$

We are going to prove that

$$c(\vartheta) = \left(1 - s_r(\mu)\right) a_\vartheta(\zeta - 0) + s_r(\mu) a_\vartheta(\zeta + 0). \tag{7}$$

For this purpose we choose a function $b \in W$ being identically 1 in a neighborhood of $\zeta$ and identically 0 in a neighborhood of $\vartheta$ and $-\vartheta$. Then, obviously, $b \cdot a_\vartheta \in W$ and we have

$$a_\vartheta(\zeta) = b(\zeta) a_\vartheta(\zeta)$$

(since $b(\zeta) = 1$)

$$= \varphi_N(\tau_p[T_n(ba_\vartheta)])$$

(because of (2))

$$= \varphi_N(\tau_p[T_n(b)]) \cdot \varphi_N(\tau_p[T_n(a_\vartheta)])$$

(because of (2) and (6))

$$= c(\vartheta) \tag{since b(\zeta) = 1}$$

Hence

$$c(\vartheta) = a_\vartheta(\zeta) = \begin{cases} 1, & \text{arg } \vartheta < \text{arg } \zeta < \text{arg } \vartheta + \pi \\ 0, & \text{arg } \vartheta - \pi < \text{arg } \zeta < \text{arg } \vartheta \end{cases}$$

$$= \left(1 - s_r(\mu)\right) a_\vartheta(\zeta - 0) + s_r(\mu) a_\vartheta(\zeta + 0),$$

since $a_\vartheta(\zeta - 0) = a_\vartheta(\zeta + 0)$ for $\vartheta \pm \zeta$. Thus by (3-7) we have expressed $\varphi_N(\tau_p[T_n(a_{\vartheta})])$ for every $\vartheta \in T$ in terms of $r$, $\mu$ and $\zeta$ such as it is desired. Considering
finite linear combinations we get
\[ \varphi_n(\tau_p[T_n(\chi)]) = (1 - s_r(\mu)) \chi(\xi - 0) + s_r(\mu) \chi(\xi + 0) \]
for every \( \chi \in PC_0 \).

Given an arbitrary function \( a \in PC_p(T) \) we can find \( a_j \in PC_0 \) such that
\[ ||T_n(a) - \{T_n(a_j)\}||_{A_p} \to 0 \quad (j \to \infty). \]

Hence, first of all,
\[ ||T_n(a) - \{T_n(a_j)\}||_{A_p} \to 0 \quad (j \to \infty). \]
Furthermore
\[ ||\{T_n(\alpha)\} - \{T_n(\alpha_j)\}||_{A_p} = \sup_n ||T_n(\alpha - \alpha_j)|| \]
\[ \geq \liminf_n ||T_n(\alpha - \alpha_j)|| = ||T(\alpha - \alpha_j)||_{\mathcal{M}_p} = ||T(\alpha - \alpha_j)||_{\mathcal{M}_p} \]
and
\[ \begin{aligned}
(1 - s_r(\mu)) (\alpha - \alpha_j) (\xi - 0) + s_r(\mu) (\alpha - \alpha_j) (\xi + 0) \\
\leq \max_{(\xi, \lambda) \in \mathbb{T} \times [0, 1]} |(1 - s_w(\lambda))(\alpha - \alpha_j) (\xi - 0) + s_w(\lambda)(\alpha - \alpha_j) (\xi + 0)| \\
= \max_{(\xi, \lambda) \in \mathbb{T} \times [0, 1]} |(1 - s_w(\lambda))(\alpha - \alpha_j) (\xi - 0) + s_w(\lambda)(\alpha - \alpha_j) (\xi + 0)| \\
= \max \{ ||T(\alpha - \alpha_j)||_{\mathcal{M}_p} ||T(\alpha - \alpha_j)||_{\mathcal{M}_p} \}
\end{aligned} \]
Thus
\[ ||T(\alpha - \alpha_j)||_{\mathcal{M}_p} = ||T(\alpha - \alpha_j)||_{\mathcal{M}_p} \]
as \( j \to \infty \). Combining (8) and (9) we get the assertion in full generality.

§ 6. Some lemmas on Banach algebras

The simple facts stated here will be applied in the Sections 8 and 9.

Let \( \mathfrak{A} \) be a Banach algebra with unit \( e \) and \( \mathfrak{F} \subseteq \mathfrak{A} \) a closed two-sided ideal. Suppose that \( \mathfrak{A}/\mathfrak{F} \) is commutative. By \( j \) we denote the canonical projection of \( \mathfrak{A} \) onto \( \mathfrak{A}/\mathfrak{F} \), by \( N \) the maximal ideal space of \( \mathfrak{A}/\mathfrak{F} \) and by \( \Gamma_N \) the Gelfand map of \( \mathfrak{A}/\mathfrak{F} \) into \( C(N) \).

For concrete examples put \( \mathfrak{A} = \mathfrak{A}_p, \mathfrak{F} = \mathfrak{F}_p \) or \( \mathfrak{A} = \mathfrak{A}_p, \mathfrak{F} = \mathfrak{I}_p \). In what follows \( \otimes \) always denotes the projective tensor product. Put \( \mathfrak{A}^2 = \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{F} \) and denote by \( j_2 \) the canonical projection of \( \mathfrak{A} \otimes \mathfrak{A} \) onto \( \mathfrak{A}^2 \). Then \( U^2 = \{e\} \otimes \mathfrak{A} \otimes \mathfrak{F} \) is naturally embedded in \( \mathfrak{A}^2 \) and let clos \( U^2 \) denote the closure of \( U^2 \) in \( \mathfrak{A}^2 \).

**Lemma 4:** clos \( U^2 \) is contained in the centre of \( \mathfrak{A}^2 \).

**Proof:** It suffices to prove that
\[ j_2(e \otimes a) j_2(b \otimes c) = j_2(b \otimes c) j_2(e \otimes a) \]
for arbitrary $a, b, c \in \mathcal{A}$. But since $\mathcal{A}/\mathfrak{I}$ has been supposed to be commutative, we have $ac - ca \in \mathfrak{I}$ from what (1) results.

**Lemma 5:** Define the map $\varphi: \mathcal{A}/\mathfrak{I} \rightarrow \text{clos } U^2$ by $\varphi: ja \mapsto j_2(e \otimes a)$, $a \in \mathcal{A}$. Then

(i) $\varphi$ is defined correctly,
(ii) $\varphi$ is an algebraic homomorphism,
(iii) $\varphi$ is continuous.

**Proof:** (i) Let $ja = jb$. Then $a - b \in \mathfrak{I}$, hence $e \otimes (a - b) \in \mathcal{A} \otimes \mathfrak{I}$, i.e. $j_2(e \otimes a) = j_2(e \otimes b)$.
(ii) is obvious.
(iii) We have

\[\|j_2(e \otimes a)\| = \inf \{\|e \otimes a + c\|: c \in \mathcal{A} \otimes \mathfrak{I}\}\]
\[\leq \inf \{\|e \otimes a + e \otimes k\|: k \in \mathfrak{I}\} = \inf \{\|a + k\|: k \in \mathfrak{I}\} = \|ja\|\]

Thus $\text{clos } U^2$ is a commutative Banach algebra with unit. Let $M$ be the maximal ideal space of $\text{clos } U^2$ and $\Gamma_M$ the Gelfand map. Then

\[
\mathcal{A} \xrightarrow{j} \mathcal{A}/\mathfrak{I} \xrightarrow{\varphi} C(N) \xrightarrow{\text{clos}} \mathcal{A} \otimes \mathfrak{I} \xrightarrow{\text{clos } U^2} \mathcal{A}
\]

**Lemma 6:** If $m \in M$ then $n = \varphi^{-1}(m) \overset{\text{def}}{=} \{a \in \mathcal{A}/\mathfrak{I}: qa \in m\}$ belongs to $N$. If $\varphi_m$ denotes the complex homomorphism on $\text{clos } U^2$ associated with $m$, i.e. $m = \text{Ker } \varphi_m$, then $\varphi_m \circ \varphi$ is a complex homomorphism on $\mathcal{A}/\mathfrak{I}$ and $\varphi^{-1}(m) = \text{Ker } (\varphi_m \circ \varphi)$.

**Remark:** It is easily shown that, vice versa, for every $n \in N$ there exists an $m \in M$ such that $n = \varphi^{-1}(m)$, but this fact is not needed for our purposes.

**Proof of Lemma 6:** Let $m \in M$, $m = \text{Ker } \varphi_m$, $\varphi = \varphi_m \circ \varphi$. By Lemma 5, $\varphi$ is a continuous algebraic homomorphism of $\mathcal{A}/\mathfrak{I}$ into $C$. From $\varphi(je) = \varphi_m(0) = \varphi_m(j_2(e \otimes e)) = 1$ we get $\varphi \equiv 0$, i.e. $n = \text{Ker } \varphi \in N$. The equality $n = \varphi^{-1}(m)$ follows from the equivalences

\[ja \in \varphi^{-1}(m) \Leftrightarrow qa \in m \Leftrightarrow \varphi_m qja = 0 \Leftrightarrow \varphi ja = 0 \Leftrightarrow ja \in \text{Ker } \varphi = n\]

**Lemma 7:** Let $a \in \mathcal{A}$. Then for every $m \in M$

\[(\Gamma_M qja)(m) = (\Gamma_N ja)(\varphi^{-1}(m)).\]

**Proof:** If $m = \text{Ker } \varphi_m$, then, by Lemma 6, $n = \varphi^{-1}(m) = \text{Ker } (\varphi_m \circ \varphi) \in N$, hence

\[(\Gamma_M qja)(m) = \varphi_m qja = (\varphi_m \circ \varphi) ja = (\Gamma_N ja)(n)\]


Our proofs of the sufficiency of the conditions of the Theorems 1 and 2 are based upon the local principle of R. G. DOUGLAS and N. YA. KRUPNIK (see [3] for the case of $C^*$-algebras and [5] for the case of Banach algebras). This local principle reads as follows:

Let $\mathcal{C}$ be the centre of a Banach algebra $\mathcal{A}$, $\mathcal{A}_0$ a closed subalgebra of $\mathcal{C}$ and $\mathfrak{R}$ the...
maximal ideal space of $\mathcal{A}$. For $M \in \mathcal{A}$ we denote by $\mathfrak{Z}_M$ the closed two-sided ideal generated by $M$ in $\mathcal{A}$, i.e.

$$\mathfrak{Z}_M = \text{clos} \{ \sum A_kX_k : A_k \in \mathcal{A}, X_k \in M \}.$$ 

Finally, let $\pi_M$ denote the canonical projection of $\mathcal{A}$ onto $\mathcal{A}/\mathfrak{Z}_M$. Then for $A \in \mathcal{A}$

$$A \in \mathcal{A} \Leftrightarrow \pi_M A \in G(\mathcal{A}/\mathfrak{Z}_M) \quad \forall M \in \mathcal{A}.$$ 

§ 8. Sufficiency of the conditions of Theorem 1

Put $B_p^0 = B_p \otimes B_p/\mathfrak{R}_p \otimes \mathfrak{R}_p$, $B_p^1 = B_p \otimes B_p/\mathfrak{R}_p \otimes \mathfrak{R}_p$, $B_p^2 = B_p \otimes B_p/\mathfrak{R}_p \otimes \mathfrak{R}_p$ and denote by $\alpha_0, \alpha_1, \alpha_2$ the canonical projections of $B_p \otimes B_p$ onto $B_p^0, B_p^1, B_p^2$ respectively. For $\alpha \in PC_p(T^2)$ we have $W(\alpha) \in B_p \otimes B_p$. In order to show that $W(\alpha) \in \Phi_p(T^2)$ it is sufficient to prove that $\alpha_0 W(\alpha) \in G B_p^0$. On the other hand, $\alpha_0 W(\alpha) \in G B_p^0$ follows from $\alpha_1 W(\alpha) \in G B_p^1$ and $\alpha_2 W(\alpha) \in G B_p^2$ (cf. [6]).

Let us, for example, prove that $\alpha_2 W(\alpha) \in G B_p^2$. We set $U_p^2 = \{ I \} \otimes B_p/\mathfrak{R}_p \otimes \mathfrak{R}_p$ and denote by $\text{clos} \ U_p^2$ the closure of $U_p^2$ in $B_p^2$. By Lemma 4 $\text{clos} \ U_p^2$ is contained in the centre of $B_p^2$ and is therefore a commutative Banach algebra. Let $\mathfrak{M}_p$ be the maximal ideal space of $\text{clos} \ U_p^2$ and $\mathfrak{M}_p$ the Gelfand map of $\text{clos} \ U_p^2$ into $C(\mathfrak{M}_p)$. Finally, we define the map $\gamma : B_p/\mathfrak{R}_p \rightarrow \text{clos} \ U_p^2$ by $\gamma : \sigma_0 A \mapsto \alpha_2(I \otimes A)$, $A \in B_p$, (cf. Lemma 5). Thus

$$\begin{aligned}
B_p &\xrightarrow{\sigma_0} B_p \otimes B_p/\mathfrak{R}_p \\
&\xrightarrow{\gamma} C(\mathfrak{M}_p)
\end{aligned}$$

For $M \in \mathfrak{M}_p$ define $J_M$ to be the closed two-sided ideal generated by $M$ in $B_p^2$, i.e.

$$J_M = \text{clos} \{ \sum A_jX_j : A_j \in B_p^2, X_j \in M \}.$$ 

Let $\pi_M$ denote the canonical projection of $B_p^2$ onto $B_p^2/J_M$. In Proposition 5 below we shall prove that

$$\alpha_2 W(\alpha) - \alpha_2(T(\sigma_0 \otimes I) \otimes I) \in J_M,$$

where $(\xi_0, \mu_0) \in T \times [0, 1]$ has to be chosen in accordance with the identification of $\bar{\mathfrak{A}}_p$ with $T \times [0, 1]$ as that point on $T \times [0, 1]$ which corresponds to $N = \gamma^{-1}(M)$ (cf. Lemma 6). Finally, in Proposition 6 it will be proved that

$$\pi_M \alpha_2(T(\sigma_0 \otimes I) \otimes I) \in G(B_p^2/J_M)$$

is a consequence of $T(\sigma_0 \otimes I) \in G(\mathfrak{A}_p)$. Application of the local principle of R. G. Douglas and N. Y. Z. Krupnik (with $\mathfrak{A} = B_p^2$, and $\mathfrak{A}_0 = \text{clos} \ U_p^2$) then gives $\alpha_2 W(\alpha) \in G B_p^2$ if only $T(\sigma_0 \otimes I) \in G(\mathfrak{A}_p)$ for all $(\xi_0, \mu_0) \in T \times [0, 1]$.

Proposition 5: Let $\alpha \in PC_p(T^2)$, $M \in \mathfrak{M}_p$ and $N = \gamma^{-1}(M) \in \mathfrak{A}_p$. Let $(\xi_0, \mu_0)$ be the point on the cylinder $T \times [0, 1]$ which corresponds to $N \in \mathfrak{A}_p$ via the homeomorphism $\mathfrak{A}_p \cong T \times [0, 1]$. Then

$$\alpha_2 W(\alpha) - \alpha_2(T(\sigma_0 \otimes I) \otimes I) \in J_M.$$ 

Proof: At first we consider the case that $\alpha$ is a finite sum of the form

$$a(\xi, \eta) = \sum b_i(\xi) c_i(\eta), \quad (\xi, \eta) \in T^2,$$
where $b_i, c_i \in PC_p(T)$. Then $W(a) = \sum_i T(b_i) \otimes T(c_i)$. By formula (4.2)
$$a^i_{\omega, \mu}(t) = \sum_i \left[ T_{g_p, \sigma_p} T(c_i)(\zeta_0, \mu_0) \right] b_i(t), \quad t \in T.$$ 
Thus (the subscript $p$ will now be dropped for the sake of convenience)
$$\alpha_2 W(a) - \alpha_2 \left( T(a^i_{\omega, \mu}) \otimes I \right) = \alpha_2 \sum_i T(b_i) \otimes T(c_i) - \alpha_2 \sum_i T(b_i) \otimes \left[ T_{g_p, \sigma_p} T(c_i)(\zeta_0, \mu_0) \right] I$$
$$= \sum_i \alpha_2(T(b_i) \otimes I) \cdot \alpha_2(I \otimes \left[ T(c_i) - (T_{g_p, \sigma_p} T(c_i)(\zeta_0, \mu_0) \right]) I].$$

Because $\alpha_2(T(b_i) \otimes I) \in \mathcal{B}_p$, it remains to show that
$$\alpha_2(I \otimes \left[ T(c_i) - (T_{g_p, \sigma_p} T(c_i)(\zeta_0, \mu_0) \right]) I}] \in M.$$

This on its hand is in view of $\alpha_2(I \otimes A) = \gamma A$ equivalent to
$$\gamma T(c_i) - (T_{g_p, \sigma_p} T(c_i)(\zeta_0, \mu_0) \right]) I] \in M,$$
i.e. to
$$(T_{g_p, \sigma_p} T(c_i)(\zeta_0, \mu_0) \right)) \in M.$$ 

But the latter equality immediately follows from Lemma 7. Thus, for functions of the form (2) the relation (1) has been proved.

For an arbitrary function $a \in PC_p(T^2)$ we can find functions $a_i(\xi, \eta) = \sum_i b_i(t) c_i(t)(\eta)$, $(\xi, \eta) \in T^2$, of the form (2) such that
$$||W(a) - W(a_j)||_{\mathcal{B}_p} \to 0 \quad (j \to \infty).$$

From (3) we get
$$||\alpha_2 W(a) - \alpha_2 W(a_j)||_{\mathcal{B}_p} \to 0 \quad (j \to \infty)$$

and in order to prove (1) for the general case it remains to show that
$$||T(a^i_{\omega, \mu}) - T(a^j_{\omega, \mu}||_{\mathcal{B}_p} \to 0 \quad (j \to \infty).$$

This follows by an argument used already in the proof of Lemma 3: with the help of Lemma 2 we can show that $(T(a^i_{\omega, \mu}))_{i=1}^\infty$ forms a Cauchy sequence in $\mathcal{B}_p$ and then from (3) we can conclude that its limit is just $T(a^i_{\omega, \mu})$.

Proposition 6: If $T(a^i_{\omega, \mu}) \in G\mathcal{Q}(L)$ then
$$\pi_M \alpha_2(T(a^i_{\omega, \mu}) \otimes I) \in G(\mathcal{B}_p^2/J_M).$$

Proof: First of all, we show that $T^{-1}(a^i_{\omega, \mu})$ belongs not only to $G(L)$, but even to $\mathcal{B}_p$. Indeed, the spectrum of $\sigma_p T(a^i_{\omega, \mu}) \in \mathcal{B}_p/\mathcal{K}_p$ is in virtue of (2.1) just the curve
$$\{(1 - s_p(\mu)) a^i_{\omega, \mu}(t - 0) + s_p(\mu) a^i_{\omega, \mu}(t + 0) : t \in T, \mu \in [0, 1]\}.$$ 

Since $T(a^i_{\omega, \mu})$ has been supposed to be invertible in $G(L)$, Theorem $G$ of Section 2 shows that the origin cannot lie on this curve, i.e. $\sigma_p T(a^i_{\omega, \mu}) \in G(\mathcal{B}_p/\mathcal{K}_p)$. Thus there
is a regularizer \( R \in \mathfrak{R}_p \) (modulo \( \mathfrak{S}_p \)). From \( T^{-1}(a_{1,\mu}) - R \in \mathfrak{R}_p \subset \mathfrak{R}_p \) we get \( T^{-1}(a_{1,\mu}) \in \mathfrak{R}_p \). Now \( (T^{-1}(a_{1,\mu}) \otimes I)(T(a_{1,\mu}) \otimes I) = I \otimes I \) implies
\[
\pi_M \alpha_2(T^{-1}(a_{1,\mu}) \otimes I) \cdot \pi_M \alpha_2(T(a_{1,\mu}) \otimes I) = \pi_M \alpha_2(I \otimes I),
\]
i.e. \( \pi_M \alpha_2(T(a_{1,\mu}) \otimes I) \in G(\mathfrak{R}_p^2/J_M) \)

§ 9. Sufficiency of the conditions of Theorem 2

The Theorems \( \Phi \) and \( G \) of Section 2 and a little geometrical consideration give the following result.

Lemma 8: Let \( a \in PC_p(T) \). If \( T(a) \in G_2(l) \) and \( T(a) \in G_2(l') \) then \( T(a) \in G_2(l) \) for every \( r \in [p, q] \). Furthermore, \( T(a) \in G_2(l) \) if and only if \( T(a) \in G_2(l') \), where \( 1/p + 1/q = 1 \) and \( a(t) = a(1/t), t \in T \).

Before proceeding to the subject of this section itself, it is necessary to prove still one auxiliary fact.

Proposition 7: Let \( a \in PC_p(T^2) \) and suppose that \( W(a), W(a_1), W(a_2), W(a_{12}) \in \Phi (\mathfrak{R} \otimes \mathfrak{R}) \). Then \( T(a_{1,\mu}) \in G_2(l) \) and \( T(a_{2,\mu}) \in G_2(l') \) for every \( r, v \in [p, q] \) \( (1/p + 1/q = 1), \zeta \in T \) and \( \mu \in [0, 1] \).

Proof: In accordance with Theorem 1, from \( W(a), W(a_1) \in \Phi (\mathfrak{R} \otimes \mathfrak{R}) \) we get \( T(a_{1,\mu}), T((a_1)_{\mu}) \in G_2(l) \). Since \( (a_{1})_{\mu} = (a_{1})_{\mu} \) (recall the notation \( a(t) = a(1/t), t \in T \)), we get \( T(a_{1,\mu}), T((a_1)_{\mu}) \in G_2(l). \) Applying Lemma 8, what results is
\[
T(a_{1,\mu}) \in G_2(l) \quad \forall \, v \in [p, q] \quad \forall \, (\xi, \mu) \in T \times [0, 1]. \tag{1}
\]
Furthermore, \( W(a_2) \in \Phi (\mathfrak{R} \otimes \mathfrak{R}) \) implies by Theorem 1, \( T((a_2)_{\mu}) \in G_2(l) \). But \( (a_2)_{\mu} = (a_{1})_{\mu} \), thus by Lemma 8
\[
T((a_2)_{\mu}) \in G_2(l) \quad \forall \, (\xi, \mu) \in T \times [0, 1]. \tag{2}
\]
Due to \( W(a_{12}) \in \Phi (\mathfrak{R} \otimes \mathfrak{R}) \) we have, again by Theorem 1, \( T((a_{12})_{\mu}) \in G_2(l) \). But \( (a_{12})_{\mu} = (a_{1})_{\mu} \), thus by Lemma 8
\[
T((a_{12})_{\mu}) \in G_2(l) \quad \forall \, (\xi, \mu) \in T \times [0, 1]. \tag{3}
\]
From (2), (3), and Lemma 8 we get
\[
T(a_{1,\mu}) \in G_2(l) \quad \forall \, v \in [p, q] \quad \forall \, (\xi, \mu) \in T \times [0, 1]. \tag{4}
\]
Analogously one can prove that
\[
T(a_{2,\mu}) \in G_2(l') \quad \forall \, v \in [p, q] \quad \forall \, (\xi, \mu) \in T \times [0, 1], \tag{5}
\]
\[
T(a_{2,\mu}) \in G_2(l') \quad \forall \, v \in [p, q] \quad \forall \, (\xi, \mu) \in T \times [0, 1]. \tag{6}
\]
Now, recalling (3,4), it is easy to show that the following equivalence holds
\[
T(a_{1,\mu}) \in G_2(l) \quad \forall \, v \in [p, q] \quad \forall \, (\xi, \mu) \in T \times [0, 1] \quad \Rightarrow \quad T(a_{2,\mu}) \in G_2(l') \quad \forall \, v \in [p, q] \quad \forall \, (\xi, \mu) \in T \times [0, 1].
\]
Thus from (1), we get
\[
T(a_{2,\mu}) \in G_2(l') \quad \forall \, v \in [p, q] \quad \forall \, (\xi, \mu) \in T \times [0, 1]. \tag{7}
\]
and from (4)
\[ T(a_{r,v},^2) \in G \mathcal{O}(p) \quad \forall \, v \in [p, q] \quad \forall \, (\zeta, \mu) \in T \times [0, 1]. \] (8)
Combining (7), (8), and Lemma 8 we arrive at
\[ T(a_{r,v},^2) \in G \mathcal{O}(p) \quad \forall \, r, v \in [p, q] \quad \forall \, (\zeta, \mu) \in T \times [0, 1]. \]
Similarly, (5) and (6) give
\[ T(a_{r,v},^2) \in G \mathcal{O}(p) \quad \forall \, r, v \in [p, q] \quad \forall \, (\zeta, \mu) \in T \times [0, 1]. \]

Now we are in position to proceed in analogy to the Fredholm theory considered in the preceding section. Put
\[ A_p,^0 = A_p \otimes A_p/I_p \otimes I_p, \quad A_p,^1 = A_p \otimes A_p/I_p \otimes A_p, \]
\[ A_p,^2 = A_p \otimes A_p/I_p \otimes I_p \]
and denote the corresponding canonical projections by \( \beta_0, \beta_1, \beta_2, \) respectively. Note that \( \{W_n(a)\} \in A_p \otimes A_p \) for \( a \in PC_p(T^2) \) \( (W_n(a) \) are the “finite sections” of \( W(a) \) defined by (3.5)). In the same way as in [2] we can prove that \( W(a) \in \mathcal{I}_p(p_n \otimes p_n) \) if only \( \beta_0(W_n(a)) \in GA_p,^0 \) and \( W(a), W(a_1), W(a_2), W(a_{12}) \in G \mathcal{O}(p \otimes p) \). The usual standard trick [6] may be applied to derive \( \beta_0(W_n(a)) \in GA_p,^0 \) from \( \beta_1(W_n(a)) \in GA_p,^1 \) and \( \beta_2(W_n(a)) \in GA_p,^2 \).

We shall prove that \( \beta_2(W_n(a)) \in GA_p,^2 \) by means of the local principle of R. G. DOUGLAS and N. YA. KRUPNIK. Lemma 4 yields that the closure \( \text{clos} \, U_p,^2 \) of \( U_p,^2 \) is contained in the centre of \( A_p,^2 \) and is therefore a commutative Banach algebra. Let \( M_p \) denote the maximal ideal space of \( \text{clos} \, U_p,^2 \), \( \mathcal{I}_{M_p} \) the Gelfand map and \( \delta \) the map of \( A_p/I_p \) into \( \text{clos} \, U_p,^2 \) defined by \( \delta : r_{p}(A_n) \to \beta_0([p_n \otimes A_n]) \) (cf. Lemma 5). Thus
\[ A_p, \delta \to A_p/I_p \]
\[ \beta \to \text{clos} \, U_p,^2, \beta \to \mathcal{C}(M_p) \]
For \( M \in M_p \) let
\[ J_M = \text{clos} \, \{ \sum \, A_j X_j : A_j \in A_p,^2, X_j \in M \}. \]
Then \( J_M \) is a closed two-sided ideal in \( A_p,^2 \). We denote by \( \pi_M \) the canonical projection of \( A_p,^2 \) onto \( A_p,^2/J_M \). For \( M \in M_p \) we have, by Lemma 6, \( N = \delta^{-1}(M) \in N_p \). Due to Proposition 4 there exist \( r \in [p, q] \) \( (1/p + 1/q = 1) \), \( \mu \in [0, 1] \) and \( \zeta \in T \) such that
\[ (f_{p,q},^2(T_n(a))) (N) = (1 - s_r(\mu)) a(\zeta - 0) + s_r(\mu) a(\zeta + 0) \]
for every \( a \in PC_p(T) \). Now in the same way as Proposition 5 was proved, one may show that
\[ \beta_2(W_n(a)) - \beta_2(T_n(a_{r,v},^2) \otimes P_n) \in J_M \]
Combining the just proved Proposition 7 with Proposition 8 proved below, we obtain that under the conditions of Theorem 2
\[ \pi_M \beta_2(T_n(a_{r,v},^2) \otimes P_n) \in G(A_p,^2/J_M) \]
for every \( r \in [p, q] \), \( \zeta \in T \), \( \mu \in [0, 1] \), \( M \in M_p \). Applying the local principle quoted in Section 7 (with \( \mathcal{H} = A_p,^2 \), \( \mathcal{H}_0 = \text{clos} \, U_p,^2 \) we get \( \beta_2(W_n(a)) \in GA_p,^2 \).
Proposition 8: Let \( a \in PC_p(T^2), \ M \in \mathbb{M}_p \) and \((r; \zeta, \mu)\) be the triplet corresponding to \( M \) by Proposition 4. If \( T(a, r, \zeta, \mu) \in G_p(l') \) for every \( v \in [p, q] \) \((1/p + 1/q = 1)\) then
\[
\tau_M \beta_2 T_n(a, r, \zeta, \mu) \otimes P_n \in G(A_p^2/J_M).
\]

Proof: By Proposition 3 we have
\[
\tau_p \{ T_n(a, r, \zeta, \mu) \} \in G(A_p/1_p). \tag{9}
\]
Furthermore, \( T(a, r, \zeta, \mu) \in G_p(l') \) for every \( v \in [p, q] \) and Lemma 8 gives that
\[
T(a, r, \zeta, \mu), T((a, r, \zeta, \mu)^{-1}) \in G_p(l'). \tag{10}
\]
From (9) and (10) we may with the method of the proof of Satz 3 in [8] derive that there exists a \((R_n, \nu) \in A_p\) such that \( R_n T(a, r, \zeta, \mu) = P_n + C_n'\), where \((C_n') \in A_p\) and \(\|C_n'\| \to 0 \ (n \to \infty)\).

Thus
\[
\beta_2 \{ P_n' \otimes P_n \} \cdot \beta_2 \{ T_n(a, r, \zeta, \mu) \otimes P_n \} = \frac{1}{2} \{ P_n + C_n' \otimes P_n \} - \frac{1}{2} \{ P_n \otimes P_n \} + \frac{1}{2} \{ C_n' \otimes P_n \}. \tag{11}
\]

In case \(\|C_n'\| > 0\) we have \( C_n' \otimes P_n = C_n' \|C_n'\|^{-1/2} \otimes \|C_n'\|^{1/2} P_n\). Put

\[
D_n = \begin{cases} 0, & \|C_n'\| = 0, \\ C_n' \|C_n'\|^{-1/2}, & \|C_n'\| > 0. \end{cases}
\]

\[
E_n = \begin{cases} 0, & \|C_n'\| = 0, \\ \|C_n'\|^{1/2} P_n, & \|C_n'\| > 0. \end{cases}
\]

Similarly as in [2] the inclusion \( G_p \subseteq A_p \) can be proved. Consequently, \(\{D_n \otimes E_n\} \in A_p \otimes A_p\). Since, moreover, \(\{D_n \otimes E_n\} \in I_p \otimes I_p \subseteq A_p \otimes I_p\) and \(C_n' \otimes P_n = D_n \otimes E_n\), we get \(\beta_2 \{ C_n' \otimes P_n \} = \beta_2 \{ D_n \otimes E_n \} = 0\). Now (11) gives the invertibility of \(\tau_M \beta_2 \{ T_n(a, r, \zeta, \mu) \otimes P_n \} \) in \( A_p^2 \), implying, of course, the invertibility of \(\tau_M \beta_2 \{ T_n(a, r, \zeta, \mu) \otimes P_n \} \) in \( A_p^2/J_M\). \(\blacksquare\)

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