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Estimates for Blow-Up Solutions to Nonlinear Elliptic Equations with p-Growth in the Gradient

V. Ferone, E. Giarrusso, B. Messano and M. R. Posteraro

Abstract. In this paper we deal with blow-up solutions to p-Laplacian equations with a nonlinear gradient term. We prove comparison results for the solutions in terms of the solutions to suitable symmetrized problems defined in a ball. We analyze two cases where the form of the term which depends on the gradient plays different roles.

Keywords. p-Laplacian equations, blow-up solutions, rearrangements

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1. Introduction

Problems involving Laplacian or p-Laplacian operators with blow-up boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, have been studied by many authors. Namely, the following typical problem:

$$\begin{cases} \Delta_p u = f(u) & \text{in } \Omega \\ u(x) \to \infty & \text{as } x \to \partial\Omega \end{cases} \tag{1.1}$$

has been considered under suitable assumptions on f. Here the p-Laplacian operator is $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, and it reduces to Laplacian for $p = 2$.

A problem of type (1.1) has first been considered by Bieberbach [12] when $n = 2$, $p = 2$ and $f(u) = e^u$. Successively, a long list of papers has been

Vincenzo Ferone, Ester Giarrusso, Maria Rosaria Posteraro: Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli Federico II, Complesso Universitario Monte S. Angelo, Via Cintia, 80126 Napoli, Italy;
ferone@unina.it, ester.giarrusso@unina.it, posterar@unina.it
Basilio Messano: Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli Federico II, Piazzale V. Tecchio 80, 80125 Napoli, Italy;
messano@unina.it

dedicated to the study of problems in the form (1.1), addressing various issues related to it.

Existence, uniqueness and asymptotic results for solutions to problem (1.1) have been given, for example, in [7, 14, 17, 21, 22, 24, 30]. Also the possibility to obtain a priori estimates for solutions to (1.1), via symmetrization techniques, has been investigated (see, for example, [7, 17, 32]). We recall that there exists a wide bibliography concerning the case of different boundary conditions such as, for example, homogeneous Dirichlet conditions (see, for instance, [1, 35]).

In the present paper we are interested in the case in which the equation (1.1) has a lower-order term which depends on the gradient. The presence of such a term has been considered in [4, 10, 15, 16, 23, 31, 37]. For example, problems with blow-up boundary condition for an equation involving the Laplacian and a term which depends on the gradient are connected with a stochastic control problem when the state of the controlled system is a diffusion process (see [23]).

We are interested in establishing a priori estimates for solutions to problem in the form

$$\begin{cases} \Delta_p u \pm |\nabla u|^p = f(u) & \text{in } \Omega \\ u(x) \to \infty & \text{as } x \to \partial\Omega, \end{cases} \tag{1.2}$$

under suitable assumptions on f which depend on the sign which appears in front of the term $|\nabla u|^p$ (see Sections 3 and 4).

In particular, we prove that any solution of (1.2) can be compared with the solution v of the "symmetrized" problem

$$\begin{cases} \Delta_p v \pm |\nabla v|^p = f(v) & \text{in } \Omega^\# \\ v(x) \to \infty & \text{as } x \to \partial\Omega^\#, \end{cases} \tag{1.3}$$

where $\Omega^\#$ is the ball centered at the origin such that $|\Omega^\#| = |\Omega|$ (we denote by $|E|$ the Lebesgue measure of a set $E \subseteq \mathbb{R}^n$). Namely, we prove that

$$\min_{\Omega} u(x) \geq \min_{\Omega^\#} v(x). \tag{1.4}$$

By some examples given in Sections 3 and 4, it is possible to show that $\min_{\Omega^\#} v(x)$ can be explicitly written in terms of $|\Omega|$ for some particular choices of f. Such a result states that one can obtain a sharp estimate from below of a solution to (1.2) in terms of the solution to a problem defined in a set (ball) having the same measure of Ω. On the other hand, our approach allows us also to estimate u from above in terms of the solution w to a problem in the form (1.3) defined in a ball B such that $\min_\Omega u(x) = \min_B w(x)$ (see Remarks 3.1 and 4.2). From this point of view one obtains an estimate of $|\Omega|$ in terms of $\min_\Omega u(x)$.

We finally remark that we have to treat probems in the form (1.2) in different ways depending on the sign of the term containing the gradient, even though the comparison results one can state in both cases appear to be essentially the same. Indeed, a natural way to obtain a comparison result consists in making use of a change of the unknown function. When the positive sign appears in (1.2) such a transformation leads to a suitable blow-up problem, while, if the negative sign appears in (1.2), then transformation leads to a homogeneous Dirichlet problem. This means that different arguments have to be used in the two cases.

2. The radial case

In this section we give some properties of radial solutions to two boundary value problems defined in a ball which will be used in the next sections.

We start with the problem with radial symmetric data

$$\begin{cases} \Delta_p v = \beta(v) & \text{in } B_R \\ v(x) \to \infty & \text{as } x \to \partial B_R, \end{cases} \tag{2.1}$$

where B_R is the ball centered at the origin with radius R, and $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following conditions:

i) $\beta(s)$ is a continuous increasing function such that $\beta(0) = 0$ and $\beta(s) > 0$, for all $s > 0$;

ii) (Keller condition)
$$\int^{\infty} \frac{1}{\left(\int_0^s \beta(\tau)d\tau\right)^{\frac{1}{p}}} \, ds < +\infty.$$

Under the hypotheses i) and ii) problem (2.1) admits a unique solution (see [14, Theorem 6.4 and Remark 3.4]) which is radial. Therefore, such a solution $v = v(r)$ satisfies a Cauchy problem in the form

$$\begin{cases} \dfrac{1}{r^{n-1}}\left(r^{n-1} \mid v'(r) \mid^{p-2} v'(r)\right)' = \beta(v(r)) & \text{in } (0, R) \\ v(0) = M, \qquad v'(0) = 0, \end{cases} \tag{2.2}$$

where $M = \min_{[0,R]} v(r)$. Using, for example, results contained in [34] (see also [27]) one can state the following result.

Proposition 2.1. *Let* $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfy conditions* i) *and* ii). *Let* $v_{R_1}(|x|) = v_{R_1}(r)$ *and* $v_{R_2}(|x|) = v_{R_2}(r)$ *be the radial solutions to problem* (2.1), *resp., in the balls* B_{R_1} *and* B_{R_2}, *with* $R_1, R_2 \in]0, +\infty[$. *Set* $M_1 = \min_{B_{R_1}} v_{R_1}(x)$, $M_2 = \min_{B_{R_2}} v_{R_2}(x)$ *and* $\bar{R} = \min\{R_1, R_2\}$. *If* $\tilde{v}(s) = v\left(\left(\frac{s}{\omega_n}\right)^{\frac{1}{n}}\right)$, *then the following statements are true:*

I) $M_1 > M_2 \Longrightarrow \tilde{v}_{R_1}(s) > \tilde{v}_{R_2}(s), \forall s \in [0, \omega_n \bar{R}^n]$;

II) $M_1 > M_2 \Longrightarrow R_1 < R_2$.

III) *The function* $M(R)$ *is a continuous function in* $]0,+\infty[$.

Remark 2.2. Let v be a solution to problem (2.1). The function $\tilde{v}(s)$ defined in Proposition 2.1 satisfies the equality

$$\tilde{v}(s) = M_R + c_{n,p} \int_0^s \tau^{(\frac{1}{n}-1)\frac{p}{p-1}} \left(\int_0^\tau \beta\big(\tilde{v}(\sigma)\big) d\sigma \right)^{\frac{1}{p-1}} d\tau, \tag{2.3}$$

where $M_R = \min_{B_R} v(|x|) = v(0) = \tilde{v}(0)$ and $c_{n,p} = \left(n\omega_n^{\frac{1}{n}}\right)^{-\frac{p}{p-1}}$.

Indeed, $v(r)$ satisfies the equation in (2.2) and, observing that $\tilde{v}(s)$ is increasing, the equation in (2.2) becomes

$$\frac{d}{ds}\left(s^{(1-\frac{1}{n})p}(\tilde{v}'(s))^{p-1}\right) = \frac{1}{\left(n\omega_n{}^{\frac{1}{n}}\right)^p}\beta(\tilde{v}(s)).$$

From here, integrating from 0 to s, we have

$$\tilde{v}'(s) = s^{(\frac{1}{n}-1)\frac{p}{p-1}} \left(\int_0^s \frac{1}{\left(n\omega_n{}^{\frac{1}{n}}\right)^p} \beta(\tilde{v}(\sigma)) d\sigma \right)^{\frac{1}{p-1}},$$

and then a further integration between 0 and s gives (2.3).

Now, let us consider the problem

$$\begin{cases} -\Delta_p v = \beta(v) & \text{in } B_R \\ v = 0 & \text{on } \partial B_R, \end{cases} \tag{2.4}$$

where B_R is the ball centered at the origin with radius R. Assume that the function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following conditions:

j) $\beta(r)$ is decreasing;

jj) $\lim_{r\to 0^+} \beta(r) < +\infty$.

According to [26] problem (2.4) has a unique solution $v \in W_0^{1,p}(\Omega)$ which is radial. Then setting $v(|x|) = v(r)$ and $\tilde{v}(s) = v\big((\frac{s}{\omega_n})^{\frac{1}{n}}\big)$ proceeding as in Remark 2.2, we have

$$\tilde{v}(s) = M_R - c_{n,p} \int_0^s \tau^{\left(\frac{1}{n}-1\right)\left(\frac{p}{p-1}\right)} \left(\int_0^\tau \beta(\tilde{v}(\sigma))\, d\sigma \right)^{\frac{1}{p-1}} d\tau, \tag{2.5}$$

where $M_R = \max_{B_R} v(x)$.

Proposition 2.3. *Let* $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfy conditions* j) *and* jj). *Let* $v_{R_1}(r)$ *and* $v_{R_2}(r)$ *be the solutions to problem* (2.4), *resp., in the balls* B_{R_1} *and* B_{R_2}, *with* $R_1, R_2 \in]0,+\infty[$. *Set* $M_1 = \max_{B_{R_1}} v_{R_1}(x), M_2 = \max_{B_{R_2}} v_{R_2}(x)$ *and* $\bar{R} = \min\{R_1, R_2\}$. *If* $\tilde{v}(s) = v\big((\frac{s}{\omega_n})^{\frac{1}{n}}\big)$ *the following statements are true:*

I) $M_1 > M_2 \Longrightarrow \tilde{v}_{R_1}(s) > v_{R_2}(s)$, $\forall s \in [0, \omega_n \bar{R}^n]$;
II) $M_1 > M_2 \Longrightarrow R_1 > R_2$.
III) *The function* $M(R)$ *is continuous in* $]0, +\infty[$.

Proof. From (2.5) we have

$$\tilde{v}_{R_1}(s) = M_1 - c_{n,p} \int_0^s \tau^{\left(\frac{1}{n}-1\right)\frac{p}{p-1}} \left(\int_0^\tau \beta\big(\tilde{v}_{R_1}(\sigma)\big) d\sigma \right)^{\frac{1}{p-1}} d\tau$$

$$\tilde{v}_{R_2}(s) = M_2 - c_{n,p} \int_0^s \tau^{\left(\frac{1}{n}-1\right)\frac{p}{p-1}} \left(\int_0^\tau \beta\big(\tilde{v}_{R_2}(\sigma)\big) d\sigma \right)^{\frac{1}{p-1}} d\tau.$$

If I) is not satisfied, being $M_1 = \tilde{v}_{R_1}(0) > M_2 = \tilde{v}_{R_2}(0)$, there exists $\bar{s} \in]0, \omega_n \bar{R}^n]$ such that $\tilde{v}_{R_1}(\bar{s}) = \tilde{v}_{R_2}(\bar{s})$, $\tilde{v}_{R_1}(s) > \tilde{v}_{R_2}(s)$, for all $s \in]0, \bar{s}[$. Consequently,

$$\begin{aligned} 0 &= \tilde{v}_{R_1}(\bar{s}) - \tilde{v}_{R_2}(\bar{s}) \\ &= M_1 - M_2 - c_{n,p} \int_0^{\bar{s}} \tau^{-\frac{p(n-1)}{n(p-1)}} \left[\left(\int_0^\tau \beta\big(\tilde{v}_{R_1}(\sigma)\big) d\sigma \right)^{\frac{1}{p-1}} - \left(\int_0^\tau \beta\big(\tilde{v}_{R_2}(\sigma)\big) d\sigma \right)^{\frac{1}{p-1}} \right] d\tau \\ &> 0. \end{aligned}$$

Thus, we have a contradiction and I) is proved.

According to the uniqueness of the solution in the ball with radius R_1, see [26], it is enough to prove that $R_1 \geq R_2$. If $R_1 < R_2$, we will have by I) $0 = \tilde{v}_{R_1}(R_1) < v_{R_2}(R_2) = 0$ that is absurdo. Then II) is proved.

Obviously, from II) we derive that the function $M(R)$ is increasing. Now, arguing by contradiction, suppose that for some R_0 the inequality $M(R_0^-) < M(R_0^+)$ holds. Hence $M(R_0^-) \leq M(R_0) \leq M(R_0^+)$, and at least one of these inequalities is strict. If $M(R_0^-) < M(R_0)$, take $M_\alpha \in]M(R_0^-), M(R_0)[$ and consider the following problem:

$$\begin{cases} \dfrac{1}{r^{n-1}} \big(r^{n-1} |v'(r)|^{p-2} v'(r)\big)' = \beta(v(r)) & \text{in } (0, R) \\ v(0) = M_\alpha, \qquad v'(0) = 0. \end{cases}$$

Let v_α be the unique (classical) solution of such a problem (see [34]). Since $M_\alpha < M_{R_0}$ by I) $v_\alpha(r) < v_{R_0}(r)$. Then there exists R_α such that $v_\alpha(R_\alpha) = 0$ and, by II), $R_\alpha < R_0$, that implies $M_\alpha < M(R_0^-) < M_\alpha$, that is absurdo. The proposition is completely proved. □

3. First comparison result

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. We consider the problem

$$\begin{cases} \operatorname{div}(a(x, u, \nabla u)) + H(x, u, \nabla u) = f(u) & \text{in } \Omega \\ u(x) \to \infty & \text{as } x \to \partial\Omega, \end{cases} \tag{3.1}$$

where $a(x,s,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ is a Carathéodory function satisfying, for some $p>1$, the following ellipticity condition:

$$a(x,s,\xi)\xi\ge|\xi|^p,\quad \text{a.e. } x\in\Omega,\ \forall(s,\xi)\in\mathbb{R}\times\mathbb{R}^n. \tag{3.2}$$

$H(x,s,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ is a Carathéodory function satisfying, for some positive constant γ, the inequality

$$a(x,s,\xi)\xi\le H(x,s,\xi)\le\gamma|\xi|^p,\quad \text{a.e. } x\in\Omega,\ \forall(s,\xi)\in\mathbb{R}\times\mathbb{R}^n, \tag{3.3}$$

and $f:\mathbb{R}\to\mathbb{R}_+$ is a continuous function.

We say that $u\in W^{1,p}_{loc}(\Omega)$ is a weak solution to problem (3.1) if $a(x,u,\nabla u)\in\left(L^{p'}(\Omega)\right)^n$, $f(u)\in L^{p'}_{loc}(\Omega)$ and

$$-\int_{\Omega'}a(x,u,\nabla u)\nabla\psi\,dx+\int_{\Omega'}H(x,u,\nabla u)\psi\,dx=\int_{\Omega'}f(u)\psi\,dx, \tag{3.4}$$

for every $\psi\in W^{1,p}_0(\Omega')\cap L^\infty(\Omega')$, with $\Omega'\subset\subset\Omega$ and $\lim_{x\to\partial\Omega}u(x)=+\infty$.

Let us consider the problem

$$\begin{cases}\Delta_p v+|\nabla v|^p=f(v) & \text{in } \Omega^\#\\ v(x)\to\infty & \text{as } x\to\partial\Omega^\#,\end{cases} \tag{3.5}$$

where $\Omega^\#$ is the ball such that $|\Omega^\#|=|\Omega|$.

Our aim is to prove a comparison result between the solutions to problems (3.1) and (3.5).

Theorem 3.1. *Let $u\in W^{1,p}_{loc}(\Omega)$ be a weak solution to problem* (3.1). *If $\beta(s)=(p-1)^{1-p}s^{p-1}f((p-1)\log s)$, $s>0$, satisfies* i) *and* ii) *in Section* 2 *and $v\in W^{1,p}_{loc}(\Omega^\#)$ is the radial solution to problem* (3.5), *then*

$$\operatorname*{ess\,inf}_{x\in\Omega}u(x)\ge\operatorname*{ess\,inf}_{x\in\Omega^\#}v(x). \tag{3.6}$$

Before giving the proof of the above theorem we briefly recall the definition of increasing rearrangement of a measurable function. For a measurable function $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}$ one can define the distribution function μ_u of u as follows:

$$\mu_u(t)=\left|\{x\in\Omega:u(x)<t\}\right|,\quad t\in\mathbb{R}.$$

The function μ_u is increasing; moreover, its generalized inverse function is the increasing rearrangement u_* of u

$$u_*(s)=\inf\{t\in\mathbb{R}:\mu_u(t)>s\},\quad s\in[0,|\Omega|].$$

The spherically symmetric increasing rearrangement of u is defined by

$$u_{\#}(x) = u_*(\omega_n |x|^n), \quad x \in \Omega^{\#},$$

where $\Omega^{\#}$ is the ball centered at the origin having the same measure as Ω and ω_n is the measure of the unit ball in $\mathbb{R}^n$. For an exhaustive treatment of the properties of rearrangements we refer to [2, 19, 20, 29].

We finally recall that for a measurable set $E \subset \mathbb{R}$ of finite measure, the following well known isoperimetric inequality holds true (see [13]):

$$P(E) \geq n\omega^{\frac{1}{n}} |E|^{1-\frac{1}{n}},$$

where $P(E)$ denotes the perimeter of E.

Proof of Theorem 3.1. Let $u \in W^{1,p}_{loc}(\Omega)$ be a weak solution of problem (3.1). Let us consider, for $t > 0$, the following function belonging to $W^{1,p}_0(\Omega') \cap L^\infty(\Omega')$, for some $\Omega' \subset\subset \Omega$:

$$\varphi(x) = \begin{cases} e^u h, & \bar{u} < t - h \\ e^u (t - \bar{u}), & t - h \leq \bar{u} < t \\ 0, & \bar{u} \geq t, \end{cases}$$

where $h > 0$ and $\bar{u} = e^{\frac{u}{p-1}}$. If we use $\varphi(x)$ as test function in (3.4), using the ellipticity condition (3.2) and inequality (3.3), we have

$$\begin{aligned} (p-1)^{p-1} \int_{t-h\leq \bar{u} < t} |\nabla \bar{u}|^p \, dx &\leq \frac{1}{p-1} \int_{t-h \leq \bar{u} < t} a(x, u, \nabla u) \nabla u \, e^{\frac{pu}{p-1}} \, dx \\ &\leq \int_{\bar{u} < t-h} f(u) \, e^u h \, dx + \int_{t-h\leq \bar{u} < t} f(u) \, e^u (t - \bar{u}) \, dx. \end{aligned}$$

Passing to the limit in h, in a standard way (see, for example, [1, 35]) we obtain

$$\frac{d}{dt} \int_{\bar{u} < t} |\nabla \bar{u}|^p dx \leq \int_{\bar{u} < t} \beta(\bar{u}) \, dx. \tag{3.7}$$

It is well known that, from the Hölder inequality, the Fleming–Rishel coarea formula and the isoperimetric inequality, for a.e. $t > M = \operatorname{ess\,inf}_{x\in\Omega} \bar{u}(x) = e^{\frac{u_*(0)}{p-1}}$, it holds

$$\frac{\left(n\omega_n^{\frac{1}{n}} (\mu_{\bar{u}}(t))^{1-\frac{1}{n}}\right)^p}{(\mu'_{\bar{u}}(t))^{p-1}} \leq \frac{d}{dt} \int_{\bar{u} < t} |\nabla \bar{u}|^p dx.$$

Consequently, by the properties of rearrangements, from (3.7) we derive the following inequality, a.e. in $[0, |\Omega|]$:

$$(U'(s))^{p-1} \left(n\omega_n^{\frac{1}{n}} s^{1-\frac{1}{n}}\right)^p \leq \int_0^s \beta(U(\sigma)) \, d\sigma, \tag{3.8}$$

where we have put $U(s) = \bar{u}_*(s) = e^{\frac{u_*(s)}{p-1}}$. After an integration, from (3.8) it is immediate to obtain

$$U(s) \leq M + c_{n,p} \int_0^s \tau^{\left(\frac{1}{n}-1\right)\frac{p}{p-1}} \left[\int_0^\tau \beta(U(\sigma)) d\sigma \right]^{\frac{1}{p-1}} d\tau, \quad \forall s \in [0, |\Omega|[.$$

where $c_{n,p} = \left(n\omega_n^{\frac{1}{n}}\right)^{-\frac{p}{p-1}}$.

For $\varepsilon > 0$, because of Proposition 2.1, there exists a ball D_ε such that the radial solution w_ε to problem

$$\begin{cases} \Delta_p w_\varepsilon = \beta(w_\varepsilon) & \text{in } D_\varepsilon \\ w_\varepsilon(x) \to \infty & \text{as } x \to \partial D_\varepsilon, \end{cases}$$

is such that $\min_{D_\varepsilon} w_\varepsilon(x) = M+\varepsilon$. Set $w_\varepsilon(x) = W_\varepsilon(\omega_n|x|^n)$, $W_\varepsilon(s)$ is the solution to problem

$$W_\varepsilon(s) = M + \varepsilon + c_{n,p} \int_0^s \tau^{(\frac{1}{n}-1)\frac{p}{p-1}} \left[\int_0^\tau \beta(W_\varepsilon(\sigma)) d\sigma \right]^{\frac{1}{p-1}} d\tau, \tag{3.9}$$

in $[0, s_\varepsilon]$, where $s_\varepsilon = |D_\varepsilon| < |\Omega|$. Being

$$M = \min_{s\in[0,s_\varepsilon]} U(s) < M + \varepsilon = \min_{s\in[0,s_\varepsilon]} W_\varepsilon(s),$$

let us prove that

$$U(s) < W_\varepsilon(s), \quad \forall s \in [0, s_\varepsilon]. \tag{3.10}$$

On the contrary, there exists $\bar{s} \in [0, s_\varepsilon]$, clearly $\bar{s} < |\Omega|$, such that $U(\bar{s}) = W_\varepsilon(\bar{s})$ and $U(s) < W_\varepsilon(s)$, for all $s \in [0, \bar{s}]$. Then, from (3.8) and (3.9) it follows

$$\begin{aligned} 0 &= U(\bar{s}) - W_\varepsilon(\bar{s}) \\ &\leq -\varepsilon + c_{n,p} \int_0^{\bar{s}} \tau^{\left(\frac{1}{n}-1\right)\frac{p}{p-1}} \left[\left(\int_0^\tau \beta(U(\sigma))\, d\sigma \right)^{\frac{1}{p-1}} - \left(\int_0^\tau \beta(W_\varepsilon(\sigma))\, d\sigma \right)^{\frac{1}{p-1}} \right] d\tau \\ &< 0, \end{aligned}$$

so we have a contradiction. Condition (3.10) is thus proved.

Let us observe that $W_\varepsilon(s)$ is monotonically convergent to a function $W(s)$, with respect to ε. Letting $\varepsilon \to 0$ in (3.10), it follows

$$U(s) \leq W(s), \quad \forall s \in [0, s_0[, \tag{3.11}$$

where $s_0 = \lim_{\varepsilon\to 0} s_\varepsilon \leq | \Omega |$ and $W(s)$ is the solution of

$$W(s) = M + c_{n,p} \int_0^s \tau^{\left(\frac{1}{n}-1\right)\frac{p}{p-1}} \left[\int_0^\tau \beta(W(\sigma)) d\sigma \right]^{\frac{1}{p-1}} d\tau, \tag{3.12}$$

in $[0, s_0[$. Actually it holds that, for every $\tilde{s} < s_0$, $W_\varepsilon(s) \to W(s)$, $s \in [0, \tilde{s}]$, hence $W(s)$ satisfies (3.12) in $[0, s_0[$.

Set $w(x) = W(\omega_n |x|^n)$, $w(x)$ is the solution of problem

$$\begin{cases} \Delta_p w = \beta(w) & \text{in } B \\ w(x) \to \infty & \text{as } x \to \partial B, \end{cases} \tag{3.13}$$

where B is the ball such that $|B| = s_0$.

Set $\nu(x) = \log(w^{p-1}(x))$, $x \in B$, $\nu(x)$ is the unique solution of problem

$$\begin{cases} \Delta_p \nu + |\nabla \nu|^p = f(\nu) & \text{in } B \\ \nu(x) \to \infty & \text{as } x \to \partial B. \end{cases} \tag{3.14}$$

Let us observe that $W(x) = e^{\frac{\nu(x)}{p-1}}$ so $W(s) = e^{\frac{\nu_*(s)}{p-1}}$. Then, being $U(s) = e^{\frac{u_*(s)}{p-1}}$, from (3.11) it follows $u_*(s) \le \nu_*(s)$, for all $s \in [0, |B|]$, where $|B| = s_0 \le |\Omega|$.

Finally, considering the radial solution $v(x)$ to problem (3.5), the inequality (3.6) follows from condition $|B| \le |\Omega|$ because of II) of Proposition 2.1. The theorem is thus proved. □

Remark 3.2. Let us observe that in the proof of Theorem 3.1 it is contained the comparison result

$$u_*(s) \le \nu_*(s), \quad \forall s \in [0, |B|],$$

where ν is the solution to the problem (3.14) in a ball B, such that

$$\operatorname*{ess\,inf}_{x \in \Omega} u = \operatorname*{ess\,inf}_{x \in B} \nu.$$

Remark 3.3. We observe that, in order to prove the comparison result in Theorem 3.1 it is necessary to know that for problem (3.13) there is a unique solution and the properties established in Proposition 2.1 hold true. Every time f is such that the above properties are satisfied one can prove Theorem 3.1. The assumptions we have made on f are an example of a sufficient condition.

Example 3.4. In some cases it is possible to write the solution to problem (3.5) and then to make explicit the lower bound in (3.6). For example, for $1 < p < n$, one can choose

$$f(s) = (p-1)^{p-1} e^{\frac{pp'}{n-p} s}, \tag{3.15}$$

where, as usual, $p' = \frac{p}{p-1}$. We then consider the problem (3.5) in the form

$$\begin{cases} \Delta_p v + |\nabla v|^p = (p-1)^{p-1} e^{\frac{pp'}{n-p} v} & \text{in } B_R \\ v(x) \to \infty & \text{as } x \to \partial B_R, \end{cases} \tag{3.16}$$

where B_R is the ball centered at the origin with radius R. Clearly $\beta(s) = (p-1)^{1-p} s^{p-1} f((p-1)\log s) = s^{\frac{np}{n-p}-1}$ and the assumptions of Theorem 3.1 are satisfied. The function

$$v(x) = (n-p)\log\left(\left[\left(\frac{n-p}{p-1}\right)^{p-1} n\right]^{\frac{1}{pp'}} R^{\frac{1}{p}} \left(\frac{1}{R^{p'} - |x|^{p'}}\right)^{\frac{1}{p'}}\right),$$

is the radial solution of problem (3.16). Hence, for a solution u to problem (3.1), with $f(u)$ as in (3.15), in any domain Ω with $|\Omega| = |B_R|$, the following inequality holds:

$$\operatorname*{ess\,inf}_{x\in\Omega} u \geq \frac{n-p}{p'} \log\left(\left[\left(\frac{n-p}{p-1}\right)^{p-1} n\right]^{\frac{1}{p}} \frac{1}{R}\right).$$

4. Second comparison result

We consider the problem

$$\begin{cases} \operatorname{div}(a(x,u,\nabla u)) = G(x,u,\nabla u) + f(u) & \text{in } \Omega \\ u(x) \to \infty & \text{as } x \to \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded subset of $\mathbb{R}^n$, $n \geq 2$, p is a real number with $p > 1$, and $a(x,s,\xi) : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function satisfying the ellipticity condition (3.2), $G(x,s,\xi) : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function satisfying the inequality

$$|G(x,s,\xi)| \leq |\xi|^p, \qquad \text{a.e. } x \in \Omega,\ \forall (s,\xi) \in \mathbb{R}\times\mathbb{R}^n,$$

and $f : \mathbb{R} \to \mathbb{R}_+$ is a continuous function.

Let us consider the problem

$$\begin{cases} \Delta_p v - |\nabla v|^p = f(v) & \text{in } \Omega^\# \\ v(x) \to \infty & \text{as } x \to \partial\Omega^\#. \end{cases} \tag{4.2}$$

Theorem 4.1. *Let $u \in W^{1,p}_{loc}(\Omega)$ be a weak solution to problem* (4.1) *and let $F(r) = (p-1)^{1-p}\, r^{p-1} f(\log r^{1-p})$, $r > 0$, be a decreasing function such that $\lim_{r\to 0^+} F(r) < +\infty$. If $v \in W^{1,p}_{loc}(\Omega^\#)$ is a solution to problem* (4.2)*, then*

$$\operatorname*{ess\,inf}_{x\in\Omega} u(x) \geq \operatorname*{ess\,inf}_{x\in\Omega^\#} v(x). \tag{4.3}$$

Moreover if $F(r) \in C^2(]0,+\infty[)$, then

$$\int_0^s f(u_*(r))\, e^{-u_*(r)}\, dr \geq \int_0^s f(v_*(r))\, e^{-v_*(r)}\, dr, \quad r \in [0,|\Omega|[. \tag{4.4}$$

Remark 4.2. We observe explicitly that problem (4.2) has a unique radial solution. Indeed, by a suitable change of unknown function (see also problems (4.10) and (4.11) in the proof below) it is possible to write problem (4.2) in the form (2.4). The hypotheses made in Theorem 4.1 are such that conditions j) and jj) in Section 2 are satisfied and, taking into account the results in [26], the uniqueness of the solution follows.

Proof of Theorem 4.1. We use, for $t > 0$, the following test function in the weak formulation of problem (4.1),

$$\varphi(x) = \begin{cases} e^{-u}h, & \tilde{u} > t + h \\ e^{-u}(\tilde{u} - t), & t < \tilde{u} \leq t + h \\ 0, & \tilde{u} \leq t, \end{cases}$$

where $h > 0$ and $\tilde{u} = e^{-\frac{u}{p-1}}$. Proceeding as in the proof of Theorem 3.1 we obtain, instead of (3.8),

$$(-U'(s))^{p-1}\left(n\omega_n^{\frac{1}{n}} s^{1-\frac{1}{n}}\right)^p \leq \int_0^s F(U(\sigma))\, d\sigma, \tag{4.5}$$

where we have used the definition of $F(r)$ and we have put $U(s) = e^{-\frac{u_*(s)}{p-1}}$. After an integration one immediately obtains

$$U(s) \geq M - c_{n,p} \int_0^s \tau^{\left(\frac{1}{n}-1\right)\left(\frac{p}{p-1}\right)} \left(\int_0^\tau F(U(\sigma))\, d\sigma\right)^{\frac{1}{p-1}} d\tau, \tag{4.6}$$

where $M = U(0) = e^{-\frac{u_*(0)}{p-1}}$ and $c_{n,p} = \left(n\omega_n^{\frac{1}{n}}\right)^{-\frac{p}{p-1}}$.

For $\varepsilon > 0$, because of Proposition 2.3, there exists $s_\varepsilon > 0$ and a function $w_\varepsilon(x)$ such that $\sup_{x \in D_\varepsilon} w_\varepsilon(x) = M - \varepsilon$ and $w_\varepsilon(x)$ is solution to problem

$$\begin{cases} -\Delta_p w_\varepsilon = F(w_\varepsilon) & \text{in } D_\varepsilon \\ w_\varepsilon = 0 & \text{on } \partial D_\varepsilon, \end{cases}$$

where D_ε is the ball centered at the origin such that $|D_\varepsilon| = s_\varepsilon$. The function $w_\varepsilon(x)$ is radial and $W_\varepsilon(\omega_n |x|^n) = w_\varepsilon(x)$ satisfies, for $s \in [0, s_\varepsilon]$,

$$W_\varepsilon(s) = M - \varepsilon - c_{n,p} \int_0^s \tau^{\left(\frac{1}{n}-1\right)\left(\frac{p}{p-1}\right)} \left(\int_0^\tau F(W_\varepsilon(\sigma))\, d\sigma\right)^{\frac{1}{p-1}} d\tau. \tag{4.7}$$

Being $U(0) = \sup_{s \in [0, |\Omega|[} U(s) > W_\varepsilon(0) = \sup_{s \in [0, s_\varepsilon[} W_\varepsilon(s)$, there exists $\delta > 0$ such that $U(s) > W_\varepsilon(s)$ in $[0, \delta[$. We claim that

$$U(s) > W_\varepsilon(s) \quad \forall s \in \left[0, \min\{s_\varepsilon, |\Omega|\}\right[. \tag{4.8}$$

Otherwise there exists $\bar{s} > 0$ such that $U(s) > W_\varepsilon(s)$ in $[0, \bar{s}[$ and $U(\bar{s}) = W_\varepsilon(\bar{s})$. Then by (4.6) and (4.7) we have

$$\begin{aligned} 0 &= U(\bar{s}) - W_\varepsilon(\bar{s}) \\ &= \varepsilon - c_{n,p} \int_0^{\bar{s}} \frac{1}{\tau^{p'\left(1-\frac{1}{n}\right)}} \left[\left(\int_0^\tau F(U(\sigma))\, d\sigma \right)^{\frac{1}{p-1}} - \left(\int_0^\tau F(W_\varepsilon(\sigma))\, d\sigma \right)^{\frac{1}{p-1}} \right] d\tau > 0, \end{aligned}$$

that is a contradiction.

Let us observe that (4.8) implies also $s_\varepsilon \le |\Omega|$; otherwise we would have $W_\varepsilon(|\Omega|) > \lim_{s \to |\Omega|} U(s) = 0$ against (4.8). Letting ε go to 0 in (4.7) we have

$$U(s) \ge W(s), \quad \forall s \in [0, s_0[, \tag{4.9}$$

where $s_0 = \lim_{\varepsilon \to 0} s_\varepsilon \le |\Omega|$ and $W(s)$ is solution of

$$W(s) = M - c_{n,p} \int_0^s \tau^{\left(\frac{1}{n}-1\right)\left(\frac{p}{p-1}\right)} \left(\int_0^\tau F(W(\sigma))\, d\sigma \right)^{\frac{1}{p-1}} d\tau.$$

Set $w(x) = W(\omega_n |x|^n)$, $w(x)$ is a solution of

$$\begin{cases} -\Delta_p w = F(w) & \text{in } B \\ \lim\limits_{|x| \to \partial B} w(x) = 0 & \text{on } \partial B, \end{cases} \tag{4.10}$$

where B is the ball centered at the origin such that $|B| = s_0$.

If we consider the function $\nu(x)$ such that $w(x) = e^{-\frac{\nu(x)}{p-1}}$, $x \in B$, $\nu(x)$ is the unique solution of

$$\begin{cases} \Delta_p \nu - |\nabla \nu|^p = f(\nu) & \text{in } B \\ \nu(x) \to \infty & \text{as } x \to \partial B. \end{cases} \tag{4.11}$$

Let us observe that $W(s) = e^{-\frac{\nu_*(s)}{p-1}}$. Then being $U(s) = e^{-\frac{u_*(s)}{p-1}}$ from (4.9) we have $u_*(s) \le \nu_*(s)$, for all $s \in [0, |B|[$, that implies $\operatorname{ess\,inf}_{x \in \Omega} u(x) = \operatorname{ess\,inf}_{x \in B} \nu(x)$ and

$$|B| \le |\Omega|. \tag{4.12}$$

To prove (4.3) we argue by contradiction. If $\operatorname{ess\,inf}_{x \in \Omega} u(x) = \operatorname{ess\,inf}_{x \in B} \nu(x) < \operatorname{ess\,inf}_{x \in B} v(x)$, then by Proposition 2.3 it would follow $|B| \ge |\Omega^\#|$. By uniqueness equality sign cannot occur and this contradicts (4.12). This means

$$\operatorname*{ess\,inf}_{x \in \Omega} u(x) \ge \operatorname*{ess\,inf}_{x \in B} \nu(x).$$

To prove (4.4) we consider (4.5) and we set $U_1(s) = \int_0^s F\big(e^{-\frac{u_*(\tau)}{p-1}}\big)\,d\tau$ obtaining

$$\begin{cases} U_1'' + \dfrac{F'\big(F^{-1}(U_1')\big)U_1^{\frac{1}{p-1}}}{\big(n\omega_n^{\frac{1}{n}} s^{1-\frac{1}{n}}\big)^{\frac{p}{p-1}}} \geq 0 \\ U_1(0) = 0, \quad U_1'(0) = F\left(e^{-\frac{u_*(0)}{p-1}}\right). \end{cases}$$

If $v(x)$ is a solution of (4.2), then setting $V_1(s) = \int_0^s F\big(e^{-\frac{v_*(\tau)}{p-1}}\big)\,d\tau$ we have

$$\begin{cases} V_1'' + \dfrac{F'\big(F^{-1}(V_1')\big)V_1^{\frac{1}{p-1}}}{\big(n\omega_n^{\frac{1}{n}} s^{1-\frac{1}{n}}\big)^{\frac{p}{p-1}}} = 0 \\ V_1(0) = 0, \quad V_1'(0) = F\left(e^{-\frac{v_*(0)}{p-1}}\right). \end{cases}$$

Since $F'(r) \leq 0$, a maximum principle gives $U_1(s) \geq V_1(s)$, that is (4.4). □

Remark 4.3. Let us observe that in the proof of Theorem 4.1 it is contained the comparison result

$$u_*(s) \leq \nu_*(s), \quad \forall s \in [0, |B|],$$

where ν is the solution to the problem (4.11) in B, such that

$$\operatorname*{ess\,inf}_{x\in\Omega} u = \operatorname*{ess\,inf}_{x\in B} \nu.$$

Example 4.4. As in the previous section, in some cases, it is possible to write the solution to problem (4.2) and then to make explicit the lower bound in (4.3). We will limit ourself to the simple case $n = p = 2$, and we choose

$$f(s) = e^{s-e^{-s}}. \tag{4.13}$$

We then consider the problem (4.2) in the form

$$\begin{cases} \Delta v - |\nabla v|^2 = e^{v-e^{-v}} & \text{in } B_R \\ v(x) \to \infty & \text{as } x \to \partial B_R, \end{cases} \tag{4.14}$$

where B_R is the ball centered at the origin with radius R. Clearly $F(r) = r\,f(-\log r) = e^{-r}$ and the assumptions of Theorem 4.1 are satisfied. If we put $L_R = 4 + 2\sqrt{4 + 2R^2}$, the function

$$v(x) = -\log\left(2\log\frac{L_R^2 - 8|x|^2}{8L_R}\right),$$

is the radial solution of problem (4.14). Hence, for a solution u to problem (4.1) with $f(u)$ as in (4.13), in any domain Ω with $|\Omega| = |B_R|$, the following inequality holds:

$$\operatorname*{ess\,inf}_{x\in\Omega} u \geq -\log\left(2\log\frac{L_R}{8}\right).$$

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