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Littlewood-Paley Theory for the Differential Operator $\frac{\partial^2}{\partial {x_1}^2}\frac{\partial^2}{\partial {x_2}^2} - \frac{\partial^2}{\partial {x_3}^2}$

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Abstract. Littlewood–Paley theory for the differential operator, $\triangle_{\mathbb{D}} = \partial^2_{x_1}\partial^2_{x_2} - \partial^2_{x_3}$, is developed. This study leads to the introduction of a new class of Triebel–Lizorkin spaces $\dot{F}^{\alpha,q}_p(\mathbb{D})$ associated with the dilation $(x_1, x_2, x_3) \to (2^{\nu_1}x_1, 2^{\nu_2}x_2, 2^{\nu_1+\nu_2}x_3)$, $(\nu_1, \nu_2) \in \mathbb{Z}^2$. The corresponding atomic and molecular decompositions are obtained. A frame generated by modulations, dilations and translations is also studied. Using this result, we show that $\triangle_{\mathbb{D}}$ is a linear isomorphism from $\dot{F}^{\alpha,q}_p(\mathbb{D})$ to $\dot{F}^{\alpha-2,q}_p(\mathbb{D})$.

Keywords. Littlewood–Paley theory, Triebel–Lizorkin Spaces

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1. Introduction and motivation

In this paper, we develop a Littlewood–Paley theory for the operator

$$\triangle_{\mathbb{D}} = \frac{\partial^2}{\partial {x_1}^2}\frac{\partial^2}{\partial {x_2}^2} - \frac{\partial^2}{\partial {x_3}^2}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

We see that this theory is related to the function spaces associated with the following dilation group on $\mathbb{R}^3$:

$$\mathbb{D} = \left\{(x_1, x_2, x_3) \to (2^{\nu_1}x_1, 2^{\nu_2}x_2, 2^{\nu_1+\nu_2}x_3) : (\nu_1, \nu_2) \in \mathbb{Z}^2\right\}. \tag{1}$$

Before introducing the Littlewood–Paley theory for $\triangle_{\mathbb{D}}$ and the function spaces for the above dilation group, let us recall some ideas and notions concerning the function spaces associated with the dilations

$$(x_1, x_2, \ldots, x_n) \to (2^j x_1, 2^j x_2, \ldots, 2^j x_n), \quad \text{where } j \in \mathbb{Z}, \tag{2}$$

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and its relation with the Laplacian $\triangle_n = \sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2}$. Frazier and Jawerth [3] established a framework for the study of the function spaces associated with (2). A fundamental tool for studying these function spaces is the Littlewood–Paley analysis: if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies

$$\sum_{j\in\mathbb{Z}} \hat{\varphi}(2^j\xi) = 1, \quad \xi \neq 0, \tag{3}$$

then, for any $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ ($\mathcal{P}(\mathbb{R}^n)$ denotes the set of polynomial),

$$f = \sum_{j\in\mathbb{Z}} \varphi_j * f \quad \text{in } \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n),$$

where $\hat{\varphi}(\xi) = \int e^{-ix\cdot\xi}\varphi(x)dx$ denotes the Fourier transform of φ and $\varphi_j(x) = 2^{jn}\varphi(2^jx)$ (see [8]). Using this identity, we can define the well known Triebel–Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$, $0 < p, q < \infty$. They are defined via the Littlewood–Paley function $g_\alpha(f)(\xi) = g_\alpha^\varphi(f)(\xi) = \big(\sum_{j\in\mathbb{Z}}(2^{j\alpha}|\varphi_j * f|)^q\big)^{\frac{1}{q}}$, where φ is a Schwartz function satisfying

$$\operatorname{supp}\hat{\varphi} \subset \big\{\xi \in \mathbb{R}^n : \tfrac{1}{2} \leq |\xi| \leq 2\big\} \tag{4}$$

and $|\hat{\varphi}(\xi)| \geq C > 0$ if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$. The Triebel–Lizorkin space consists of those $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that $g_\alpha(f) \in L_p(\mathbb{R}^n)$. (The definition of $\dot{F}_p^{\alpha,q}(\mathbb{R}^n)$ is independent of φ.) Moreover, celebrated atomic and molecular decompositions for these function spaces are known. One of the important properties between the Laplacian $\triangle_n$ and the Triebel–Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^n)$ is the following pair of inequalities: there exist $C_1 \geq C_2 > 0$ such that

$$C_2\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^n)} \leq \|\triangle_n f\|_{\dot{F}_p^{\alpha-2,q}} \leq C_1\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^n)}, \quad \forall f \in \dot{F}_p^{\alpha,q}(\mathbb{R}^n). \tag{5}$$

Recall that when $q = 2$, the Triebel–Lizorkin space $\dot{F}_p^{\alpha,2}$ is the well-known homogeneous Sobolev space and, the analogy of (5) on the inhomogeneous Triebel–Lizorkin spaces is one of the fundamental properties for the elliptic theory of differential equation (see [9]).

The proof of (5) is based on the homogeneity properties of the symbol of the Laplacian, $L(\xi) = \xi_1^2 + \cdots + \xi_n^2$, $\xi = (\xi_1, \ldots, \xi_n)$, under the dilation (2). That is, $L(2^j\xi) = 2^{2j}L(\xi)$. If φ satisfies (4), then on the support of $\hat{\varphi}(2^{-j}\xi)$, we have $L(\xi) \sim 2^{2j}$. Furthermore, the function ϕ defined by $\hat{\phi} = L\hat{\varphi}$ also satisfies (4). Thus, we have $g_{\alpha-2}^\varphi(\triangle_n f)(\xi) \sim g_\alpha^\phi(f)(\xi)$ and (5) follows from this estimation.

On the other hand, there exists a Littlewood–Paley theory for a non-hypoelliptic differential operator, namely, the bi-Laplacian on $\mathbb{R}^{n_1+n_2}$, $n_1, n_2 \in \mathbb{N}$, $\triangle_p = \big(\sum_{i=1}^{n_1} \frac{\partial^2}{\partial {x_i}^2}\big)\big(\sum_{i=n_1+1}^{n_2} \frac{\partial^2}{\partial {x_i}^2}\big)$. We see that the family of function spaces associated with it are generated by the following dilations on $\mathbb{R}^{n_1+n_2}$:

$$(x_1, x_2, \ldots, x_n) \to (2^{j_1}x_1, \ldots, 2^{j_1}x_{n_1}, 2^{j_2}x_{n_1+1}, \ldots, 2^{j_2}x_{n_2}),$$

where $(j_1, j_2) \in \mathbb{Z}^2$. In this case, (3) should be modified as follows:

$$\sum_{(j_1,j_2)\in\mathbb{Z}^2} \hat{\varphi}(2^{j_1}\xi_1, \ldots, 2^{j_1}\xi_{n_1}, 2^{j_2}\xi_{n_1+1}, \ldots 2^{j_2}\xi_{n_2}) = 1, \quad \Big(\sum_{i=1}^{n_1} \xi_i^2\Big)\Big(\sum_{i=n_1+1}^{n_2} \xi_i^2\Big) \neq 0.$$

The multi-parameter Littlewood–Paley analysis can be derived based on the above identity (see [8]). With the multi-parameter Littlewood-Paly analysis, the corresponding results for the Triebel–Lizorkin spaces on product domains can be obtained, see [11]. For a study of bi-Laplacian, the reader may consult [10].

We now consider our differential operator $\triangle_{\mathbb{D}}$. Even though it is not hypoelliptic, we can obtain a pair of inequalities similar to (5) for $\triangle_{\mathbb{D}}$. The symbol of $\triangle_{\mathbb{D}}$ is $D(\xi) = \xi_1^2\xi_2^2 + \xi_3^2$, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. We see that $\triangle_{\mathbb{D}}$ is, in some extent, a hybrid of the differential operators $\triangle_n$ and $\triangle_p$.

The action of the dilation group $\mathbb{D}$ on $\mathbb{R}^3$ induces an homogeneity property on $D(\xi)$. More precisely, we have

$$D(2^{\nu_1}\xi_1, 2^{\nu_2}\xi_2, 2^{\nu_1+\nu_2}\xi_3) = 2^{2(\nu_1+\nu_2)} D(\xi_1, \xi_2, \xi_3).$$

Thus, in order to obtain an analogue of (5) for $\triangle_{\mathbb{D}}$, we have to construct a family of function spaces associated with the dilation group $\mathbb{D}$.

There exists another dilation group that induces an homogeneity property on $D(\xi)$. For example, we may consider the family of dilations

$$(x_1, x_2, x_3) \to (2^j x_1, 2^j x_2, 4^j x_3), \quad j \in \mathbb{Z}. \tag{6}$$

We cannot develop a reasonable theory based on this dilation group since the function φ used to define the function space associated with (6) has to satisfy the condition

$$|\hat{\varphi}(\xi)| > C > 0 \quad \text{if } \tfrac{3}{5} \leq |\xi| \leq \tfrac{5}{3}, \tag{7}$$

and the function ϕ defined by $\hat{\phi} = D\hat{\varphi}$ does not satisfies (7). For instance, it is equal to zero at $\xi = (1, 0, 0)$. Indeed, this is the main reason why we introduce a two-parameter dilation group $\mathbb{D}$ because that type of technical difficulty can be avoided for the function spaces associated with $\mathbb{D}$.

In order to establish a framework for studying the function spaces associated with $\mathbb{D}$, we need the corresponding Littlewood–Paley analysis for the dilation group $\mathbb{D}$. Thus, the identities (3) must be modified in order to select an "analyzing" function φ to produce the Littlewood–Paley analysis for $\mathbb{D}$. The straightforward generalization

$$\sum_{(\nu_1,\nu_2)\in\mathbb{Z}^2} \hat{\varphi}(2^{\nu_1}\xi_1, 2^{\nu_2}\xi_2, 2^{\nu_1+\nu_2}\xi_3) = 1 \tag{8}$$

is not appropriate for $\mathbb{D}$. The easiest way to observe the drawbacks of the above identity is to take a function φ, where $\operatorname{supp}\hat{\varphi} = \{(x_1, x_2, x_3) : \frac{1}{2} \le |x_i| \le 2,\ i = 1, 2, 3\}$. We see that the union of the support of the families $\{\hat{\varphi}(2^{\nu_1}\xi_1, 2^{\nu_2}\xi_2, 2^{\nu_1+\nu_2}\xi_3)\}_{(\nu_1,\nu_2)\in\mathbb{Z}^2}$ does not cover $\mathbb{R}^3$, thus, the summation on the left hand side of (8) cannot be equal to one.

There are many ways to modify (3) to make it adaptable to this context. In this study, the following modification is used:

$$\sum_{(\nu_1,\nu_2)\in\mathbb{Z}^2} \sum_{l\in\mathbb{Z}} \hat{\varphi}(2^{\nu_1}\xi_1, 2^{\nu_2}\xi_2, 2^{\nu_1+\nu_2}\xi_3 - l) = 1, \quad \xi_1\xi_2 \neq 0. \tag{9}$$

There are some remarkable features of using (9). First, it is easy to construct a function satisfying (9) (see Section 3). Second, it provides a localization for ξ_3, and this extra localization is important in extending the Littlewood–Paley analysis to functions associated with $\mathbb{D}$. Finally, it is related to the following frame generated by modulations, translations and dilations:

$$2^{\nu_1+\nu_2} e^{i(2^{\nu_1+\nu_2}x_3 - k_3)l} \varphi(2^{\nu_1}x_1 - k_1, 2^{\nu_2}x_2 - k_2, 2^{\nu_1+\nu_2}x_3 - k_3). \tag{10}$$

For the first and second variables x_1 and x_2, this is a wavelet-type frame. For the third variable, it is a Gabor-type frame. Therefore, it is a hybrid wavelet-type and Gabor-type frame for $L^2(\mathbb{R}^3)$.

We declare some notations which will be used in this paper. For any $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, define $s(\nu) = \nu_1 + \nu_2$ and

$$2^\nu x = (2^{\nu_1}x_1, 2^{\nu_2}x_2, 2^{\nu_1+\nu_2}x_3) = (2^{\nu_1}x_1, 2^{\nu_2}x_2, 2^{s(\nu)}x_3).$$

For any $\varphi \in \mathcal{S}(\mathbb{R}^3)$ and $l \in \mathbb{Z}$, define $\varphi^l(x)$ by

$$\hat{\varphi}^l(\xi_1, \xi_2, \xi_3) = \hat{\varphi}(\xi_1, \xi_2, \xi_3 - l).$$

Furthermore, let $(M_l\varphi)(x_1, x_2, x_3) = e^{ix_3l}\varphi(x_1, x_2, x_3)$ be the modulation operator with respect to the third variable. Then, $\varphi^l(x) = (M_l\varphi)(x)$.

Let $\nu \in \mathbb{Z}^2$, $k \in \mathbb{Z}^3$. Define $\varphi_\nu(x) = 2^{2s(\nu)}\varphi(2^\nu x)$, $\varphi_{\nu,k}(x) = 2^{s(\nu)}\varphi(2^\nu x - k)$ and

$$\varphi^l_\nu(x) = (\varphi^l)_\nu(x) = 2^{2s(\nu)}\varphi^l(2^\nu x).$$

Based on (9), we will present and prove the results for the Littlewood–Paley analysis of the dilation group (1) in Section 2. The convergence of the Littlewood–Paley expansion for a class of tempered distributions, Theorem 2.2, will be proved based on a convergence result for a subspace of the rapidly decreasing functions, Theorem 2.1.

Once the Littlewood–Paley analysis for the dilation group $\mathbb{D}$ has been constructed, it can be used to define and study the function spaces associated with $\mathbb{D}$ using the ideas from [3]. We define a family of function spaces in Definition 3.1 using the following *Littlewood–Paley function for* $\triangle_{\mathbb{D}}$,

$$\left(\sum_{l\in\mathbb{Z}}\sum_{\nu\in\mathbb{Z}^2}\left(2^{(\nu_1+\nu_2)\alpha}(1+|l|)^{\alpha}|f*\varphi_\nu^l|\right)^q\right)^{\frac{1}{q}}, \quad \nu=(\nu_1,\nu_2)\in\mathbb{Z}^2,$$

and a family of sequence spaces in Definition 3.2. We have an extra term $(1+|l|)^\alpha$ on the above definition because in our case $\hat{\varphi}_\nu^l$ is "located" on the region

$$2^{\nu_1}\le|\xi_1|<2^{\nu_1+1},\;\; 2^{\nu_2}\le|\xi_2|<2^{\nu_2+1},\;\; 2^{\nu_1+\nu_2}|l|\le|\xi_3|<2^{\nu_1+\nu_2}(|l|+1).$$

There is a strong connection between these function spaces and sequence spaces, and this is justified by the boundedness of the corresponding φ-ψ transforms defined in Definition 3.3. Moreover, this boundedness result can be used to prove that our function spaces are well-defined (see Theorem 3.6).

Section 4 follows ideas from [3] and introduces a class of "almost diagonal" operators on the sequence spaces. It will be shown that they are bounded on the sequence spaces in Theorem 4.2, and this will be used to produce decompositions of the functions. Molecular and atomic decompositions will be performed in Section 5. Furthermore, Theorem 5.5 will show that if φ satisfies the inequality

$$0<B<\sum_{l\in\mathbb{Z}}\sum_{\nu\in\mathbb{Z}^2}|\hat{\varphi}(2^{\nu_1}\xi_1,2^{\nu_2}\xi_2,2^{s(\nu)}\xi_3-l)|^2<A \quad \text{if } \xi_1\xi_2\neq 0,$$

then, there exists a dilation of φ that generates a frame of the form given by (10). Thus, the family of function spaces $\dot{F}_p^{\alpha,q}(\mathbb{D})$ is not only constructed to produce an analogy of (5) for $\triangle_{\mathbb{D}}$, it also has its own independent interest.

In Section 6, the main result for the differential operator $\triangle_{\mathbb{D}}$ is obtained. We prove an analogy of (5) for the differential operator $\triangle_{\mathbb{D}}$ and the family of function spaces $\dot{F}_p^{\alpha,q}(\mathbb{D})$.

This paper thus, on the one hand, introduces a new class of function spaces associated with $\triangle_{\mathbb{D}}$ that, in some ways, resemble Triebel–Lizorkin spaces. These function spaces are induced by a family of dilations given by (1). On the other hand, the results also provide a framework for studying frames generated by modulations, dilations and translations (10).

Finally, note that, for the sake of simplicity, most results reported here are not the best results under the minimal assumptions. The main purpose of this paper is to present the framework for the study of function spaces associated with the dilation group $\mathbb{D}$ and the differential operator $\triangle_{\mathbb{D}}$. There is no attempt to search for the optimal conditions for which this framework holds.

2. A Littlewood–Paley type identity

This section will establish a Littlewood–Paley type identity for the dilation group, $\mathbb{D}$. It will first show that a Littlewood–Paley type expansion is valid for a class of smooth functions. This will then be extended to the distributions by duality.

For $\tau \geq 0$ and $N > 3$, define

$$\|\varphi\|_{\tau,N} = \sup_{x\in\mathbb{R}^3,\, |\gamma|\leq\tau} (1+|x|)^N |(\partial^\gamma \varphi)(x)|. \tag{11}$$

Let $\mathcal{S}_v(\mathbb{R}^3) = \left\{\varphi \in \mathcal{S}(\mathbb{R}^3) : \int_{\mathbb{R}} x_j^\lambda \varphi(x_1,x_2,x_3)dx_j = 0,\ j=1,2,\ \forall\lambda\in\mathbb{N}\right\}$.

Theorem 2.1. *Let $\varphi \in \mathcal{S}_v(\mathbb{R}^3)$ satisfy*

$$\sum_{l\in\mathbb{Z}}\sum_{\nu\in\mathbb{Z}^2} \hat{\varphi}^l(2^\nu \xi) = 1, \quad \xi = (\xi_1,\xi_2,\xi_3),\ \xi_1\xi_2 \neq 0. \tag{12}$$

Then, for any $\psi \in \mathcal{S}_v(\mathbb{R}^3)$,

$$\psi = \sum_{l\in\mathbb{Z}}\sum_{\nu\in\mathbb{Z}^2} \psi * \varphi_\nu^l \quad \textit{in } \mathcal{S}(\mathbb{R}^3).$$

Proof. By Lemma 7.1 with $m=0$, $\mu=(0,0)$ and $k=h=(0,0,0)$, we find that for any $L>1$ there exists a constant $C(L)>0$ depending only on L such that

$$|\psi * \varphi_\nu^l(x_1,x_2,x_3)| \leq C(L)\Omega_L^{\nu,l},$$

where

$$\begin{aligned}\Omega_L^{\nu,l} = &\left(\prod_{i=1,2}\min\left(2^{\nu_i(4L+\frac12)}, 2^{-\nu_i(4L-\frac12)}\right)\right)\left(1+\frac{|2^{s(\nu)}l|}{\max(2^{s(\nu)},1)}\right)^{-L}\\ &\times\left[\left(1+\frac{|x_1|}{\max(2^{-\nu_1},1)}\right)\left(1+\frac{|x_2|}{\max(2^{-\nu_2},1)}\right)\left(1+\frac{|x_3|}{\max(2^{-s(\nu)},1)}\right)\right]^{-L}.\end{aligned}$$

We assert that

$$\begin{aligned}&\sum_{\nu\in\mathbb{Z}^2,l\in\mathbb{Z}} |\psi * \varphi_\nu^l(x_1,x_2,x_3)|\\ &\leq C(L)\sum_{l\in\mathbb{Z}}\left[\left(\sum_{\nu_1,\nu_2\geq 0} + \sum_{\substack{\nu_1\geq0,\nu_2<0\\ s(\nu)\geq0}} + \sum_{\substack{\nu_1\geq0,\nu_2<0\\ s(\nu)<0}} + \sum_{\substack{\nu_1<0,\nu_2\geq0\\ s(\nu)\geq0}} + \sum_{\substack{\nu_1<0,\nu_2\geq0\\ s(\nu)<0}} + \sum_{\nu_1,\nu_2<0}\right)\Omega_L^{\nu,l}\right]\\ &= C(L)(I+II+III+IV+V+VI).\end{aligned}$$

For VI,

$$VI \leq \sum_{l\in\mathbb{Z}} \sum_{\nu_2=-\infty}^{-1} \sum_{\nu_1=-\infty}^{-1} \frac{2^{(\nu_1+\nu_2)(4L+\frac{1}{2})}}{(1+2^{\nu_1}|x_1|)^L(1+2^{\nu_2}|x_2|)^L(1+2^{s(\nu)}|x_3|)^L(1+2^{s(\nu)}|l|)^L}.$$

Notice that $(1+2^j|z|)^{-L} \leq 2^{-jL}(2^{-j}+|z|)^{-L} \leq 2^{-jL}(1+|z|)^{-L}$ if $j \leq 0$. Thus,

$$VI \leq \sum_{l\in\mathbb{Z}} \sum_{\nu_2=-\infty}^{-1} \sum_{\nu_1=-\infty}^{-1} \frac{2^{(\nu_1+\nu_2)(4L-3L+\frac{1}{2})}}{[(1+|l|)\prod_{j=1}^{3}(1+|x_j|)]^L} \leq C(L)\frac{1}{[\prod_{j=1}^{3}(1+|x_j|)]^L}.$$

Similarly, for V,

$$\begin{aligned} V &\leq \sum_{l\in\mathbb{Z}} \sum_{\nu_2=0}^{\infty} \sum_{\nu_1=-\infty}^{-\nu_2} \frac{2^{(\nu_1-\nu_2)(4L-\frac{1}{2})}}{[(1+2^{\nu_1}|x_1|)(1+|x_2|)(1+2^{s(\nu)}|x_3|)(1+2^{s(\nu)}|l|)]^L} \\ &\leq \sum_{l\in\mathbb{Z}} \sum_{\nu_2=0}^{\infty} \sum_{\nu_1=-\infty}^{-\nu_2} \frac{2^{\nu_1(L-\frac{1}{2})}2^{-\nu_2(6L-\frac{1}{2})}}{[(1+|l|)\prod_{j=1}^{3}(1+|x_j|)]^L}. \end{aligned}$$

For IV,

$$\begin{aligned} IV &\leq \sum_{l\in\mathbb{Z}} \sum_{\nu_2=0}^{\infty} \sum_{\nu_1=-\nu_2}^{-1} \frac{2^{(\nu_1-\nu_2)(4L-\frac{1}{2})}}{[(1+2^{\nu_1}|x_1|)(1+|x_2|)(1+|x_3|)(1+|l|)]^L} \\ &\leq \sum_{l\in\mathbb{Z}} \sum_{\nu_2=0}^{\infty} \sum_{\nu_1=-\nu_2}^{-1} \frac{2^{\nu_1(3L-\frac{1}{2})}2^{-\nu_2(4L-\frac{1}{2})}}{[(1+|l|)\prod_{j=1}^{3}(1+|x_j|)]^L}. \end{aligned}$$

The estimates for III and II are similar to the estimates of V and IV, respectively. The estimate for I is easier. In conclusion, we have

$$I+II+III+IV+V+VI \leq C(L)\frac{1}{\prod_{j=1}^{3}(1+|x_j|)^L}.$$

Since for any $\psi \in \mathcal{S}(\mathbb{R}^3)$, $\partial^\gamma(\sum_{l\in\mathbb{Z}}\sum_{\nu\in\mathbb{Z}^2}\psi * \varphi_\nu^l) = \sum_{l\in\mathbb{Z}}\sum_{\nu\in\mathbb{Z}^2}(\partial^\gamma\psi) * \varphi_\nu^l$, for all $\gamma \in \mathbb{N}^3$, by the above results, $\big\|\sum_{l\in\mathbb{Z}}\sum_{\nu\in\mathbb{Z}^2}\psi * \varphi_\nu^l\big\|_{\tau,L} < \infty$, for all $\tau > 0$, and $L > 2$. Hence, $\sum_{l\in\mathbb{Z}}\sum_{\nu\in\mathbb{Z}^2}\psi * \varphi_\nu^l$ converges in $\mathcal{S}(\mathbb{R}^3)$. By (12), it is trivial that for any $\psi \in L^2(\mathbb{R}^3)$, $\sum_{l\in\mathbb{Z}}\sum_{\nu\in\mathbb{Z}^2}\psi * \varphi_\nu^l = \psi$ in $L^2(\mathbb{R}^3)$. Therefore, for any $\psi \in \mathcal{S}(\mathbb{R}^3)$, the series $\sum_{l\in\mathbb{Z}}\sum_{\nu\in\mathbb{Z}^2}\psi * \varphi_\nu^l$ converges to ψ in $\mathcal{S}(\mathbb{R}^3)$. □

Denote the set $\{f \in \mathcal{S}'(\mathbb{R}^3) : \text{supp}\hat{f} \subseteq \{\xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3 : \xi_1\xi_2 = 0\}\}$ by $\mathcal{G}(\mathbb{R}^3)$. By duality and the fact that the dual space of the function space $\mathcal{S}_v(\mathbb{R}^3)$ is $\mathcal{S}'(\mathbb{R}^3)/\mathcal{G}(\mathbb{R}^3)$, we establish the following theorem, which is a Littlewood–Paley type identity associated with the group $\mathbb{D}$.

Theorem 2.2. *Let $\varphi \in \mathcal{S}_v(\mathbb{R}^3)$ satisfy* (12). *Then, for any $f \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{G}(\mathbb{R}^3)$,*

$$f = \sum_{l\in\mathbb{Z}} \sum_{\nu\in\mathbb{Z}^2} f * \varphi_\nu^l \quad \textit{in } \mathcal{S}'(\mathbb{R}^3)/\mathcal{G}(\mathbb{R}^3).$$

Note that it has not be assumed that the analyzing function φ in Theorem 2.2 is a band-limited function. Therefore, Bernstein's inequality cannot be used to conclude our result. Furthermore, our method provides a new approach to the original Littlewood–Paley analysis without the compact support assumption for the analyzing function. The reader is referred to [8] for detail.

3. Function spaces and sequence spaces

In this section, we will define and study some basic properties of the function spaces and sequence spaces associated with $\mathbb{D}$. These function and sequence spaces will be defined in Section 3.1. The φ-ψ transforms for $\mathbb{D}$ will be introduced along with a proof that they are bounded in Section 3.2.

3.1. Some definitions. Let $\varphi_d, \psi_d, \varphi_t, \psi_t \in \mathcal{S}(\mathbb{R})$ satisfy

$$\operatorname{supp}\hat{\varphi}_d,\ \operatorname{supp}\hat{\psi}_d,\ \operatorname{supp}\hat{\varphi}_t,\ \operatorname{supp}\hat{\psi}_t \subseteq \left\{\eta \in \mathbb{R} : \tfrac{1}{2} \le |\eta| \le 2\right\}; \tag{13}$$

$$\sum_{j\in\mathbb{Z}} \overline{\hat{\varphi}_d(2^j\eta)}\hat{\psi}_d(2^j\eta) = 1, \quad \eta \neq 0; \tag{14}$$

$$\sum_{l\in\mathbb{Z}} \overline{\hat{\varphi}_t(\eta - l)}\hat{\psi}_t(\eta - l) = 1, \quad \forall \eta \in \mathbb{R}. \tag{15}$$

The functions

$$\varphi(x_1, x_2, x_3) = \varphi_d(x_1)\varphi_d(x_2)\varphi_t(x_3), \quad \psi(x_1, x_2, x_3) = \psi_d(x_1)\psi_d(x_2)\psi_t(x_3) \tag{16}$$

then satisfy

$$\sum_{\nu_1,\nu_2,l\in\mathbb{Z}} \overline{\hat{\varphi}(2^{\nu_1}\xi_1, 2^{\nu_2}\xi_2, 2^{\nu_1+\nu_2}\xi_3 - l)}\hat{\psi}(2^{\nu_1}\xi_1, 2^{\nu_2}\xi_2, 2^{\nu_1+\nu_2}\xi_3 - l) = 1, \ \text{ if } \xi_1\xi_2 \neq 0. \tag{17}$$

Identity (17) can be rewritten as

$$\sum_{l\in\mathbb{Z}} \sum_{\nu\in\mathbb{Z}^2} \overline{\hat{\varphi}^l(2^\nu\xi)}\hat{\psi}^l(2^\nu\xi) = 1, \quad \xi = (\xi_1, \xi_2, \xi_3),\ \xi_1\xi_2 \neq 0.$$

Let $\tilde{\varphi}(x) = \overline{\varphi(-x)}$. Theorem 2.2 ensures that if $f \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{G}(\mathbb{R}^3)$, then

$$f = \sum_{l\in\mathbb{Z}} \sum_{\nu\in\mathbb{Z}^2} f * \tilde{\varphi}_\nu^l * \psi_\nu^l \quad \text{in } \mathcal{S}'(\mathbb{R}^3)/\mathcal{G}(\mathbb{R}^3). \tag{18}$$

Let $\mathcal{I} = \mathcal{Q} \times \mathbb{Z}$, where $\mathcal{Q} = \{Q_{\nu,k}\}_{\nu\in\mathbb{Z}^2,k\in\mathbb{Z}^3}$, and the *dyadic box* $Q_{\nu,k}$, is defined by

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : k_j \leq 2^{\nu_j} x_j \leq k_j + 1, j = 1, 2,\ k_3 \leq 2^{\nu_1+\nu_2} x_3 \leq k_3 + 1\}.$$

Given $I = (Q_{\nu,k}, l) \in \mathcal{I}$, where $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$ and $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$, define the *measures* of I by $|I| = |Q_{\nu,k}| = 2^{-2s(\nu)}$, $|I|_1 = 2^{-\nu_1}$, $|I|_2 = 2^{-\nu_2}$ and $|I|_3 = 2^{-s(\nu)}$. The *Fourier translation* of I is defined by $t(I) = l$ and

$$\varphi_I(x) = \varphi^l_{\nu,k}(x) = 2^{s(\nu)} \varphi^l(2^\nu x - k), \quad \psi_I(x) = \psi^l_{\nu,k}(x) = 2^{s(\nu)} \psi^l(2^\nu x - k).$$

Moreover, for any $I = (Q_{\nu,k}, l) \in \mathcal{I}$, let $c_{Q_{\nu,k}} = c_I = (c_{I,1}, c_{I,2}, c_{I,3})$, where $c_{I,1} = 2^{-\nu_1} k_1$, $c_{I,2} = 2^{-\nu_2} k_2$, $c_{I,3} = 2^{-s(\nu)} k_3$, $f_I = 2^{s(\nu)} l$, and $\chi_I(x)$ is the characteristic function of $Q_{\nu,k}$.

Using Shannon's formula on each term on the right hand side of (18) yields

$$f = \sum_{l\in\mathbb{Z}} \sum_{\nu\in\mathbb{Z}^2,k\in\mathbb{Z}^3} \langle f, \varphi^l_{\nu,k}\rangle \psi^l_{\nu,k} = \sum_{I\in\mathcal{I}} \langle f, \varphi_I\rangle \psi_I, \tag{19}$$

for $f \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{G}(\mathbb{R}^3)$.

The function spaces and sequences spaces associated with the dilation group $\mathbb{D}$ can now be defined. Let $\alpha \in \mathbb{R}$, $0 < q < \infty$ and φ satisfy (13)-(17). For any $f \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{G}(\mathbb{R}^3)$, define the *Littlewood–Paley function associated with* $\triangle_{\mathbb{D}}$ by

$$d^q_\alpha(f) = \left(\sum_{l\in\mathbb{Z}} \sum_{\nu\in\mathbb{Z}^2} \left(2^{s(\nu)\alpha}(1 + |l|)^\alpha |f * \varphi^l_\nu|\right)^q \right)^{\frac{1}{q}}.$$

Definition 3.1. Let $\alpha \in \mathbb{R}$ and $0 < p, q < \infty$. The function space $\dot{F}^{\alpha,q}_p(\mathbb{D})$ consists of those distributions $f \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{G}(\mathbb{R}^3)$ which satisfy

$$\|f\|_{\dot{F}^{\alpha,q}_p(\mathbb{D})} = \|d^q_\alpha(f)\|_{L^p(\mathbb{R}^3)} < \infty, \tag{20}$$

where φ satisfies (17) for some $\psi \in \mathcal{S}(\mathbb{R}^3)$ and

$$\operatorname{supp} \hat{\varphi},\ \operatorname{supp} \hat{\psi} \subseteq \left\{\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \tfrac{3}{8} \leq |\xi_j| \leq \tfrac{8}{3},\ j = 1, 2, 3\right\}.$$

Definition 3.2. Let $\alpha \in \mathbb{R}$ and $0 < p, q < \infty$. The sequence space $\dot{f}^{\alpha,q}_p(\mathbb{D})$ consists of those sequences $s = \{s_I\}_{I\in\mathcal{I}}$ which satisfy

$$\|s\|_{\dot{f}^{\alpha,q}_p(\mathbb{D})} = \left\| \left(\sum_{I\in\mathcal{I}} \left(|I|^{-\frac{\alpha}{2}}(1 + |t(I)|)^\alpha |s_I \tilde{\chi}_I|\right)^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^3)} < \infty,$$

where $\tilde{\chi}_I(x) = |I|^{-\frac{1}{2}} \chi_I(x)$.

Comparing these formulations with the original Triebel–Lizorkin spaces (for example, the one in [3]), note the extra term $2^{(\nu_1+\nu_2)\alpha}(1 + |l|)^\alpha$ in the definition of $\dot{F}^{\alpha,q}_p(\mathbb{D})$. It comes from the derivatives of the function φ^l_ν. This can be better illustrated by using the expression

$$\varphi^l_\nu(x_x, x_2, x_3) = 2^{2\nu_1+2\nu_2} e^{i2^{(\nu_1+\nu_2)} l x_3} \varphi\left(2^{\nu_1}x_1, 2^{\nu_2}x_2, 2^{\nu_1+\nu_2}x_3\right).$$

3.2. The φ-ψ transforms. Based on the identity (19), we can introduce the φ-ψ transforms for $\dot{F}_p^{\alpha,q}(\mathbb{D})$ and $\dot{f}_p^{\alpha,q}(\mathbb{D})$. The "original" φ-ψ transforms for the Triebel–Lizorkin spaces were first introduced by Frazier and Jawerth in [3]. Using their ideas, the corresponding φ-ψ transforms for $\dot{F}_p^{\alpha,q}(\mathbb{D})$ and $\dot{f}_p^{\alpha,q}(\mathbb{D})$ can be defined as follows.

Definition 3.3. Let φ, ψ be the Schwartz functions given by (16). Define the operators S_φ and T_ψ by

$$S_\varphi(f) = \{\langle f, \varphi_I \rangle\}_{I\in\mathcal{I}} \quad \text{and} \quad T_\psi(s) = \sum_{I\in\mathcal{I}} s_I \psi_I,$$

where $f \in \mathcal{S}'(\mathbb{R}^3)$ and $s = \{s_I\}$ is a sequence indexed by $\mathcal{I}$.

We will show that S_φ is a bounded linear operator from $\dot{F}_p^{\alpha,q}(\mathbb{D})$ to $\dot{f}_p^{\alpha,q}(\mathbb{D})$, and that T_ψ is a bounded linear operator from $\dot{f}_p^{\alpha,q}(\mathbb{D})$ to $\dot{F}_p^{\alpha,q}(\mathbb{D})$. From (19), the composition $T_\psi \circ S_\varphi$ is the identity operator in $\dot{F}_p^{\alpha,q}(\mathbb{D})$. Call S_φ and T_ψ the φ-ψ transforms for $\mathbb{D}$. Some notation and theorems will now presented for proving of the boundedness of S_φ and T_ψ.

Let M_S be the strong maximal operator on $\mathbb{R}^3$. It is obvious that $\mathrm{M}_S(f) \le (\mathrm{M}_{x_1} \circ \mathrm{M}_{x_2} \circ \mathrm{M}_{x_3})(f)$ for any locally integrable function f, where M_{x_i} is the (ordinary) maximal operator corresponding to the variable x_i, $i = 1, 2, 3$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. So, an iteration of the Fefferman–Stein vector-valued maximal inequalities produces the following theorem for the strong maximal operator.

Theorem 3.4. *Suppose* $1 < p, q < \infty$. *Then,*

$$\left\| \left(\sum_{i\in\mathbb{Z}} |\mathrm{M}_S f_i|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^3)} \le C_{p,q} \left\| \left(\sum_{i\in\mathbb{Z}} |f_i|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^3)}.$$

A simple modification of Lemma A.2 and Remark A.3 of [3] establishes the following lemma.

Lemma 3.5. *Suppose* $l \in \mathbb{Z}$, $0 < a \le r < \infty$ *and* $\lambda > \frac{r}{a}$. *There exists a constant* $C > 0$ *which depends on* $\lambda - \frac{r}{a}$ *only, such that for each* $I = (Q_{\nu,k}, h) \in \mathcal{I}$ *and each* $x \in Q_{\nu,k}$,

$$\left(\sum_{\substack{|J|_1 = 2^{-\mu_1}, |J|_2 = 2^{-\mu_2} \\ t(J) = l}} |\beta_J|^r \prod_{\sigma=1}^{3} \left(1 + \frac{|c_{I,\sigma} - c_{J,\sigma}|}{\max(|I|_\sigma, |J|_\sigma)} \right)^{-\lambda} \right)^{\frac{1}{r}}$$

$$\le C 2^{(\mu_1-\nu_1)_+ \frac{2}{a}} 2^{(\mu_2-\nu_2)_+ \frac{2}{a}} \left(\mathrm{M}_S \left(\sum_{\substack{|J|_1 = 2^{-\mu_1}, |J|_2 = 2^{-\mu_2} \\ t(J) = l}} |\beta_J|^a \chi_J \right)(x) \right)^{\frac{1}{a}},$$

where $\mu_+ = \max(\mu, 0)$.

This leads to an analogue of [3, Lemma 2.3] for our setting. Let $r > 0$ and $\lambda > 1$. For any $s = \{s_I\}_{I\in\mathcal{I}}$, define $s^*_{r,\lambda}$ by

$$(s^*_{r,\lambda})_I = \sum_{\substack{|J|_1=|I|_1,|J|_2=|I|_2\\ t(J)=t(I)}} |s_J|^r \prod_{\sigma=1}^{3} \big(1+|I|_\sigma^{-1}|c_{I,\sigma}-c_{J,\sigma}|\big)^{-\lambda}.$$

Using Lemma 3.5 with $r = \min(p,q)$, $a = \frac{2r}{1+\lambda}$ and $\mu = \nu$, Theorem 3.4 with indices $\frac{p}{a}$ and $\frac{q}{a}$ yields $\|s^*_{r,\lambda}\|_{\dot{f}^{\alpha,q}_p(\mathbb{D})} \le C\|s\|_{\dot{f}^{\alpha,q}_p(\mathbb{D})}$ for some constant $C > 0$ independent of $s \in \dot{f}^{\alpha,q}_p(\mathbb{D})$. Notice that the above inequality still holds if $(c_{I,1}, c_{I,2}, c_{I,3})$ in the definition of $(s^*_{r,\lambda})_I$ is replaced by any $(x_1,x_2,x_3) \in Q$ where $I = (Q,l)$.

The proof of the following theorem is based on ideas from [3]. Thus, for simplicity, only an outline of the proof is provided.

Theorem 3.6. *Let $\alpha \in \mathbb{R}$ and $0 < p,q < \infty$. The definition for the function space $\dot{F}^{\alpha,q}_p(\mathbb{D})$ is independent of the function φ in (20). The operator S_φ is a bounded linear operator from $\dot{F}^{\alpha,q}_p(\mathbb{D})$ to $\dot{f}^{\alpha,q}_p(\mathbb{D})$. The operator T_ψ is bounded from $\dot{f}^{\alpha,q}_p(\mathbb{D})$ to $\dot{F}^{\alpha,q}_p(\mathbb{D})$.*

Proof. We use $\|f\|_{\dot{F}^{\alpha,q}_p(\mathbb{D},\varphi)}$ to denote the norm for $\dot{F}^{\alpha,q}_p(\mathbb{D})$ by using the function φ. It will now be shown that if φ and θ both satisfy the conditions in Theorem 3.1, then $\|\cdot\|_{\dot{F}^{\alpha,q}_p(\mathbb{D},\varphi)}$ and $\|\cdot\|_{\dot{F}^{\alpha,q}_p(\mathbb{D},\theta)}$ are equivalent quasi-norms.

Prove first the boundedness of T_ψ. It will be shown that for any φ satisfying the conditions in Definition 3.1, there is a constant $C > 0$ such that $\|T_\psi(s)\|_{\dot{F}^{\alpha,q}_p(\mathbb{D},\varphi)} \le C\|s\|_{\dot{f}^{\alpha,q}_p(\mathbb{D})}$.

Let $f = T_\psi(s) = \sum_{I\in\mathcal{I}} s_I\psi_I$. Taking the convolution on both sides with φ^l_ν, by the compactness of the support of $\hat{\varphi}$, we assert that

$$\varphi^l_\nu * f = \sum_{(\mu,l)\in P} \sum_{|J|_1=2^{-\mu_1},|J|_2=2^{-\mu_2}} s_J(\psi_J * \varphi^l_\nu),$$

where $P = \{(\mu,l) \in \mathbb{Z}^3 : \nu_j - 2 \le \mu_j \le \nu_j + 2, j = 1,2,\ l-4 \le t(J) \le l+4\}$. Similar to [3, Theorem 2.2], for any $I \in \mathcal{I}$ with $t(I) = l$, we find that there exists a collection $\{I_\gamma\}_{\gamma\in P} \subset \mathcal{I}$ such that $I \subseteq I_\gamma$, $|I|_1 \le |I_\gamma|_1 \le 4|I|_1$, $|I|_2 \le |I_\gamma|_2 \le 4|I|_2$ and $l-4 \le t(I_\gamma) \le l+4$. Moreover,

$$|(\varphi^l_\nu * f)(x)| \le C \sum_{\gamma\in P} \sum_{|I|_1=2^{-\nu_1},|I|_2=2^{-\nu_2}} (s^*_{r,\lambda})_{I_\gamma}\tilde{\chi}_I(x), \tag{21}$$

where $r = \min(p,q)$ and $\lambda > 1$. Multiply $|I|^{-\frac{\alpha}{2}}(1+|t(I)|)^\alpha = 2^{s(\nu)\alpha}(1+|l|)^\alpha$ on both sides of (21); then, take the l^q norm and apply the $L^p(\mathbb{R}^3)$ norm. Taking the summation for ν and l, by Minkowski's inequality,

$$\begin{aligned}\|T_\psi(s)\|_{\dot{F}_p^{\alpha,q}(\mathbb{D},\varphi)} &\le C\sum_{\gamma\in P}\left\|\left(\sum_{I\in\mathcal{I}}\left(|I_\gamma|^{-\frac{\alpha}{2}}(1+|t(I_\gamma)|)^\alpha|(s^*_{r,\lambda})_{I_\gamma}\tilde{\chi}_{I_\gamma}|\right)^q\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^3)}\\ &\le 225C\|s\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})},\end{aligned}$$

because $\chi_I \le \chi_{I_\gamma}$ and $\mathrm{card}(P) \le 225$.

The boundedness of S_φ will now be established. Let $I=(Q,l)\in\mathcal{I}$. Recall that $r=\min(p,q)$. Using Peetre's inequality,

$$\sum_{|I|_1=2^{-\nu_1},|I|_2=2^{-\nu_2}}[|(S_\varphi f)_I|\tilde{\chi}_I(x)]^q \le \left[2^{s(\nu)\alpha}[\mathrm{M}_S(\tilde{\varphi}^l_\nu * f)^r]^{\frac{1}{r}}\right]^q.$$

Multiplying $|I|^{-\frac{\alpha}{2}}(1+t(I))^\alpha = 2^{s(\nu)\alpha}(1+|l|)^\alpha$, summing over ν and l, taking the $\frac{1}{q}$ power and applying the $L^{p/r}$ norm on both sides; then using Theorem 3.4 with indices $\frac{p}{r}$ and $\frac{q}{r}$ yields $\|S_\varphi(f)\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})} \le \|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{D},\tilde{\varphi})}$.

The independence of φ of the definition of $\dot{F}_p^{\alpha,q}(\mathbb{D})$ can now be demonstrated. Once this is proved, the boundedness of S_φ follows. Suppose φ,ψ and θ,ζ are two pairs of functions satisfying the conditions in Theorem 3.1. Then,

$$\begin{aligned}\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{D},\varphi)} &= \left\|\sum_{I\in\mathcal{I}}(S_\theta(f))_I\zeta_I\right\|_{\dot{F}_p^{\alpha,q}(\mathbb{D},\varphi)}\\ &= \|T_\zeta(S_\theta(f))\|_{\dot{F}_p^{\alpha,q}(\mathbb{D},\varphi)}\\ &\le C\|(S_\theta(f))\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})}\\ &\le C\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{D},\tilde{\theta})}.\end{aligned}$$

The results then follow obviously from the above inequalities. □

Let φ satisfy the conditions of Definition 3.1 with

$$B \le \sum_{l\in\mathbb{Z},\nu\in\mathbb{Z}^2}|\hat{\varphi}^l(2^\nu\xi)|^2 \le A, \quad \text{if } \xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3,\ \xi_1\xi_2\ne 0,$$

for some constants $A \ge B > 0$. Theorem 3.6 then allows using φ to define $\dot{F}_p^{\alpha,q}(\mathbb{D})$. By using this function to define $\dot{F}_p^{\alpha,q}(\mathbb{D})$, it is easy to see that $\dot{F}_2^{0,2}(\mathbb{D}) = L^2(\mathbb{R}^3)$.

4. Almost diagonal matrices

In [3], there is an important class of operators acting on the little Triebel–Lizorkin spaces. This section presents the corresponding class of operators on $\dot{f}_p^{\alpha,q}(\mathbb{D})$ and proves that they are bounded operators. Begin with the definition of this class of operators.

Definition 4.1. Let $\beta > 0$ and $L > 1$. The class of almost diagonal matrices $\omega(\beta, L)$ associated with $\mathbb{D}$ consists of those sequence $A = \{a_{IJ}\}_{I,J\in\mathcal{I}}$ which satisfies

$$|a_{IJ}| \le C\omega_{IJ}(\beta, L), \tag{22}$$

where

$$\begin{aligned}\omega_{IJ}(\beta, L) = &\left[\prod_{\sigma=1}^{3} \min\left(\frac{|I|_\sigma}{|J|_\sigma}, \frac{|J|_\sigma}{|I|_\sigma}\right)\right]^{\beta} \left(1 + \frac{|f_I - f_J|}{\max(|I|_3^{-1}, |J|_3^{-1})}\right)^{-L} \\ &\times \left[\prod_{\sigma=1}^{3}\left(1 + \frac{|c_{I,\sigma} - c_{J,\sigma}|}{\max(|I|_\sigma, |J|_\sigma)}\right)\right]^{-L},\end{aligned}$$

and $C > 0$ is a constant independent of $I, J \in \mathcal{I}$. The norm $\|\{a_{IJ}\}\|_{\omega(\beta,L)}$ is the infimum of the constant $C > 0$ for which (22) holds.

Compared to the almost diagonal matrices in [3], the one used in this paper has an extra decay for the Fourier translation. This extra decay is used to assert the boundedness of these operators on the sequence spaces $\dot{f}_p^{\alpha,q}(\mathbb{D})$.

The main purpose of this section is to show the boundedness of the almost diagonal matrix in the sequence spaces $\dot{f}_p^{\alpha,q}(\mathbb{D})$. The precise statement of this result is given in Theorem 4.2.

Again note, however, that for the sake of brevity, no attempt has been made to present the best result under the minimal assumptions for the following theorem.

Theorem 4.2. *Suppose $\alpha \in \mathbb{R}$ and $0 < p, q < \infty$. Let $\mathcal{J} = \frac{1}{\min(1,p,q)}$. If β, L satisfy $\beta > 5\mathcal{J} + 4|\alpha|$ and $L > 2|\alpha| + 2\mathcal{J}$, then an almost diagonal matrix, $A = \{a_{IJ}\} \in \omega(\beta, L)$ is a bounded operator on $\dot{f}_p^{\alpha,q}(\mathbb{D})$ and $\|A(s)\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})} \le C\|\{a_{IJ}\}\|_{\omega(\beta,L)}\|s\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})}$ for some $C > 0$.*

Proof. Without loss of generality, assume that $\|\{a_{IJ}\}\|_{\omega(\beta,L)} = 1$. We deal first with the case where $\alpha \ge 0$, $r = \min(p, q) > 1$. The case $\alpha < 0$ will follow similarly. It will be shown that, in this case, it is sufficient to assume that $\beta > 4 + 4\alpha$.

Take $A = \sum_{j=1}^{6} A_j$ where

$$(A_1 s)_I = \sum_{|I|_1\ge|J|_1, |I|_2\ge|J|_2} a_{IJ}s_J, \quad (A_2 s)_I = \sum_{\substack{|I|_1\ge|J|_1, |I|_2<|J|_2 \\ |I|_3\ge|J|_3}} a_{IJ}s_J$$

$$(A_3 s)_I = \sum_{\substack{|I|_1\ge|J|_1, |I|_2<|J|_2 \\ |I|_3<|J|_3}} a_{IJ}s_J, \quad (A_4 s)_I = \sum_{\substack{|I|_1<|J|_1, |I|_2\ge|J|_2 \\ |I|_3\ge|J|_3}} a_{IJ}s_J$$

$$(A_5 s)_I = \sum_{\substack{|I|_1<|J|_1, |I|_2\ge|J|_2 \\ |I|_3<|J|_3}} a_{IJ}s_J, \quad (A_6 s)_I = \sum_{|I|_1<|J|_1, |I|_2<|J|_2} a_{IJ}s_J.$$

We will estimate A_2 and A_3. The others follow similarly. Let $|I|_1 = 2^{-\nu_1}$, $|I|_2 = 2^{-\nu_2}$, $|J|_1 = 2^{-\mu_1}$ and $|J|_2 = 2^{-\mu_2}$. Moreover, let $f_I = 2^{s(\nu)}k$ and $f_J = 2^{s(\mu)}l$.

By the Hölder inequality, $\sum_i |a_i b_i| \le \big(\sum_i |a_i|^q |b_i|^{\frac{q}{2}}\big)^{\frac{1}{q}} \big(\sum_i |b_i|^{\frac{q'}{2}}\big)^{\frac{1}{q'}}$, with indices q, q' satisfying $\frac{1}{q} + \frac{1}{q'} = 1$ and Definition 4.1, there is a constant $C > 0$ independent of $I, J \in \mathcal{I}$ such that

$$|(A_3 s)_I| \le C \Bigg[\sum_{\substack{|I|_1 \ge |J|_1, |I|_2 < |J|_2 \\ |I|_3 < |J|_3, l \in \mathbb{Z}}} |s_J|^q \left(\frac{2^{\nu_1} 2^{\mu_2} 2^{s(\mu)}}{2^{\mu_1} 2^{\nu_2} 2^{s(\nu)}} \right)^{\frac{q\beta}{2}} \big(1 + 2^{-s(\nu)} |2^{s(\nu)}k - 2^{s(\mu)}l|\big)^{-\frac{qL}{2}}$$
$$\times \left[\prod_{\sigma=1}^{3} \left(1 + \frac{|c_{I,\sigma} - c_{J,\sigma}|}{\max(|I|_\sigma, |J|_\sigma)} \right) \right]^{-\frac{qL}{2}} \Bigg]^{\frac{1}{q}},$$

because $L > 2$, $\beta > 2$ and

$$\sum_{\mu_1=\nu_1}^{\infty} \sum_{|J|_1 = 2^{-\mu_1}} \left(\frac{|J|_1}{|I|_1} \right)^{\frac{q'\beta}{2}} \big(1 + |I_1|^{-1} |c_{I,1} - c_{J,1}|\big)^{-\frac{q'L}{2}} \le C \sum_{\mu_1=\nu_1}^{\infty} 2^{(\frac{q'\beta}{2} - 1)(\nu_1 - \mu_1)},$$

for some constants $C > 0$ independent of μ_1 and ν_1.

It is now permitted to apply Lemma 3.5 with $\beta_J = |s_J|^q$, $a = \frac{1}{q}$ and $r = 1$, since $L > 2$. This gives

$$|(A_3 s)_I|^q \le C \sum_{l \in \mathbb{Z}} \sum_{\mu_1=\nu_1}^{\infty} \sum_{\mu_2=-\infty}^{\nu_2} \Bigg[\left(\frac{2^{\nu_1} 2^{\mu_2} 2^{s(\mu)}}{2^{\mu_1} 2^{\nu_2} 2^{s(\nu)}} \right)^{\frac{q\beta}{2}} \big(1 + 2^{-s(\nu)} |2^{s(\nu)}k - 2^{s(\mu)}l|\big)^{-\frac{qL}{2}}$$
$$\times \frac{2^{2q\mu_1}}{2^{2q\nu_1}} \Bigg(\mathrm{M}_S \Big(\sum_{\substack{|J|_1 = 2^{-\mu_1}, |J|_2 = 2^{-\mu_2} \\ t(J) = l}} |s_J \chi_J| \Big)(x) \Bigg)^q \Bigg]$$

for $x \in Q$, where $I = (Q, k)$. Let $\mathbb{M}(x) = \mathrm{M}_S\big(\sum_{|J|_1 = 2^{-\mu_1}, |J|_2 = 2^{-\mu_2}, t(J) = l} |s_J \tilde{\chi}_J|\big)(x)$, so that

$$\|A_3 s\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})} \le C \Bigg\| \Bigg[\sum_{\substack{\nu \in \mathbb{Z}^2 \\ k \in \mathbb{Z}}} \sum_{l \in \mathbb{Z}} \sum_{\mu_1=\nu_1}^{\infty} \sum_{\mu_2=-\infty}^{\nu_2} \left(\frac{2^{\nu_1} 2^{\mu_2}}{2^{\mu_1} 2^{\nu_2}} \right)^{q\left(\frac{\beta}{2} - 2\right)} \left(\frac{2^{s(\mu)}}{2^{s(\nu)}} \right)^{q\left(\frac{\beta}{2} - 1\right)}$$
$$\times (\mathbb{M}(x))^q \frac{2^{s(\nu)q\alpha} (1 + |k|)^{q\alpha}}{(1 + 2^{-s(\nu)} |2^{s(\nu)}k - 2^{s(\mu)}l|)^{\frac{qL}{2}}} \Bigg]^{\frac{1}{q}} \Bigg\|_{L^p}, \tag{23}$$

because $|I|^{-\frac{1}{2}} = 2^{s(\nu) - s(\mu)} |J|^{-\frac{1}{2}}$ and $|I| = 2^{-2s(\nu)}$.

We have

$$\frac{2^{s(\nu)}(1+|k|)}{(1+2^{-s(\nu)}|2^{s(\nu)}k-2^{s(\mu)}l|)} \leq \left(\frac{2^{s(\nu)}}{2^{s(\mu)}}\right)^2 \frac{2^{s(\mu)}(1+|k|)}{(1+2^{-s(\mu)}|2^{s(\nu)}k-2^{s(\mu)}l|)} \leq \frac{2^{2s(\nu)}}{2^{s(\mu)}}(1+|l|) \tag{24}$$

when $s(\nu) \geq s(\mu)$. Using $q > 1$ and $\beta > 4\alpha + 2$, yields

$$\|A_3 s\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})} \leq C \left\| \left[\sum_{\nu\in\mathbb{Z}^2,\, k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \sum_{\mu_1=\nu_1}^{\infty} \sum_{\mu_2=-\infty}^{\nu_2} \left(\frac{2^{\nu_1}2^{\mu_2}}{2^{\mu_1}2^{\nu_2}}\right)^{q\left(\frac{\beta}{2}-2\right)} \times (\mathbb{M}(x))^q \frac{2^{s(\mu)q\alpha}(1+|l|)^{q\alpha}}{(1+2^{-s(\nu)}|2^{s(\nu)}k-2^{s(\mu)}l|)^{q\left(\frac{L}{2}-\alpha\right)}} \right]^{\frac{1}{q}} \right\|_{L^p}.$$

Interchanging the order of summations of ν, k and μ, l, we assert that

$$\begin{aligned}\|A_3 s\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})} &\leq C \left\| \left[\sum_{\mu\in\mathbb{Z}^2,\, l\in\mathbb{Z}} \sum_{\nu_1=-\infty}^{\mu_1} \sum_{\nu_2=\mu_2}^{\infty} \sum_{k\in\mathbb{Z}} \left(\left(\frac{2^{\nu_1}2^{\mu_2}}{2^{\mu_1}2^{\nu_2}}\right)^{\frac{\beta}{2}-2} \times \mathbb{M}(x) \frac{2^{s(\mu)\alpha}(1+|l|)^{\alpha}}{(1+2^{-s(\nu)}|2^{s(\nu)}k-2^{s(\mu)}l|)^{\frac{L}{2}-\alpha}} \right)^q \right]^{\frac{1}{q}} \right\|_{L^p} \\ &\leq C \left\| \left[\sum_{\mu\in\mathbb{Z}^2,\, l\in\mathbb{Z}} \left[\mathrm{M}_S \left(2^{s(\mu)\alpha}(1+|l|)^{\alpha} \sum_{\substack{|J|_1=2^{-\mu_1},|J|_2=2^{-\mu_2}\\ t(J)=l}} |s_J \tilde{\chi}_J| \right)(x) \right]^q \right]^{\frac{1}{q}} \right\|_{L^p},\end{aligned}$$

because $\beta > 4$ and $L > 2\alpha + 2$. Using Theorem 3.4, we obtain $\|A_3 s\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})} \leq C\|s\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})}$.

The estimate for A_2 is similar. Instead of (23), the formulation is

$$\|A_2 s\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})} \leq C \left\| \left[\sum_{I\in\mathcal{I}} \sum_{l\in\mathbb{Z}} \sum_{\mu_1=\nu_1}^{\infty} \sum_{\mu_2=-\infty}^{\nu_2} \left(\frac{2^{\nu_1}2^{\mu_2}}{2^{\mu_1}2^{\nu_2}}\right)^{q\left(\frac{\beta}{2}-2\right)} \left(\frac{2^{s(\nu)}}{2^{s(\mu)}}\right)^{\frac{q\beta}{2}} \times (\mathbb{M}(x))^q \frac{2^{s(\nu)q\alpha}(1+|k|)^{q\alpha}}{(1+2^{-s(\mu)}|2^{s(\nu)}k-2^{s(\mu)}l|)^{\frac{qL}{2}}} \right]^{\frac{1}{q}} \right\|_{L^p}.$$

In this case,

$$\frac{2^{s(\nu)}(1+|k|)}{(1+2^{-s(\mu)}|2^{s(\nu)}k-2^{s(\mu)}l|)} \leq 2^{s(\mu)}(1+|l|) \tag{25}$$

can be used when $s(\nu) < s(\mu)$. This leads to

$$\|A_2 s\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})} \le C \left\| \left[\sum_{\mu\in\mathbb{Z}^2,\, l\in\mathbb{Z}} \sum_{\nu_1=-\infty}^{\mu_1} \sum_{\nu_2=\mu_2}^{\infty} \sum_{k\in\mathbb{Z}} \left(\left(\frac{2^{\nu_1}2^{\mu_2}}{2^{\mu_1}2^{\nu_2}}\right)^{\frac{\beta}{2}-2} \left(\frac{2^{s(\nu)}}{2^{s(\mu)}}\right)^{\frac{\beta}{2}} \times \mathbb{M}(x) \frac{2^{s(\mu)\alpha}(1+|l|)^{\alpha}}{(1+2^{-s(\mu)}|2^{s(\nu)}k - 2^{s(\mu)}l|)^{\frac{L}{2}-\alpha}} \right)^q \right]^{\frac{1}{q}} \right\|_{L^p},$$

since

$$\sum_{k\in\mathbb{Z}} \frac{1}{(1+2^{-s(\mu)}|2^{s(\nu)}k - 2^{s(\mu)}l|)^N} \le C_N 2^{s(\mu)-s(\nu)} \tag{26}$$

for some constant C_N depending only on $N > 1$. This yields the desired result, because $\beta > 4\alpha + 4$. The estimates for the other operators, A_1, A_4, A_5 and A_6, follow similarly.

The case for $r \le 1$ can be proved through an argument similar to the one presented in [3, p. 55]. According to the definition of an almost diagonal matrix, for any $\{a_{IJ}\} \in \omega(\beta, L)$, there exists $\tilde{r} < r = \min(p,q)$ such that $\tilde{r}$ satisfies $\{|a_{IJ}|^{\tilde{r}}\} \in \omega(\tilde{r}\beta, \tilde{r}L)$, $\tilde{r}\beta > 5 + 4\tilde{r}\alpha$ and $\tilde{r}L > 2\alpha\tilde{r} + 2$. Thus, $\tilde{A} = \{|a_{IJ}|^{\tilde{r}}(|I|/|J|)^{\frac{1}{2}-\frac{\tilde{r}}{2}}\} \in \omega\big(\tilde{r}\beta - \frac{1}{2} + \frac{\tilde{r}}{2}, \tilde{r}L\big)$. Furthermore, for any $s = \{s_I\}_{I\in\mathcal{I}}$, define $t = \{t_I\}_{I\in\mathcal{I}}$ by $t_I = |I|^{\frac{1}{2}-\frac{\tilde{r}}{2}}|s_I|^{\tilde{r}}$. Then, $\|s\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})} = \|t\|^{\frac{1}{\tilde{r}}}_{\dot{f}_{p/\tilde{r}}^{\alpha\tilde{r},q/\tilde{r}}(\mathbb{D})}$. We have

$$\|As\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})} \le \big\|\tilde{A}t\big\|^{\frac{1}{\tilde{r}}}_{\dot{f}_{p/\tilde{r}}^{\alpha\tilde{r},q/\tilde{r}}(\mathbb{D})}$$

by the $\tilde{r}$-triangle inequality, since $\tilde{r}\beta - \frac{1}{2} + \frac{\tilde{r}}{2} > \tilde{r}\beta - \frac{1}{2} > 4 + 4\alpha\tilde{r}$ and $\tilde{r}L > 2\alpha\tilde{r} + 2$. The boundedness of $\tilde{A}$ follows from the boundedness of A. □

5. The main results for the function spaces

The atomic, molecular and frame decompositions are some of the important results for the Triebel–Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^n)$. This section presents the corresponding decompositions for $\dot{F}_p^{\alpha,q}(\mathbb{D})$.

5.1. Molecular estimate. We begin with the definition of molecules.

Definition 5.1. Let $\beta > 0$. The space of molecules of order β for $\dot{F}_p^{\alpha,q}(\mathbb{D})$, $\mathcal{M}_\beta$, consists of $\{m_I\}_{I\in\mathcal{I}}$ that satisfy, for some constant $C > 0$,

$$|(\partial^\gamma m_I)(x_1,x_2,x_3)| \le C|I|^{-\frac{1}{2}}(1+|t(I)|)^{\gamma_3} \prod_{\sigma=1}^{3} \frac{|I|_\sigma^{-\gamma_\sigma}}{(1+|I|_\sigma^{-1}|x_\sigma - c_{I,\sigma}|)^\beta} \tag{27}$$

$$|\hat{m}_I(\xi_1,\xi_2,\xi_3)| \le \frac{C|I|^{\frac{1}{2}}}{[(1+|I|_1|\xi_1|)(1+|I|_2|\xi_2|)(1+|I|_3|\xi_3 - f_I|)]^\beta}, \tag{28}$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{N}^3$ and $0 \le \gamma_i \le [\beta]$; $i = 1, 2, 3$; and m_I satisfies the vanishing moment conditions

$$\int_{\mathbb{R}} m_I(x_1, x_2, x_3) x_1^{\lambda} dx_1 = 0 \quad \text{and} \quad \int_{\mathbb{R}} m_I(x_1, x_2, x_3) x_2^{\lambda} dx_2 = 0,$$

where $\lambda \in \mathbb{N}$ and $\lambda \le [\beta]$. Define $\|\{m_I\}\|_{\mathcal{M}_\beta}$ to be the infimum of constant C satisfying (27) and (28).

The molecules in $\mathcal{M}_\beta$ have a decay in the third variable of the Fourier domain. This is introduced on purpose because it is essential on the molecular decompositions for $\dot{F}_p^{\alpha,q}(\mathbb{D})$. The utility of this decay cannot be seen explicitly in the following theorem. The use of this decay is absorbed in Lemma 7.1.

Theorem 5.2. *Let $\alpha \in \mathbb{R}$, $0 < p, q < \infty$ and $\beta > 20\mathcal{J} + 16|\alpha| + 6$. Suppose that $s = \{s_I\}_{I \in \mathcal{I}} \in \dot{f}_p^{\alpha,q}(\mathbb{D})$ and let $\{m_I\}_{I \in \mathcal{I}}$ be a family of molecules of order β for $\dot{F}_p^{\alpha,q}(\mathbb{D})$.*

1. *If $f = \sum_{I \in \mathcal{I}} s_I m_I$, then $f \in \dot{F}_p^{\alpha,q}(\mathbb{D})$ and $\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{D})} \le C\|s\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})}$ for some constant $C > 0$ independent of s.*
2. *For any $f \in \dot{F}_p^{\alpha,q}(\mathbb{D})$, $\|\{\langle f, m_I\rangle\}\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})} \le C\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{D})}$ for some constant $C > 0$ independent of f.*

Proof. Using the φ-ψ transforms for $\mathbb{D}$ to m_I, we have

$$f = \sum_{I \in \mathcal{I}} s_I m_I = \sum_{I \in \mathcal{I}} s_I \sum_{J \in \mathcal{I}} a_{IJ} \psi_J = \sum_{J \in \mathcal{J}} (As)_J \psi_J = (T_\psi \circ A)(s),$$

where $a_{IJ} = \langle m_I, \psi_J \rangle$ and $A = \{a_{IJ}\}$. From the assumption for β, Lemma 7.1 and the inequality

$$\min\left(\frac{2^{\mu_1}}{2^{\nu_1}}, \frac{2^{\nu_1}}{2^{\mu_1}}\right) \min\left(\frac{2^{\mu_2}}{2^{\nu_2}}, \frac{2^{\nu_2}}{2^{\mu_2}}\right) \le \min\left(\frac{2^{s(\nu)}}{2^{s(\mu)}}, \frac{2^{s(\mu)}}{2^{s(\nu)}}\right), \tag{29}$$

A is an almost diagonal matrix for $\dot{f}_p^{\alpha,q}(\mathbb{D})$, hence $\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{D})} \le C\|s\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})}$, where C is the product of the operator norms of T_ψ and A.

The second part is straightforward. The pairing $\langle f, m_I \rangle$ is interpreted by $\langle f, m_I \rangle = \sum_{J \in \mathcal{I}} S_\varphi(f) \langle \psi_J, m_I \rangle$. This is the desired result, because $\{\langle \psi_J, m_I \rangle\}$ is an almost diagonal matrix for $\dot{f}_p^{\alpha,q}(\mathbb{D})$ and S_φ is bounded. □

5.2. The generalized φ-ψ transforms. Observe that the functions φ and ψ for the φ-ψ transforms are band-limited. This section will show that, in some ways, this condition can be relaxed. Furthermore, we generalize the results of the φ-ψ transforms to the functions φ and ψ that also depend on l. That type of generalization seems unnecessary, but it is crucial on the estimate for the differential operator $\triangle_{\mathbb{D}}$ in Section 6. The following theorem asserts that if a family $\{\varphi_{[l]}\}_{l \in \mathbb{Z}}$ of Schwartz functions parameterized by l, $l \in \mathbb{Z}$, satisfies (30) and (31), then a dilation of $\{\varphi_{[l]}\}_{l \in \mathbb{Z}}$ can generate an identity similar to (19).

Theorem 5.3. *Let $\{\varphi_{[l]}\}_{l\in\mathbb{Z}} \subset \mathcal{S}_v(\mathbb{R}^3)$ satisfy the discrete Littlewood–Paley inequality:*

$$0 < B < \sum_{\nu\in\mathbb{Z}^2,\, l\in\mathbb{Z}} \left|\hat{\varphi}_{[l]}(2^{\nu_1}\xi_1, 2^{\nu_2}\xi_2, 2^{s(\nu)}\xi_3 - l)\right|^2 < A \quad \text{if } \xi_1\xi_2 \neq 0, \tag{30}$$

where A and B are constants and

$$\|\varphi_{[l]}\|_{\tau,N} \leq C_{\tau,N}, \tag{31}$$

where $C_{\tau,N} > 0$ is a constant depending on τ and N only and $\|\cdot\|_{\tau,N}$ is the semi-norm of the Schwartz functions defined in (11)*. For any $\eta \in \mathbb{Z}^2$, let*

$$\varphi^{l,\eta}_{\nu,k}(x) = 2^{s(\nu)+2s(\eta)}\varphi^{l}_{[l]}(2^{\eta}(2^{\nu}x - k)), \quad l \in \mathbb{Z},\ \nu \in \mathbb{Z}^2,\ k \in \mathbb{Z}^3.$$

There exist an $\eta \in \mathbb{Z}^2$ and a family $\{\psi^{l,\eta}_{\nu,k}\} \in \bigcap_{\beta>0}\mathcal{M}_\beta$ such that for any $f \in \mathcal{S}_v(\mathbb{R}^3)$,

$$f = \sum_{\nu\in\mathbb{Z}^2,\, l\in\mathbb{Z}} \langle f, \psi^{l,\eta}_{\nu,k}\rangle \varphi^{l,\eta}_{\nu,k} = \sum_{\nu\in\mathbb{Z}^2,\, l\in\mathbb{Z}} \langle f, \varphi^{l,\eta}_{\nu,k}\rangle \psi^{l,\eta}_{\nu,k} \quad \text{in } \mathcal{S}_v(\mathbb{R}^3). \tag{32}$$

In addition, for any $f \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{G}(\mathbb{R}^3)$, we also have

$$f = \sum_{\nu\in\mathbb{Z}^2,\, l\in\mathbb{Z}} \langle f, \psi^{l,\eta}_{\nu,k}\rangle \varphi^{l,\eta}_{\nu,k} = \sum_{\nu\in\mathbb{Z}^2,\, l\in\mathbb{Z}} \langle f, \varphi^{l,\eta}_{\nu,k}\rangle \psi^{l,\eta}_{\nu,k} \quad \text{in } \mathcal{S}'(\mathbb{R}^3)/\mathcal{G}(\mathbb{R}^3), \tag{33}$$

where $\mathcal{S}'(\mathbb{R}^3)/\mathcal{G}(\mathbb{R}^3)$ is endowed with the weak topology induced from $\mathcal{S}_v(\mathbb{R}^3)$.

Proving Theorem 5.3 requires some preparation. For any $\{\varphi_{[l]}\}_{l\in\mathbb{Z}} \subset \mathcal{S}_v(\mathbb{R}^3)$ satisfying (30) and (31), define the function $\Phi(\xi)$ by

$$\Phi(\xi) = \sum_{\nu\in\mathbb{Z}^2,\, l\in\mathbb{Z}} \left|\hat{\varphi}_{[l]}(2^{\nu_1}\xi_1, 2^{\nu_2}\xi_2, 2^{s(\nu)}\xi_3 - l)\right|^2 \quad \text{when } \xi_1\xi_2 \neq 0.$$

It is obvious that $\Phi(\xi)$ satisfies $B < \Phi(\xi) < A$. We define $\phi_{[l]}(x)$ by

$$\hat{\phi}_{[l]}(\xi) = \frac{\overline{\hat{\varphi}_{[l]}(\xi)}}{\Phi(\xi_1, \xi_2, \xi_3 + l)} \quad \text{when } \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \text{ and } \xi_1\xi_2 \neq 0,$$

and $\hat{\phi}_{[l]}(\xi_1, 0, \xi_3) = \hat{\phi}_{[l]}(0, \xi_2, \xi_3) = 0$. Then, the function $\phi_{[l]}(x)$ is well defined, and $\phi_{[l]}(x), \varphi_{[l]}(x)$ satisfy

$$\sum_{\nu\in\mathbb{Z}^2,\, l\in\mathbb{Z}} \hat{\varphi}_{[l]}(2^{\nu_1}\xi_1, 2^{\nu_2}\xi_2, 2^{s(\nu)}\xi_3 - l)\overline{\hat{\phi}_{[l]}(2^{\nu_1}\xi_1, 2^{\nu_2}\xi_2, 2^{s(\nu)}\xi_3 - l)} = 1 \quad \text{if } \xi_1\xi_2 \neq 0, \tag{34}$$

because $\Phi(2^{\nu}\xi) = \Phi(\xi)$ for any $\nu \in \mathbb{Z}^2$.

It will now be shown that $\phi_{[l]}$ belongs to $\mathcal{S}_v(\mathbb{R}^3)$. Estimate first the partial derivatives of $\Phi(\xi)$. Since $\hat{\varphi}_{[l]} \in \mathcal{S}(\mathbb{R}^3)$,

$$|(\partial^\gamma \Phi)(\xi_1, \xi_2, \xi_3)| \leq C_{\gamma,N} \sum_{\substack{\nu\in\mathbb{Z}^2 \\ l\in\mathbb{Z}}} \frac{2^{(\gamma_1+\gamma_3)\nu_1} 2^{(\gamma_2+\gamma_3)\nu_2}}{[(1+2^{\nu_1}|\xi_1|)(1+2^{\nu_2}|\xi_2|)(1+|2^{\nu_1+\nu_2}\xi_3 - l|)]^N}$$

for all $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{N}^3$ and $N > 0$. As

$$\sum_{j\in\mathbb{Z}} \frac{2^{j\lambda}}{(1+2^j|y|)^M} \leq \sum_{j=-\infty}^{-[\log_2|y|]} 2^{j\lambda} + \sum_{j=-[\log_2|y|]}^{\infty} \frac{2^{j\lambda}}{(2^j|y|)^M} \leq C|y|^{-\lambda},$$

when $M > \lambda$. Thus, if we take $N > \gamma_1 + \gamma_2 + \gamma_3 + 2$, we have

$$|(\partial^\gamma \Phi)(\xi_1, \xi_2, \xi_3)| \leq C_\gamma |\xi_1|^{-\gamma_1-\gamma_3} |\xi_2|^{-\gamma_2-\gamma_3}.$$

By the product rule and the fact that $0 < B < \Phi(\xi)$,

$$|(\partial^\gamma \Phi^{-1})(\xi_1, \xi_2, \xi_3 + l)| \leq C_\gamma |\xi_1|^{-\gamma_1-\gamma_3} |\xi_2|^{-\gamma_2-\gamma_3}. \tag{35}$$

Since $\varphi_{[l]} \in \mathcal{S}_v(\mathbb{R}^3)$, for any $\lambda, N > 0$ and $\gamma \in \mathbb{N}^3$, there exists a constant $C_{\lambda,\gamma,N} > 0$ independent of l, such that

$$|(\partial^\gamma \hat{\varphi}_{[l]})(\xi_1, \xi_2, \xi_3)| \leq C_{\lambda,\gamma,N} \frac{|\xi_1|^\lambda |\xi_2|^\lambda}{[(1+|\xi_1|)(1+|\xi_2|)(1+|\xi_3|)]^N}. \tag{36}$$

By (35) and (36), we assert that $\hat{\phi}_{[l]}(\xi)$ belongs to $\mathcal{S}(\mathbb{R}^3)$ and, hence, $\{\phi_{[l]}\}_{l\in\mathbb{Z}} \subset \mathcal{S}_v(\mathbb{R}^3)$ and satisfies (31). With the family $\{\phi_{[l]}\}$, we are now ready to prove Theorem 5.3.

Proof of Theorem 5.3. Ideas from [3, Theorem 4.2 and Theorem 4.4] will be used to prove this theorem. For brevity, for the families $\{\varphi_{[l]}\}_{l\in\mathbb{Z}} \subset \mathcal{S}_v(\mathbb{R}^3)$ and $\{\phi_{[l]}\}_{l\in\mathbb{Z}} \subset \mathcal{S}_v(\mathbb{R}^3)$, we write $(\varphi_{[l]})^l_\nu = \varphi^l_\nu$ and $(\phi_{[l]})^l_\nu = \phi^l_\nu$. For any fixed $\eta \in \tilde{\mathbb{Z}}^2 = \{(\kappa, \kappa) : \kappa \in \mathbb{Z}, \kappa < 0\}$ and any $\{m_J\} \in \bigcap \mathcal{M}_\beta$, by (34) and Theorem 2.1, we have

$$m_J = \sum_{\nu\in\mathbb{Z}^2,\, l\in\mathbb{Z}} \varphi^l_{\nu+\eta} * \tilde{\phi}^l_{\nu+\eta} * m_J,$$

where

$$\begin{aligned}\varphi^l_{\nu+\eta} * \tilde{\phi}^l_{\nu+\eta} * m_J = \sum_k \bigg\{ &\int_{Q_{\nu,k}} \big[\varphi^l_{\nu+\eta}(x-y) - \varphi^l_{\nu+\eta}(x-c_{Q_{\nu,k}})\big](\tilde{\phi}^l_{\nu+\eta} * m_J)(y)\,dy \\ &+ \varphi^l_{\nu+\eta}(x-c_{Q_{\nu,k}})\bigg[\int_{Q_{\nu,k}} (\tilde{\phi}^l_{\nu+\eta} * m_J)(y) - (\tilde{\phi}^l_{\nu+\eta} * m_J)(c_{Q_{\nu,k}})\,dy\bigg] \\ &+ 2^{-2s(\nu)} \varphi^l_{\nu+\eta}(x - c_{Q_{\nu,k}})(\tilde{\phi}^l_{\nu+\eta} * m_J)(c_{Q_{\nu,k}})\bigg\}.\end{aligned}$$

For any $\eta \in \tilde{\mathbb{Z}}^2$ and $f \in \mathcal{S}_v(\mathbb{R}^3)$, define

$$\begin{aligned}\mathcal{F}_\eta(f) &= \sum_{\nu\in\mathbb{Z}^2,\, l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^3} 2^{-2s(\nu)} \varphi^l_{\nu+\eta}(x-c_{Q_{\nu,k}})(\tilde{\phi}^l_{\nu+\eta} * f)(c_{Q_{\nu,k}}) \\ &= \sum_{\nu\in\mathbb{Z}^2,\, l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^3} \langle f, \phi^{l,\eta}_{\nu,k}\rangle \varphi^{l,\eta}_{\nu,k}\,.\end{aligned}$$

Since $(\tilde{\phi}^l_{\nu+\eta} * m_J)(c_{Q_{\nu,k}}) = \int m_J(y)\overline{\phi^l_{\nu+\eta}(y-c_{Q_{\nu,k}})}dy = 2^{s(\nu)}\langle m_J, \phi^{l,\eta}_{\nu,k}\rangle$, we assert that

$$\begin{aligned}&(I-\mathcal{F}_\eta)(m_J) \\ &= \sum_{l,\nu,k} \sum_{Q_{\nu-\eta,h}\subset Q_{\nu,k}} \Bigg\{ \int_{Q_{\nu-\eta,h}} \big[(\varphi^l_\nu)(x-y) - \varphi^l_\nu(x-c_{Q_{\nu-\eta,h}})\big](\tilde{\phi}^l_\nu * m_J)(y)\,dy \\ &\quad + \varphi^l_\nu(x-c_{Q_{\nu-\eta,h}})\Bigg[\int_{Q_{\nu-\eta,h}} (\tilde{\phi}^l_\nu * m_J)(y)dy - 2^{-2s(\nu)+2s(\eta)}(\tilde{\phi}^l_\nu * m_J)(c_{Q_{\nu-\eta,h}})\Bigg]\Bigg\} \\ &= \sum_{I\in\mathcal{I}} (s_{IJ}u_I + t_{IJ}v_I),\end{aligned} \tag{37}$$

where, for $I = (Q_{\nu,k}, l)$, $s_{IJ} = |Q_{\nu,k}|^{-\frac{1}{2}} \sum_{Q_{\nu-\eta,h}\subset Q_{\nu,k}} \int_{Q_{\nu-\eta,h}} |\tilde{\phi}^l_\nu * m_J(y)|dy$, when $s_{IJ} \neq 0$,

$$u_I(x) = s^{-1}_{IJ}\left(\sum_{Q_{\nu-\eta,h}\subset Q_{\nu,k}} \int_{Q_{\nu-\eta,h}} \big[(\varphi^l_\nu)(x-y) - \varphi^l_\nu(x-c_{Q_{\nu-\eta,h}})\big](\tilde{\phi}^l_\nu * m_J)(y)dy \right);$$

$$t_{IJ} = |Q_{\nu,k}|^{-\frac{1}{2}} \sum_{Q_{\nu-\eta,h}\subset Q_{\nu,k}} \left[\int_{Q_{\nu-\eta,h}} \big|(\tilde{\phi}^l_\nu * m_J)(y) - (\tilde{\phi}^l_\nu * m_J)(c_{Q_{\nu-\eta,h}})\big| dy \right];$$

and when $t_{IJ} \neq 0$,

$$v_I(x) = t^{-1}_{IJ} \sum_{Q_{\nu-\eta,h}\subset Q_{\nu,k}} \varphi^l_\nu(x-c_{Q_{\nu-\eta,h}}) \left[\int_{Q_{\nu-\eta,h}} ((\tilde{\phi}^l_\nu * m_J)(y) - (\tilde{\phi}^l_\nu * m_J)(c_{Q_{\nu-\eta,h}}))dy \right].$$

Since $\{m_J\} \in \bigcap \mathcal{M}_\beta$, using arguments from [3, Theorem 4.2 and Theorem 4.4] (especially, inequalities [3, (4.19) and (4.26)]), for any $\beta, L > 0$ there exists a constant $C > 0$ such that $\|\{u_I\}\|_{\mathcal{M}_\beta} \leq C2^{\frac{s(\eta)}{2}}$ and $\|\{v_I\}\|_{\mathcal{M}_\beta} \leq C$. Moreover, by Lemma 7.1, $\|\{s_{IJ}\}\|_{\omega(\tilde{\beta},L)} \leq C$ and $\|\{t_{IJ}\}\|_{\omega(\tilde{\beta},L)} \leq C2^{\frac{s(\eta)}{2}}$, where $\tilde{\beta} > 6\beta + 4$, $L > \beta + 1$. Thus, for any $\beta > 0$, by Theorem 7.3, there exists a $\tilde{\beta}$ such that

$$\|\{(I-\mathcal{F}_\eta)(m_J)\}\|_{\mathcal{M}_\beta} \leq C2^{\frac{s(\eta)}{2}} \|\{m_J\}\|_{\mathcal{M}_{\tilde{\beta}}}$$

for some constant $C > 0$ independent of η. Let θ, ζ satisfy the condition of the φ-ψ transforms. As the φ-ψ transforms are bounded on $L^2(\mathbb{R}^3)$ and l^2 and

$T_\zeta \circ S_\theta$ is the identity operator in $L^2(\mathbb{R}^3)$, for any $f \in L^2(\mathbb{R}^3)$, $f = \sum_{I\in\mathcal{I}} \langle f, \theta_I\rangle \zeta_I$. Hence, by the molecular estimate,

$$\|(I-\mathcal{F}_\eta)(f)\|_{L^2(\mathbb{R}^3)} = \left\| \sum_{I\in\mathcal{I}} \langle f, \theta_I\rangle (I-\mathcal{F}_\eta)(\zeta_I) \right\|_{L^2(\mathbb{R}^3)} \le C 2^{\frac{s(\eta)}{2}} \|f\|_{L^2(\mathbb{R}^3)}$$

for some constant $C > 0$ independent of f and η.

Since $s(\eta)$ goes to minus infinity as $\kappa \to -\infty$, $\eta = (\kappa, \kappa)$. This guarantees that $\mathcal{F}_\eta$ is an invertible operator on $L^2(\mathbb{R}^3)$ for some η. Our results follow by taking

$$\psi^{l,\eta}_{\nu,k} = \mathcal{F}_\eta^{-1}\left(\phi^{l,\eta}_{\nu,k}\right) = \sum_{j=0}^{\infty} (I-\mathcal{F}_\eta)^j \left(\phi^{l,\eta}_{\nu,k}\right).$$

The family $\{\psi^{l,\eta}_{\nu,k}\}$ belongs to $\mathcal{M}_\beta$ because

$$\left\| \left\{ (I-\mathcal{F}_\eta)^j \left(\phi^{l,\eta}_{\nu,k}\right) \right\} \right\|_{\mathcal{M}_\beta} \le \left(C 2^{\frac{s(\eta)}{2}} \right)^j \left\| \{\phi^{l,\eta}_{\nu,k}\} \right\|_{\mathcal{M}_{\tilde{\beta}}}, \quad j \in \mathbb{N}, \tag{38}$$

for some sufficiently large $\tilde{\beta}$, and $\mathcal{M}_\beta$ is a Banach space.

Inequality (38) is valid because, for instance when $j = 2$, we find that by (37) there exist almost diagonal matrices $\{s_{IJ}\}$ and $\{t_{IJ}\}$, and families of molecules $\{u_I\}$ and $\{v_I\}$ such that

$$(I-\mathcal{F}_\eta)^2(\phi^{l,\eta}_{\nu,k}) = \sum_{I\in\mathcal{I}} s_{IJ}(I-\mathcal{F}_\eta)u_I + t_{IJ}(I-\mathcal{F}_\eta)v_I, \quad J = (Q_{\nu,k}, l).$$

Applying (37) to $(I-\mathcal{F}_\eta)u_I$ and $(I-\mathcal{F}_\eta)v_I$ yields almost diagonal matrices $\{s^{[m]}_{IK}\}$ and $\{t^{[m]}_{IK}\}$, $m = 1, 2$, and families of molecules $\{u^{[m]}_K\}$ and $\{v^{[m]}_K\}$, $m = 1, 2$, such that

$$\begin{aligned}
(I-\mathcal{F}_\eta)^2(\phi^{l,\eta}_{\nu,k}) &= \sum_{I\in\mathcal{I}} s_{IJ} \left(\sum_{K\in\mathcal{I}} s^{[1]}_{IK} u^{[1]}_K + s^{[2]}_{IK} u^{[2]}_K \right) + t_{IJ} \left(\sum_{K\in\mathcal{I}} t^{[1]}_{IK} v^{[1]}_K + t^{[2]}_{IK} v^{[2]}_K \right) \\
&= \sum_{K\in\mathcal{I}} \Bigg\{ \left(\sum_{I\in\mathcal{I}} s_{IJ} s^{[1]}_{IK} \right) u^{[1]}_K + \left(\sum_{I\in\mathcal{I}} s_{IJ} s^{[2]}_{IK} \right) u^{[2]}_K \\
&\quad + \left(\sum_{I\in\mathcal{I}} t_{IJ} t^{[1]}_{IK} \right) v^{[1]}_K + \left(\sum_{I\in\mathcal{I}} t_{IJ} t^{[2]}_{IK} \right) v^{[2]}_K \Bigg\}.
\end{aligned}$$

By Theorem 7.4, the matrices $\{\sum_{I\in\mathcal{I}} s_{IJ} s^{[1]}_{IK}\}$, $\{\sum_{I\in\mathcal{I}} s_{IJ} s^{[2]}_{IK}\}$, $\{\sum_{I\in\mathcal{I}} t_{IJ} t^{[1]}_{IK}\}$ and $\{\sum_{I\in\mathcal{I}} t_{IJ} t^{[2]}_{IK}\}$ are almost diagonal matrices of order $\tilde{\beta}, L$. Therefore, the desired result follows for $j = 2$ by using Theorem 7.3. The general result for $j \in \mathbb{N}$ is concluded by using Theorem 7.5. The convergence of the expansion in (32) is guaranteed by Theorem 7.3 again. The identities (33) follow from the duality of $\mathcal{S}_v(\mathbb{R}^3)$ and $\mathcal{S}'(\mathbb{R}^3)/\mathcal{G}(\mathbb{R}^3)$. □

5.3. Atomic decomposition and frames. The atomic decompositions of $\dot{F}^{\alpha,q}_p(\mathbb{D})$ will now be presented using Theorem 5.3. Say that the family $\{a_I\}_{I\in\mathcal{I}}$ is a family of *smooth atoms for* $\dot{F}^{\alpha,q}_p(\mathbb{D})$ if, for each $I=(Q,l)$, there exist constants $K, C_L, C_\gamma > 0$ such that

$$\operatorname{supp} a_I \subseteq KQ;$$

$$\int_{\mathbb{R}} a_I(x_1,x_2,x_3)x_1^\lambda dx_1 = 0, \quad \int_{\mathbb{R}} a_I(x_1,x_2,x_3)x_2^\lambda dx_2 = 0, \quad \forall \lambda \in \mathbb{N};$$

$$|\hat{a}_I(\xi_1,\xi_2,\xi_3)| \le C_L \frac{|I|^{\frac{1}{2}}}{[(1+|I|_1|\xi_1|)(1+|I|_2|\xi_2|)(1+|I|_3|\xi_3-f_I|)]^L} \quad \forall L>0;$$

and

$$|(\partial^\gamma a_I)(x)| \le C_\gamma |I|^{-\frac{1}{2}}|I|_1^{-\gamma_1}|I|_2^{-\gamma_2}|I|_3^{-\gamma_3}(1+|t(I)|)^{\gamma_3},$$

where $\gamma=(\gamma_1,\gamma_2,\gamma_3)\in\mathbb{N}^3$.

As with the molecules for $\dot{F}^{\alpha,q}_p(\mathbb{D})$, there is an extra decay for the atoms for $\dot{F}^{\alpha,q}_p(\mathbb{D})$ in the third variable of the Fourier domain. With this definition, we can establish the atomic decomposition for $\dot{F}^{\alpha,q}_p(\mathbb{D})$.

Theorem 5.4. *Let $\alpha\in\mathbb{R}$, $0<p,q<\infty$. There exists a family of smooth atoms $\{a_I\}_{I\in\mathcal{I}}$ such that, for each $f\in\dot{F}^{\alpha,q}_p(\mathbb{D})$, we have the atomic decomposition $f=\sum_{I\in\mathcal{I}} s_I a_I$, where $s=\{s_I\}\in\dot{f}^{\alpha,q}_p(\mathbb{D})$ and $\|s\|_{\dot{f}^{\alpha,q}_p(\mathbb{D})}\le C\|f\|_{\dot{F}^{\alpha,q}_p(\mathbb{D})}$ for some constant $C>0$ independent of f.*

Proof. It is obvious that there exists a function $\varphi\in\mathcal{S}_v(\mathbb{R}^3)$ satisfying (30) and $\operatorname{supp}\varphi\subset[\frac{1}{2},2]^3$. According to Theorem 5.3, there exist an $\eta\in\tilde{\mathbb{Z}}^2$ and a family of molecules $\{\psi^{l,\eta}_{\nu,k}\}\in\cap_{\beta>0}\mathcal{M}_\beta$ such that $f=\sum_{\nu\in\mathbb{Z}^2,\, l\in\mathbb{Z},k\in\mathbb{Z}^3}\langle f,\psi^{l,\eta}_{\nu,k}\rangle\varphi^{l,\eta}_{\nu,k}$. Let $a_I=\varphi^{l,\eta}_{\nu,k}$ and $s_I=\langle f,\psi^{l,\eta}_{\nu,k}\rangle$ when $I=(Q_{\nu,k},l)$. The desired decomposition then resorts and $\{a_I\}$ satisfies the requirement for being a family of smooth atoms. Furthermore, $\|\{s_I\}\|_{\dot{f}^{\alpha,q}_p(\mathbb{D})}=\|\{\langle f,\psi^{l,\eta}_{\nu,k}\rangle\}\|_{\dot{f}^{\alpha,q}_p(\mathbb{D})}\le C\|f\|_{\dot{F}^{\alpha,q}_p(\mathbb{D})}$, because the family $\{\psi^{l,\eta}_{\nu,k}\}$ satisfies the condition in Theorem 5.2. □

Recall that the family of functions $\{\varphi_\gamma\}_{\gamma\in\Gamma}$ is a frame for $L^2(\mathbb{R}^3)$ if and only if there exist constants $A>B>0$ such that

$$B\|f\|^2_{L^2(\mathbb{R}^3)} \le \sum_{\gamma\in\Gamma}|\langle f,\varphi_\gamma\rangle|^2 \le A\|f\|^2_{L^2(\mathbb{R}^3)}.$$

The frame used should reflect the translation on the Fourier domain. Therefore, we study the following family of functions:

$$2^{\nu_1+\nu_2}e^{i(2^{\nu_1+\nu_2}x_3-k_3)l}\varphi\big(2^{\nu_1}x_1-k_1, 2^{\nu_2}x_2-k_2, 2^{\nu_1+\nu_2}x_3-k_3\big),$$

where $l\in\mathbb{Z}$, $\nu=(\nu_1,\nu_2)\in\mathbb{Z}^2$ and $k=(k_1,k_2,k_3)\in\mathbb{Z}^3$. If the above family is a frame, call it a *Wavelet–Gabor type frame* for $L^2(\mathbb{R}^3)$. Theorem 5.3 provides a condition for a φ that generates a Wavelet–Gabor type frame for $L^2(\mathbb{R}^3)$.

Theorem 5.5. *If $\varphi \in \mathcal{S}_v(\mathbb{R}^3)$ satisfies*

$$0 < B < \sum_{\nu\in\mathbb{Z}^2,\, l\in\mathbb{Z}} \left|\hat{\varphi}(2^{\nu_1}\xi_1, 2^{\nu_2}\xi_2, 2^{s(\nu)}\xi_3 - l)\right|^2 < A \quad \text{if } \xi_1\xi_2 \neq 0, \tag{39}$$

for some constants $A > B > 0$, then there exists an $\eta_0 \in \mathbb{Z}$ such that for any $\eta = (\kappa, \kappa)$, $\kappa \le \eta_0$, the family

$$2^{\nu_1+\nu_2} e^{i2^{2\eta}(2^{\nu_1+\nu_2}x_3-k_3)l}\varphi\big(2^{\eta}(2^{\nu_1}x_1 - k_1), 2^{\eta}(2^{\nu_2}x_2 - k_2), 2^{2\eta}(2^{\nu_1+\nu_2}x_3 - k_3)\big),$$

where $l \in \mathbb{Z}$, $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$ and $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$, is a frame for $L^2(\mathbb{R}^3)$.

Theorem 5.6. *Let $\{\varphi_{[l]}\}_{l\in\mathbb{Z}} \subset \mathcal{S}_v(\mathbb{R}^3)$ satisfy (30). For any $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, there exist an $\eta_0 \in \mathbb{Z}$ and constants $C_1, C_2 > 0$ such that for any $\eta = (\kappa, \kappa)$, $\kappa \le \eta_0$ and $f \in \dot{F}_p^{\alpha,q}(\mathbb{D})$,*

$$C_1\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{D})} \le \left\|\left\{\langle f, \varphi_{\nu,k}^{l,\eta}\rangle\right\}\right\|_{\dot{f}_p^{\alpha,q}(\mathbb{D})} \le C_2\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{D})}.$$

Furthermore, for any $\beta > 0$ there exists a family of molecules of order β, $\{\psi_{\nu,k}^{l,\eta}\}$, such that

$$f = \sum_{l,\nu,k} \langle f, \psi_{\nu,k}^{l,\eta}\rangle \varphi_{\nu,k}^{l,\eta} = \sum_{l,\nu,k} \langle f, \varphi_{\nu,k}^{l,\eta}\rangle \psi_{\nu,k}^{l,\eta}, \quad \forall f \in \dot{F}_p^{\alpha,q}(\mathbb{D}).$$

The proofs of Theorem 5.5 and Theorem 5.6 are straightforward and, thus, omitted here.

6. The main result for the differential operator $\triangle_{\mathbb{D}}$

In this section, we prove the analogy of (5) for the differential operator $\triangle_{\mathbb{D}}$.

Theorem 6.1. *Let $0 < p, q < \infty$, $\alpha \in \mathbb{R}$ and $m \in \mathbb{Z}$. There exist constants $C_1 > C_2 > 0$ such that for any $f \in \dot{F}_p^{\alpha,q}(\mathbb{D})$,*

$$C_2\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{D})} \le \|\triangle_{\mathbb{D}}^m f\|_{\dot{F}_p^{\alpha-2m,q}(\mathbb{D})} \le C_1\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{D})}. \tag{40}$$

In particular, $\triangle_{\mathbb{D}} : \dot{F}_p^{\alpha,q}(\mathbb{D}) \to \dot{F}_p^{\alpha-2,q}(\mathbb{D})$ has both trivial kernel and closed range. Moreover, the operator $\triangle_{\mathbb{D}}$ is a linear topological isomorphism.

Proof. For brevity, we just prove inequalities (40) for $m = 1$ as the other cases follow similarly and the conclusion follows easily from this special case.

Let $\varphi \in \mathcal{S}_v(\mathbb{R}^3)$ satisfy $\mathrm{supp}\hat{\varphi} \subset [\frac{1}{2}, 2]^2 \times [-\frac{2}{3}, \frac{2}{3}]$ and (39). For brevity, we assume that the η_0 associated with φ in Theorem 5.6 equals to zero. For any $f \in \dot{F}_p^{\alpha,q}(\mathbb{D})$, we have

$$\langle \triangle_{\mathbb{D}} f, \varphi_{\nu,k}^{l}\rangle = \langle f, \triangle_{\mathbb{D}}\varphi_{\nu,k}^{l}\rangle \tag{41}$$

by the duality of Schwartz function and Schwartz distribution. We find that

$$\triangle_{\mathbb{D}}\varphi^l_{\nu,k} = 2^{2s(\nu)}\left((\partial^2_{x_1}\partial^2_{x_2}\varphi)^l_{\nu,k} + l^2\varphi^l_{\nu,k} - 2il(\partial_{x_3}\varphi)^l_{\nu,k} - (\partial^2_{x_3}\varphi)^l_{\nu,k}\right).$$

Let

$$\Phi_{[l]} = \frac{1}{1+l^2}\left((\partial^2_{x_1}\partial^2_{x_2}\varphi) + l^2\varphi - 2il(\partial_{x_3}\varphi) - (\partial^2_{x_3}\varphi)\right),$$

therefore, $\Phi_{[l]} \in \mathcal{S}_v(\mathbb{R}^3)$ satisfies condition (31) and

$$2^{2s(\nu)}(1+l^2)(\Phi_{[l]})^l_{\nu,k} = \triangle_{\mathbb{D}}\varphi^l_{\nu,k}. \tag{42}$$

Moreover, we find that $\hat{\Phi}_{[l]}(\xi_1,\xi_2,\xi_3) = \frac{\xi_1^2\xi_2^2+(\xi_3+l)^2}{1+l^2}\hat{\varphi}(\xi_1,\xi_2,\xi_3)$, and hence

$$\hat{\Phi}_{[l]}(2^{\nu_1}\xi_1, 2^{\nu_2}\xi_2, 2^{s(\nu)}\xi_3 - l) = \frac{\xi_1^2\xi_2^2+\xi_3^2}{2^{-2s(\nu)}(1+l^2)}\hat{\varphi}(2^{\nu_1}\xi_1, 2^{\nu_2}\xi_2, 2^{s(\nu)}\xi_3 - l).$$

We assert that on the support of $\hat{\varphi}(2^{\nu_1}\xi_1, 2^{\nu_2}\xi_2, 2^{s(\nu)}\xi_3 - l)$, there exist constants, $C_1 > C_2 > 0$ that are independent of ν and l, such that $C_2 < \left|\frac{\xi_1^2\xi_2^2+\xi_3^2}{2^{-2s(\nu)}(1+l^2)}\right| < C_1$. Thus, $\Phi_{[l]}$ also satisfies (30). Without loss of generality, we assume that the η_0 associated with the family $\{\Phi_{[l]}\}$ is equal to zero. By (41) and applying Theorem 5.6 to the family $\{\varphi^l_{\nu,k}\}$ we have a constant $C > 0$ such that

$$C\left\|\left\{\langle f, \triangle_{\mathbb{D}}\varphi^l_{\nu,k}\rangle\right\}\right\|_{\dot{f}^{\alpha-2,q}_p(\mathbb{D})} = C\left\|\left\{\langle \triangle_{\mathbb{D}}f, \varphi^l_{\nu,k}\rangle\right\}\right\|_{\dot{f}^{\alpha-2,q}_p(\mathbb{D})} \le \|\triangle_{\mathbb{D}}f\|_{\dot{F}^{\alpha-2,q}_p(\mathbb{D})}.$$

By (42) and applying Theorem 5.6 to the family $\{(\Phi_{[l]})^l_{\nu,k}\}$ there exists $C > 0$ such that

$$C\|f\|_{\dot{F}^{\alpha,q}_p(\mathbb{D})} \le C\left\|\left\{\langle f, (\Phi_{[l]})^l_{\nu,k}\rangle\right\}\right\|_{\dot{f}^{\alpha,q}_p(\mathbb{D})} \le \|\triangle_{\mathbb{D}}f\|_{\dot{F}^{\alpha-2,q}_p(\mathbb{D})}.$$

The proof of the second inequality in (40) follows similarly. □

7. Technical results

The first lemma of this section asserts that the "inner product" of two families of molecules generates an almost diagonal operator. It comes from iterating the result from [3, Lemma B.1] but with some major modifications.

Lemma 7.1. *Let $2N > 2M+3$, $l \in \mathbb{Z}$, $\nu = (\nu_1,\nu_2) \in \mathbb{Z}^2$ and $k = (k_1,k_2,k_3) \in \mathbb{Z}^3$. Suppose that $g(x)$ satisfies*

$$|(\partial^\gamma g)(x_1,x_2,x_3)| \le \frac{C2^{\gamma_1\nu_1+\gamma_2\nu_2+s(\nu)}}{[(1+|2^{\nu_1}x_1-k_1|)(1+|2^{\nu_2}x_2-k_2|)(1+|2^{s(\nu)}x_3-k_3|)]^{2N}}, \tag{43}$$

for $\gamma = (\gamma_1, \gamma_2, 0) \in \mathbb{N}^3$ *and* $\gamma_i \le [2M] + 2$, $i = 1, 2$*;*

$$\int_{\mathbb{R}} g(x_1, x_2, x_3) x_1^{\lambda} dx_1 = 0, \quad \int_{\mathbb{R}} g(x_1, x_2, x_3) x_2^{\lambda} dx_2 = 0, \tag{44}$$

for any $\lambda \in \mathbb{N}$*, and* $\lambda \le [2M] + 1$ *and*

$$|\hat{g}(\xi_1, \xi_2, \xi_3)| \le C 2^{-s(\nu)} \left((1+|2^{-\nu_1}\xi_1|)(1+|2^{-\nu_2}\xi_2|)(1+|2^{-s(\nu)}\xi_3 - l|)\right)^{-2N} \tag{45}$$

for some constant $C > 0$*. Suppose that* $h(x)$ *satisfies* (43)–(45) *with* l, ν *and* k *replaced by* $m \in \mathbb{Z}$, $\mu = (\mu_1, \mu_2) \in \mathbb{Z}^2$ *and* $h = (h_1, h_2, h_3) \in \mathbb{Z}^3$*, respectively. Then, there is a constant* $\tilde{C} > 0$ *independent of* l, m, ν, μ, k, h *and* $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ *such that*

$$\begin{aligned} &|(g * h)(x_1, x_2, x_3)| \\ &\le \tilde{C} \prod_{j=1}^{2} \min\left(\frac{2^{\nu_j}}{2^{\mu_j}}, \frac{2^{\mu_j}}{2^{\nu_j}}\right)^M \left(1 + \frac{|2^{s(\nu)} l - 2^{s(\mu)} m|}{\max(2^{s(\nu)}, 2^{s(\mu)})}\right)^{-N} \\ &\quad \times \left[\prod_{j=1}^{2}\left(1 + \frac{|2^{-\nu_j} k_j - 2^{-\mu_j} h_j - x_j|}{\max(2^{-\nu_j}, 2^{-\mu_j})}\right)\left(1 + \frac{|2^{-s(\nu)} k_3 - 2^{-s(\mu)} h_3 - x_3|}{\max(2^{-s(\nu)}, 2^{-s(\mu)})}\right)\right]^{-N}. \end{aligned} \tag{46}$$

Proof. Without loss of generality, assume that $m = 0$, $\mu = (\mu_1, \mu_2) = (0, 0)$ and $h = (h_1, h_2, h_3) = (0, 0, 0)$.

If $\nu_1 \ge 0$, $\nu_2 \le 0$ and $s(\nu) = \nu_1 + \nu_2 < 0$, then

$$\begin{aligned} |(g * h)(x_1, x_2, x_3)| = \frac{1}{2^{\nu_2}} \bigg| \int_{\mathbb{R}^3} & g(x_1 - y_1 + 2^{-\nu_1} k_1, 2^{-\nu_2} y_2, y_3) \\ & \times h(y_1 - 2^{-\nu_1} k_1, x_2 - 2^{-\nu_2} y_2, x_3 - y_3)\, dy_1 dy_2 dy_3 \bigg|. \end{aligned}$$

For any fixed but arbitrary y_1, x_1 and x_3, let $\breve{g}(y_2) = g(x_1 - y_1 + 2^{-\nu_1} k_1, 2^{-\nu_2} y_2, y_3)$ be a function of y_2. Similarly, for any fixed but arbitrary y_2, x_2 and x_3, define $\breve{h}(y_1) = h(y_1 - 2^{-\nu_1} k_1, x_2 - 2^{-\nu_2} y_2, x_3 - y_3)$.

Let $\breve{g}^{(r)}$ and $\breve{h}^{(s)}$, $r, s \in \mathbb{N}$, denote the ordinary derivatives of the single variable functions $\breve{g}$ and $\breve{h}$, respectively. We assert that

$$\begin{aligned} |(g * h)(x_1, x_2, x_3)| = 2^{-\nu_2} \bigg| \int_{\mathbb{R}^3} & \left[\breve{g}(y_2) - \sum_{0 \le r \le [2M]+1} \frac{\breve{g}^{(r)}(2^{\nu_2} x_2)}{r!} (y_2 - 2^{\nu_2} x_2)^r\right] \\ & \times \left[\breve{h}(y_1) - \sum_{0 \le s \le [2M]+1} \frac{\breve{h}^{(s)}(x_1)}{s!} (y_1 - x_1)^s\right] dy_1 dy_2 dy_3 \bigg|. \end{aligned}$$

The above identity is valid because of the vanishing moment conditions (44).

Let $\delta_1 = 1$ and $\delta_2 = 2^{\nu_2}$. Decompose $\mathbb{R}$ into three regions:

$$\begin{aligned} D_{j,1} &= \{y_j \in \mathbb{R} : |y_j - \delta_j x_j| \le 3\} \\ D_{j,2} &= \{y_1 \in \mathbb{R} : |y_j - \delta_j x_j| > 3 \text{ and } |y_j| \le \tfrac{1}{2}|\delta_j x_j|\} \\ D_{j,3} &= \{y_j \in \mathbb{R} : |y_j - \delta_j x_j| > 3 \text{ and } |y_j| > \tfrac{1}{2}|\delta_j x_j|\}. \end{aligned}$$

We have $|(g * h)(x_1, x_2, x_3)| \le \sum_{u,v=1,2,3} I_{u,v}$, where

$$\begin{aligned} I_{u,v} = 2^{-\nu_2}\Bigg| \int_{D_{1,u}\times D_{2,v}\times\mathbb{R}} &\left[\breve{g}(y_2) - \sum_{0\le r\le[2M]+1} \frac{\breve{g}^{(r)}(2^{\nu_2}x_2)}{r!}(y_2 - 2^{\nu_2}x_2)^r\right] \\ &\times \left[\breve{h}(y_1) - \sum_{0\le s\le[2M]+1} \frac{\breve{h}^{(s)}(x_1)}{s!}(y_1 - x_1)^s\right] dy_1 dy_2 dy_3\Bigg|. \end{aligned}$$

When $y_1 \in D_{1,1}$ and $y_2 \in D_{2,1}$, (43) yields

$$\begin{aligned} &\left|\breve{g}(y_2) - \sum_{0\le r\le[2M]+1} \frac{\breve{g}^{(r)}(2^{\nu_2}x_2)(y_2 - 2^{\nu_2}x_2)^r}{r!}\right| \\ &\le C2^{s(\nu)} \frac{|2^{\nu_2}x_2 - y_2|^{[2M]+2}}{[(1 + |2^{\nu_1}(x_1 - y_1)|)(1 + |2^{\nu_2}x_2 - k_2|)(1 + |2^{s(\nu)}y_3 - k_3|)]^{2N}}. \end{aligned}$$

Similarly,

$$\begin{aligned} &\left|\breve{h}(y_1) - \sum_{0\le s\le[2M]+1} \frac{\breve{h}^{(s)}(x_1)}{s!}(y_1 - x_1)^s\right| \\ &\le C \frac{|x_1 - y_1|^{[2M]+2}}{[(1 + |x_1 - 2^{-\nu_1}k_1|)(1 + |x_2 - 2^{-\nu_2}y_2|)(1 + |x_3 - y_3|)]^{2N}}. \end{aligned}$$

Therefore, $I_{1,1}$ can be estimated as

$$I_{1,1} \le C\left(\frac{2^{\nu_2}}{2^{\nu_1}}\right)^{2M+\frac{1}{2}} \left(\frac{1}{(1 + |x_1 - 2^{-\nu_1}k_1|)(1 + |2^{\nu_2}x_2 - k_2|)\left(1 + \frac{|x_3 - k_3|}{\max(2^{-s(\nu)},1)}\right)}\right)^{2N}.$$

The estimate for x_3 comes from [3, Lemma B.2].

Notice that $I_{1,1}$ was estimated by iterating the estimate for $\int_A$ in [3, Lemma B.1]. Similarly, if (y_1, y_2) belongs to the other domains $D_{1,u} \times D_{2,v}$, use the corresponding results for $\int_A$, $\int_B$ and $\int_C$ in [3, Lemma B.1] ($\int_A$, $\int_B$ and $\int_C$ are notations in [3, Lemma B.1]). Thus, when $\nu_1 \ge 0$ and $\nu_2 \le 0$, we obtain our desired result.

Similarly, when $\nu_1 \le 0$ and $\nu_2 \ge 0$, we have

$$|(g*h)(x_1,x_2,x_3)| \le C\left(\frac{2^{\nu_1}}{2^{\nu_2}}\right)^{2M+\frac{1}{2}}\left(\frac{1}{(1+|2^{\nu_1}x_1-k_1|)(1+|x_2-2^{-\nu_2}k_2|)\left(1+\frac{|x_3-k_3|}{\max(2^{-s(\nu)},1)}\right)}\right)^{2N}.$$

For $\nu_1 \le 0$ and $\nu_2 \le 0$, by the vanishing moment conditions,

$$|(g*h)(x_1,x_2,x_3)| = 2^{-\nu_1-\nu_2}\bigg|\int_{\mathbb{R}^2} g(2^{-\nu_1}y_1, 2^{-\nu_2}y_2, y_3) \times h(x_1-2^{-\nu_1}y_1, x_2-2^{-\nu_2}y_2, x_3-y_3)dy_1dy_2dy_3\bigg|.$$

In this case, let $\breve{g}(y_1,y_2,y_3) = 2^{-\nu_1-\nu_2}g(2^{-\nu_1}y_1, 2^{-\nu_2}y_2, y_3)$, we have

$$|(g*h)(x_1,x_2,x_3)| = \bigg|\int_{\mathbb{R}^2}\bigg[\breve{g}(y_1,y_2,y_3) - \sum_{0\le r\le[2M]+1}\frac{(\partial_{x_1}^r\breve{g})(2^{\nu_1}x_1,y_2,y_3)}{r!}(y_1-2^{\nu_1}x_1)^r\bigg] \times h(x_1-2^{-\nu_1}y_1, x_2-2^{-\nu_2}y_2, x_3-y_3)dy_1dy_2dy_3\bigg|.$$

For any fixed but arbitrary x_1, let

$$R(y_1,y_2,y_3) = \frac{1}{([2M]+1)!}\int_{2^{\nu_1}x_1}^{y_1}(y_1-t)^{[2M]+1}\left(\partial_{x_1}^{[2M]+2}\breve{g}\right)(t,y_2,y_3)\,dt.$$

The right-hand side of the above identity is the remainder term of Taylor's expansion of $\breve{g}(y_1,y_2,y_3)$ on the first variable y_1 in integral form. Use the integral form of the remainder term to define $R(y_1,y_2,y_3)$ instead of using the differential form:

$$\tilde{R}(y_1,y_2,y_3) = \frac{\left(\partial_{x_1}^{[2M]+2}\breve{g}\right)(w,y_2,y_3)}{([2M]+2)!}(y_1-2^{\nu_1}x_1)^{[2M]+2}$$

for some $|w-y_1| \le |2^{\nu_1}x_1-y_1|$ because, in general, w depends on y_2. The existence of $\partial_{x_2}^s\tilde{R}$ relies on the differentiability of w as a function of y_2. Since w is not necessarily a smooth function of y_2, the Taylor expansion of $\tilde{R}(y_1,y_2,y_3)$ cannot be used to establish the following identity (47). We have

$$\begin{aligned}|(g*h)(x_1,x_2,x_3)| &= \bigg|\int_{\mathbb{R}^2}R(y_1,y_2,y_3)h(x_1-2^{-\nu_1}y_1, x_2-2^{-\nu_2}y_2, y_3)\,dy_1dy_2dy_3\bigg| \\ &= \bigg|\int_{\mathbb{R}^2}\bigg[R(y_1,y_2,y_3) - \sum_{0\le s\le[2M]+1}\frac{(\partial_{x_2}^sR)(y_1,2^{\nu_2}x_2,y_3)}{s!}(y_2-2^{\nu_2}x_2)^s\bigg] \\ &\quad \times h(x_1-2^{-\nu_1}y_1, x_2-2^{-\nu_2}y_2, x_3-y_3)\,dy_1dy_2dy_3\bigg| \end{aligned} \tag{47}$$

because of Fubini's Theorem and the vanishing moment conditions. Applying the differential form of the remainder term for Taylor's expansion and the mean-value theorem for integrals shows that there exists a constant $C > 0$ such that

$$\begin{aligned} &\left| R(y_1, y_2, y_3) - \sum_{0\le s\le [2M]+1} \frac{(\partial_{x_2}^s R)(y_1, 2^{\nu_2}x_2, y_3)}{s!}(y_2 - 2^{\nu_2}x_2)^s \right| \\ &\le C\frac{|y_1 - 2^{\nu_1}x_1|^{[2M]+2}|y_2 - 2^{\nu_2}x_2|^{[2M]+2}}{[(1+|2^{\nu_1}x_1 - k_1|)(1+|2^{s(\nu)}x_3 - k_3|)(1+|2^{\nu_2}x_2 - k_2|)]^{2N}}. \end{aligned} \tag{48}$$

By using (47), (48) and the argument from [3, Lemma B.1],

$$\begin{aligned} &|(g*h)(x_1, x_2, x_3)| \\ &\le C\, 2^{(\nu_1+\nu_2)(2M+\frac{1}{2})}\left((1+|2^{\nu_1}x_1-k_1|)(1+|2^{\nu_2}x_2-k_2|)\left(1+\frac{|x_3-k_3|}{\max(2^{-s(\nu)},1)}\right)\right)^{-2N}. \end{aligned}$$

The estimate for the case $\nu_1 \ge 0$ and $\nu_2 \ge 0$ follows similarly. As a conclusion, there is a constant $C > 0$ independent of ν, μ, k, h and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that

$$\begin{aligned} &|(g*h)(x_1, x_2, x_3)| \\ &\le C\left(\prod_{j=1}^{2} \min\left(\frac{2^{\nu_j}}{2^{\mu_j}}, \frac{2^{\mu_j}}{2^{\nu_j}}\right)\right)^{2M+\frac{1}{2}} \\ &\quad\times\left[\prod_{j=1}^{2}\left(1+\frac{|2^{-\nu_j}k_j - 2^{-\mu_j}h_j - x_j|}{\max(2^{-\nu_j}, 2^{-\mu_j})}\right)\left(1+\frac{|2^{-s(\nu)}k_3 - 2^{-s(\mu)}h_3 - x_3|}{\max(2^{-s(\nu)}, 2^{-s(\mu)})}\right)\right]^{-2N}. \end{aligned} \tag{49}$$

Finally, estimate $g*h$ by using (45) and [3, Lemma B.2] on the variable ξ. We assert that

$$|(g*h)(x)| \le C\int_{\mathbb{R}^3} |\hat{g}(\xi)\hat{h}(\xi)|d\xi \le C\left(1+\frac{|2^{s(\nu)}l - 2^{s(\mu)}m|}{\max(2^{s(\nu)}, 2^{s(\mu)})}\right)^{-2N}. \tag{50}$$

Multiplying inequality (49) with inequality (50) and taking the square root yields (46). □

A second result is that, roughly speaking, the family of molecules is "invariant" under the mapping of the almost diagonal operator. The precise statement is given in Theorem 7.3. Some notation and basic results will be introduced before presenting Theorem 7.3.

Let $\mathcal{Q} = \{2^{-j}[0,1] + k : j, k \in \mathbb{Z}\}$ be the set of dyadic intervals in $\mathbb{R}$ and $Q_{j,k} = 2^j[0,1] + k$. For each $Q = Q_{j,k} \in \mathcal{Q}$, let $x_Q = 2^{-j}k$ and $l(Q) = 2^{-j}$.

Let $M > 1$. This leads to the following inequalities which are extensions of [3, Lemma D.1]:

$$\sum_{l(P)=2^{-j}} \left(1+\frac{|x_P - x_Q|}{\max(l(P),l(Q))}\right)^{-M} \left(1+\frac{|x-x_P|}{\max(l(R),l(P))}\right)^{-M}$$
$$\leq C\left(1+\frac{|x-x_Q|}{\max(l(P),l(Q),l(R))}\right)^{-M} \max\left(1, \frac{\min(l(R),l(Q))}{l(P)}\right) \qquad (51)$$
$$\leq C\left(1+\frac{|x-x_Q|}{\max(l(Q),l(R))}\right)^{-M} \max\left(1, \frac{l(P)}{\max(l(Q),l(R))}\right)^{M} \times \max\left(1, \frac{\min(l(R),l(Q))}{l(P)}\right) \qquad (52)$$

for a constant $C > 0$ dependent on $M > 1$ only. The last inequality results because $1+\frac{|x-x_R|}{\max(l(Q),l(R))} \leq \frac{l(P)}{\max(l(Q),l(R))}(1+\frac{|x-x_R|}{l(P)})$, provided that $l(P) > \max(l(Q),l(R))$.

Lemma 7.2. *If $\tau > \gamma$ and $1+\tau > N$, then*

$$\sum_{P\in\mathcal{Q}} \left(1+\frac{|x_Q-x_P|}{\max(l(Q),l(P))}\right)^{-N} \left(\min\left(\left(\frac{l(Q)}{l(P)}\right),\left(\frac{l(P)}{l(Q)}\right)\right)\right)^{\tau+\frac{1}{2}}$$
$$\times\, l(P)^{-\gamma-\frac{1}{2}}\left(1+\frac{|x-x_P|}{l(P)}\right)^{-N}$$
$$\leq Cl(Q)^{-\gamma-\frac{1}{2}}\left(1+\frac{|x-x_Q|}{l(Q)}\right)^{-N}.$$

Proof. Let $l(Q) = 2^{-i}$ and $l(P) = 2^{-j}$. Then,

$$\sum_{l(P)\leq l(Q)} \left(1+\frac{|x_Q-x_P|}{l(Q)}\right)^{-N}\left(\frac{l(P)}{l(Q)}\right)^{\tau+\frac{1}{2}} l(P)^{-\gamma-\frac{1}{2}}\left(1+\frac{|x-x_P|}{l(P)}\right)^{-N}$$
$$\leq C|Q|^{-\frac{1}{2}-\gamma}\left(1+\frac{|x-x_Q|}{l(Q)}\right)^{-N}$$

by (51). Moreover,

$$\sum_{l(P)>l(Q)} \left(1+\frac{|x_Q-x_P|}{l(P)}\right)^{-N}\left(\frac{l(Q)}{l(P)}\right)^{\tau+\frac{1}{2}} l(P)^{-\gamma-\frac{1}{2}}\left(1+\frac{|x-x_P|}{l(P)}\right)^{-N}$$
$$\leq C2^{\frac{i}{2}+i\gamma}\sum_{j=-\infty}^{i-1} 2^{(j-i)(1+|\gamma|+\tau-N)}\left(\frac{l(Q)}{l(Q)+|x-x_Q|}\right)^{N}$$
$$= C2^{\frac{i}{2}+i\gamma}\left(\frac{1}{1+l(Q)^{-1}|x-x_Q|}\right)^{N}$$

by using $l(P) > l(Q)$ and $1+\tau > N$. $\square$

The technical result used in the proof of Theorem 5.3 can now be stated and proved. In order to simplify the notation, we write

$$\min_{I,J}^{\sigma} = \min\left(\frac{|I|_\sigma}{|J|_\sigma}, \frac{|J|_\sigma}{|I|_\sigma}\right), \quad \sigma = 1,2,3,$$

where $I, J \in \mathcal{I}$. The proof of the following theorem is inspired by the ideas in [7, Theorem 6.4].

Theorem 7.3. *Let $\tau > 1$, $L > \tau + 1$ and $\beta > \max(1+2\tau, L)$. Suppose that $\{m_J\} \in \mathcal{M}_\tau$ and $\{a_{IJ}\} \in \omega(\beta, L)$. Then, the family $\{n_I\}$, where $n_I = \sum_J a_{IJ} m_J$ belongs to $\mathcal{M}_\tau$. Moreover, there is a constant $C > 0$ such that*

$$\|\{n_I\}\|_{\mathcal{M}_\tau} \le C\|\{a_{IJ}\}\|_{\omega(\beta,L)}\|\{m_I\}\|_{\mathcal{M}_\tau}.$$

Proof. Without loss of generality, assume $\|\{m_I\}\|_{\mathcal{M}_\tau} = 1$ and $\|\{a_{IJ}\}\|_{\omega(\beta,L)} = 1$. For any $\gamma = (\gamma_1,\gamma_2,\gamma_3) \in \mathbb{N}^3$ and $|\gamma_i| \le \tau$, $i = 1,2,3$,

$$|\partial^\gamma n_I| \le \sum_{|J|_3 \le |I|_3} a_{IJ}|\partial^\gamma m_J| + \sum_{|J|_3 > |I|_3} a_{IJ}|\partial^\gamma m_J| = X + Y.$$

The reasoning for estimating X and Y are similar, so, for brevity, only Y will be treated in detail. According to Definition 4.1, inequality (29) and Definition 5.1, for any $\beta > \max(1+2\tau, L)$ and $L > \tau + 1$,

$$\begin{aligned} Y \le & \sum_{|J|_3 \le |I|_3} \left[\prod_{\sigma=1}^{3} \min_{I,J}^{\sigma}\right]^\beta \left(1 + \frac{|f_I - f_J|}{\max(|I|_3^{-1}, |J|_3^{-1})}\right)^{-L} \\ & \times \left[\prod_{\sigma=1}^{3}\left(1 + \frac{|c_{I,\sigma} - c_{J,\sigma}|}{\max(|I|_\sigma, |J|_\sigma)}\right)\right]^{-L} \frac{|J|_1^{-\gamma_1-\gamma_3-1}|J|_2^{-\gamma_2-\gamma_3-1}(1+|t(J)|)^{\gamma_3}}{\left[\prod_{\sigma=1}^{3}(1+|J|_\sigma^{-1}|x_\sigma - c_{J,\sigma}|)\right]^\tau} \\ = & \sum_{j\in\mathbb{Z}} \sum_{|J|_1 = 2^{-j}} \left(1 + \frac{|c_{I,1} - c_{J,1}|}{\max(|I|_1, |J|_1)}\right)^{-L} (\min_{I,J}^1)^\beta \frac{|J|_1^{-\gamma_1-\gamma_3-1}}{(1+|J|_1^{-1}|x_1 - c_{J,1}|)^\tau} Y_2, \end{aligned}$$

because $|J|_2^{-1} = |J|_1|J|_3^{-1}$, where

$$Y_2 = \sum_{j\in\mathbb{Z}} \sum_{|J|_2 = 2^{-j}} \left(1 + \frac{|c_{I,2} - c_{J,2}|}{\max(|I|_2, |J|_2)}\right)^{-L} (\min_{I,J}^2)^\beta \frac{|J|_2^{-\gamma_2-\gamma_3-1}}{(1+|J|_2^{-1}|x_2 - c_{J,2}|)^\tau} Y_3,$$

and

$$Y_3 = \sum_{|J|_3 = 2^{-j}} \left(\frac{1}{1+|I|_3^{-1}|c_{I,3} - c_{J,3}|}\right)^L \left(\frac{1}{1+|J|_3^{-1}|x_3 - c_{J,3}|}\right)^\tau Y_4.$$

Recall also that $f_J = |J|_3^{-1} t(J)$, by (25) and (26), we assert that

$$Y_4 = \sum_{f_J} \left(\min{}^3_{I,J}\right)^{\beta} \frac{(1+|J|_3|f_J|)^{\gamma_3}}{(1+|J|_3|f_I - f_J|)^L} \le C(1+|t(I)|)^{\gamma_3}.$$

Estimate Y_3 using inequality (51), and then estimate Y and Y_2 with Lemma 7.2 (because $\beta > 2\tau + 1$ and $\beta > L$), we find that

$$Y \le C|I|_1^{-1-\gamma_1-\gamma_3}|I|_2^{-1-\gamma_2-\gamma_3}(1+|t(I)|)^{\gamma_3}\left[\prod_{\sigma=1}^{3}\left(1+\frac{|x-c_{I,\sigma}|}{|I|_\sigma}\right)\right]^{-\tau}.$$

For the estimate for X, we also have

$$X \le \sum_{j\in\mathbb{Z}} \sum_{|J|_1=2^{-j}} \left(1+\frac{|c_{I,1}-c_{J,1}|}{\max(|I|_1,|J|_1)}\right)^{-L} \left(\min{}^1_{I,J}\right)^{\beta-\tau} \frac{|J|_1^{-\gamma_1-\gamma_3-1}}{(1+|J|_1^{-1}|x_1-c_{J,1}|)^{\tau}} X_2.$$

X_2 is defined by

$$X_2 = \sum_{j\in\mathbb{Z}} \sum_{|J|_2=2^{-j}} \left(1+\frac{|c_{I,2}-c_{J,2}|}{\max(|I|_2,|J|_2)}\right)^{-L} \left(\min{}^2_{I,J}\right)^{\beta-\tau} \frac{|J|_2^{-\gamma_2-\gamma_3-1}}{(1+|J|_2^{-1}|x_2-c_{J,2}|)^{\tau}} X_3.$$

The estimate for X_2 is thus the same as for Y_2. For X_3,

$$\begin{aligned} X_3 &= \left(\min{}^3_{I,J}\right)^{\tau} \sum_{|J|_3=2^{-j}} \left(\frac{1}{1+|J|_3^{-1}|c_{I,3}-c_{J,3}|}\right)^{L} \left(\frac{1}{1+|J|_3^{-1}|x_3-c_{J,3}|}\right)^{\tau} X_4 \\ &\le C\left(\min{}^3_{I,J}\right)^{\tau} \left(\frac{|J|_3}{|J|_3+|c_{I,3}-x|}\right)^{\tau} X_4 \\ &\le C\left(\frac{|I|_3}{|I|_3+|c_{I,3}-x|}\right)^{\tau} X_4. \end{aligned}$$

Finally, using (24), we find that

$$X_4 = \sum_{f_J} \left(\min{}^3_{I,J}\right)^{\beta} \frac{(1+|J|_3|f_J|)^{\gamma_3}}{(1+|I|_3|f_I - f_J|)^L} \le C(1+|t(I)|)^{\gamma_3}.$$

Since $|I|_3 = |I|_1|I|_2$, we have

$$|\partial^\gamma n_I| \le C\prod_{j=1}^{3}|I|_j^{-\frac{1}{2}-\gamma_j}(1+|t(I)|)^{\gamma_3}\left[\prod_{\sigma=1}^{3}\left(1+\frac{|x-c_{I,\sigma}|}{|I|_\sigma}\right)\right]^{-\tau}.$$

To check (28), observe that $|\hat{n}_I(\xi)| \le \sum_{J\in\mathcal{I}} a_{IJ}|\hat{m}_J(\xi)|$. Demonstrating the result follows the same reasoning as the estimate of n_I. Therefore, for the sake of brevity, the estimate for $|\hat{n}_I(\xi)|$ is left to the reader. Finally, the vanishing moment conditions for the family $\{n_I\}_{I\in\mathcal{I}}$ are inherited from the corresponding conditions for $\{m_J\}_{J\in\mathcal{I}}$. □

By some modifications of the method used in [7, Theorem 6.2], we show that the composition of the almost diagonal matrices $\mathcal{A} \in \omega(\beta, M)$ and $\mathcal{B} \in \omega(\tilde{\beta}, \tilde{M})$ is an almost diagonal matrix belonging to $\omega(\min(\beta, \tilde{\beta}), \min(M, \tilde{M}))$.

Theorem 7.4. *Let $\beta, \tilde{\beta}, M, \tilde{M} > 0$ satisfy $\beta \neq \tilde{\beta}$ and $\beta + \tilde{\beta} > \min(M, \tilde{M})$. Suppose that $\mathcal{A} = \{a_{IJ}\} \in \omega(\beta, M)$ and $\mathcal{B} = \{b_{JK}\} \in \omega(\tilde{\beta}, \tilde{M})$ are almost diagonal matrices. Then, the matrix $\mathcal{A} \circ \mathcal{B} = \mathcal{C} = \{c_{IK}\}$, where*

$$c_{IK} = \sum_J a_{IJ} b_{JK} \tag{53}$$

is an almost diagonal matrix and $\mathcal{C} = \{c_{IK}\} \in \omega(\min(\beta, \tilde{\beta}), \min(M, \tilde{M}))$. Moreover, there is a constant $C > 0$, depending continuously on $M, \tilde{M}$ only, such that $\|\mathcal{C}\|_{\omega(\min(\beta,\tilde{\beta}),\min(M,\tilde{M}))} \leq C\|\mathcal{A}\|_{\omega(\beta,M)}\|\mathcal{B}\|_{\omega(\tilde{\beta},\tilde{M})}$.

Proof. We only provide the estimate for the case where $|I|_1 < |K|_1$, $|I|_2 > |K|_2$ and $|I|_3 < |K|_3$ since the estimates for the other cases follow similarly. Without loss of generality, assume that $\beta > \tilde{\beta}$, $M \geq \tilde{M}$, $\|\{a_{IJ}\}\|_{\omega(\beta,M)} = 1$ and $\|\{b_{JK}\}\|_{\omega(\tilde{\beta},\tilde{M})} = 1$. Let $|I| = 2^{-2s(\mu)}$, $|J| = 2^{-2s(\nu)}$, $|K| = 2^{-2s(\kappa)}$,

$$W = \left(1 + \frac{|f_I - f_J|}{\max(|I|_3^{-1}, |J|_3^{-1})}\right)^{-\tilde{M}} \left[\prod_{\sigma=1}^{3}\left(1 + \frac{|c_{I,\sigma} - c_{J,\sigma}|}{\max(|I|_\sigma, |J|_\sigma)}\right)\right]^{-\tilde{M}}.$$

Decompose the summation in (53) into twenty seven summations,

$$|c_{IK}| = \sum_{\substack{k=(k_1,k_2,k_3)\in\mathbb{N}^3\\ 1\leq k_1,k_2,k_3\leq 3}} \sum_{J\in A_k} a_{IJ} b_{JK} = \sum_{\substack{k=(k_1,k_2,k_3)\in\mathbb{N}^3\\ 1\leq k_1,k_2,k_3\leq 3}} Z_k,$$

where $A_k = A_{k_1,k_2,k_3} = U_{k_1,1} \cap U_{k_2,2} \cap U_{k_3,3}$, $U_{1,m} = \{J : |K|_m \leq |J|_m\}$, $U_{2,m} = \{J : |I|_m \leq |J|_m \leq |K|_m\}$ and $U_{3,m} = \{J : |J|_m < |I|_m\}$, for $m = 1, 3$; and $U_{1,2} = \{J : |I|_2 \leq |J|_2\}$, $U_{2,2} = \{J : |K|_2 \leq |J|_2 \leq |I|_2\}$ and $U_{3,2} = \{J : |J|_2 < |K|_2\}$.

Set $Z_k = 0$ when $A_k = \emptyset$. We illustrate the estimates by considering the terms $Z_{1,1,1}$ and $Z_{3,2,3}$. For $Z_{1,1,1}$, by inequality (52) and the Cauchy-Schwartz inequality,

$$\begin{aligned}
|Z_{1,1,1}| &\leq \sum_{J\in A_{1,1,1}} \left(\frac{|I|_1}{|J|_1}\right)^{\beta}\left(\frac{|I|_2}{|J|_2}\right)^{\beta-\tilde{M}}\left(\frac{|I|_3}{|J|_3}\right)^{\beta}\left(\frac{|K|_1}{|J|_1}\right)^{\tilde{\beta}-\tilde{M}}\left(\frac{|K|_2}{|J|_2}\right)^{\tilde{\beta}}\left(\frac{|K|_3}{|J|_3}\right)^{\tilde{\beta}-\tilde{M}-1} W \\
&\leq \left(\sum_{J\in A_{1,1,1}} \left(\frac{|I|_1}{|J|_1}\right)^{2\beta}\left(\frac{|K|_1}{|J|_1}\right)^{2\tilde{\beta}-2\tilde{M}}\left(\frac{|I|_2}{|J|_2}\right)^{\beta-\tilde{M}}\left(\frac{|K|_2}{|J|_2}\right)^{\tilde{\beta}}\right)^{\frac{1}{2}} \\
&\quad\times \left(\sum_{J\in A_{1,1,1}} \left(\frac{|I|_2}{|J|_2}\right)^{\beta-\tilde{M}}\left(\frac{|K|_2}{|J|_2}\right)^{\tilde{\beta}}\left(\frac{|I|_3}{|J|_3}\right)^{2\beta}\left(\frac{|K|_3}{|J|_3}\right)^{2\tilde{\beta}-2\tilde{M}-2}\right)^{\frac{1}{2}} W \\
&= U_1 U_2 W.
\end{aligned}$$

For U_1, note that $\nu_1 \le \kappa_1 \le \mu_1$ and $\nu_2 \le \mu_2 \le \kappa_2$, hence,

$$\begin{aligned}
U_1 &= |I|_1^{\beta}|K|_1^{\tilde\beta-\tilde M}|I|_2^{\frac{\beta-\tilde M}{2}}|K|_2^{\frac{\tilde\beta}{2}}\left(\sum_{\nu_1=-\infty}^{\kappa_1} 2^{\nu_1(2\beta+2\tilde\beta-2\tilde M)}\sum_{\nu_2=-\infty}^{\mu_2} 2^{\nu_2(\beta+\tilde\beta-\tilde M)}\right)^{\frac12}\\
&= 4\left(\frac{|I|_1}{|K|_1}\right)^{\beta}\left(\frac{|K|_2}{|I|_2}\right)^{\frac{\tilde\beta}{2}}\\
&\le 4\left(\frac{|I|_1}{|K|_1}\right)^{\tilde\beta}\left(\frac{|K|_2}{|I|_2}\right)^{\frac{\tilde\beta}{2}},
\end{aligned}$$

because $\beta > \tilde\beta$ and $|I|_1 < |K|_1$. For U_2, use the following range for ν_1 and ν_2: $\nu_2 \le \mu_2 \le \kappa_2$ and $\nu_1+\nu_2 \le \kappa_1+\kappa_2 \le \mu_1+\mu_2$. Introduce the substitution, $\nu_3 = \nu_1+\nu_2$ to obtain

$$\begin{aligned}
U_2 &= |I|_2^{\frac{\beta-\tilde M}{2}}\,|K|_2^{\frac{\tilde\beta}{2}}\,|I|_3^{\beta}\,|K|_3^{\tilde\beta-\tilde M-1}\left(\sum_{\nu_2=-\infty}^{\mu_2} 2^{\nu_2(\beta+\tilde\beta-\tilde M)}\sum_{\nu_3=-\infty}^{\kappa_1+\kappa_2} 2^{2\nu_3(\beta+\tilde\beta-\tilde M-1)}\right)^{\frac12}\\
&= 4\left(\frac{|K|_2}{|I|_2}\right)^{\frac{\tilde\beta}{2}}\left(\frac{|I|_3}{|K|_3}\right)^{\beta}\\
&\le 4\left(\frac{|K|_2}{|I|_2}\right)^{\frac{\tilde\beta}{2}}\left(\frac{|I|_3}{|K|_3}\right)^{\tilde\beta}.
\end{aligned}$$

Hence, we assert that $|Z_{1,1,1}| \le U_1U_2W = 16\big(\frac{|I|_1}{|K|_1}\big)^{\tilde\beta}\big(\frac{|K|_2}{|I|_2}\big)^{\tilde\beta}\big(\frac{|I|_3}{|K|_3}\big)^{\tilde\beta}W$.

For $Z_{3,2,3}$, by inequality (52),

$$|Z_{3,2,3}| \le \sum_{J\in A_{3,2,3}}\left(\frac{|J|_1}{|I|_1}\right)^{\beta-1}\left(\frac{|J|_2}{|I|_2}\right)^{\beta}\left(\frac{|J|_3}{|I|_3}\right)^{\beta-\tilde M-1}\left(\frac{|J|_1}{|K|_1}\right)^{\tilde\beta}\left(\frac{|K|_2}{|J|_2}\right)^{\tilde\beta}\left(\frac{|J|_3}{|K|_3}\right)^{\tilde\beta}W$$

and then, by the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
|Z_{3,2,3}| &\le \left(\sum_{J\in A_{3,2,3}}\left(\frac{|J|_1}{|I|_1}\right)^{2(\beta-1)}\left(\frac{|J|_1}{|K|_1}\right)^{2\tilde\beta}\left(\frac{|J|_2}{|I|_2}\right)^{\beta}\left(\frac{|K|_2}{|J|_2}\right)^{\tilde\beta}\right)^{\frac12}\\
&\quad\times\left(\sum_{J\in A_{3,2,3}}\left(\frac{|J|_2}{|I|_2}\right)^{\beta}\left(\frac{|K|_2}{|J|_2}\right)^{\tilde\beta}\left(\frac{|J|_3}{|I|_3}\right)^{2(\beta-\tilde M-1)}\left(\frac{|J|_3}{|K|_3}\right)^{2\tilde\beta}\right)^{\frac12}W\\
&= V_1V_2W.
\end{aligned}$$

Analyze V_1 using the fact that $\kappa_1 \le \mu_1 \le \nu_1$ and $\mu_2 \le \nu_2 \le \kappa_2$. This yields

$$V_1 = |I|_1^{-\beta+1}\,|K|_1^{-\tilde\beta}\,|I|_2^{-\frac{\beta}{2}}\,|K|_2^{\frac{\tilde\beta}{2}}\left(\sum_{\nu_1=\mu_1}^{\infty} 2^{-2\nu_1(\beta+\tilde\beta-1)}\sum_{\nu_2=\mu_2}^{\kappa_2} 2^{\nu_2(\tilde\beta-\beta)}\right)^{\frac12}$$
$$\le 4\left(\frac{|I|_1}{|K|_1}\right)^{\tilde\beta}\left(\frac{|K|_2}{|I|_2}\right)^{\frac{\tilde\beta}{2}}.$$

For V_2, $\mu_2 \le \nu_2 \le \kappa_2$ and $\kappa_1+\kappa_2 \le \mu_1+\mu_2 \le \nu_1+\nu_2$, hence

$$V_2 = |I|_2^{-\frac{\beta}{2}}\,|K|_2^{\frac{\tilde\beta}{2}}\,|I|_3^{-\beta+\tilde M+1}\,|K|_3^{-\tilde\beta}\left(\sum_{\nu_2=\mu_2}^{\kappa_2} 2^{\nu_2(\tilde\beta-\beta)}\sum_{\nu_3=\mu_1+\mu_2}^{\infty} 2^{-2\nu_3(\beta+\tilde\beta-\tilde M-1)}\right)^{\frac12}$$
$$\le 4\left(\frac{|K|_2}{|I|_2}\right)^{\frac{\tilde\beta}{2}}\left(\frac{|I|_3}{|K|_3}\right)^{\tilde\beta}.$$

This produces the desired result for $Z_{3,2,3}$. The estimates for the other twenty five terms follow in the same manner. □

Iterating the result in Theorem 7.4 leads to the following theorem.

Theorem 7.5. *Let $\beta, M > 0$. Suppose $\mathcal{A}_i$, $1 \le i \le m$, are almost diagonal matrices with order β, M. Then, for any $\beta', M' > 0$ satisfying $\beta > \beta'$, $M > M'$, and $\beta+\beta' > M'$, the composition of $\mathcal{A}_i$, $\mathcal{A}_1 \circ \mathcal{A}_2 \cdots \circ \mathcal{A}_m$ is an almost diagonal matrix with order β', M' and*

$$\|\mathcal{A}_1 \circ \mathcal{A}_2 \cdots \circ \mathcal{A}_m\|_{\omega(\beta',M')} \le C^{m-1}\|\mathcal{A}_1\|_{\omega(\beta,M)}\|\mathcal{A}_2\|_{\omega(\beta,M)}\cdots\|\mathcal{A}_m\|_{\omega(\beta,M)}$$

for a constant $C > 0$ depending only on M, M'.

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