Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 29 (2010), 149–162
DOI: 10.4171/ZAA/1403

Delayed Quasilinear Evolution Equations with *BV*-Coefficients

Tomáš Bárta

Abstract. In this paper we investigate local and global existence as well as asymptotic behavior of the solution for a class of abstract (hyperbolic) quasilinear equations perturbed by bounded delay operators. We assume that the leading operator is of bounded variation in time. In the last section, the abstract results are applied on a heat conduction model.

Keywords. Evolution equations, delay equations, integrodifferential equations
Mathematics Subject Classification (2000). Primary 35L90, secondary 45N05, 80A20

1. Introduction

In this paper we are interested in the following quasilinear initial value problem

$$\begin{aligned} \dot{u}(t) &= A(t, u(t))u(t) + f(t, u_t), \quad t \geq 0 \\ u(0) &= u_0. \end{aligned} \tag{1}$$

Here u_t is the so called history function:

$$u_t(s) = \begin{cases} u(t+s), & s \in [-t, 0] \\ 0, & s \in (-\infty, -t). \end{cases}$$

The example we have in mind is

$$f(t, u_t) = \int_0^t k(t-s)u(s)\,\mathrm{d}s + g(t). \tag{2}$$

In [3] we have shown existence and uniqueness for (1) with A Lipschitz continuous in t. In particular, we have proven local existence, and in case

T. Bárta: Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University of Prague, Sokolovska 83, 18675 Praha 8, Czech Republic; barta@karlin.mff.cuni.cz

that $A(t, v(t))$ generate exponentially stable evolution families we have shown existence of global solutions on bounded intervals and existence of bounded solutions on $\mathbb{R}_+$ for f in the special form (2) with k, g and u_0 small enough. In this paper, we prove local existence and global existence for A of bounded variation in t (based on the generation theorem from [2]), and existence of a stable solution (resp. an exponentially stable solution) on $\mathbb{R}_+$ if f is of the form (2) with k, $g \in L^1$ (resp. k, g exponentially stable) small enough and $A(t, v(t))$ generating exponentially stable evolution families.

Equations of the form (1), (2) come, e.g., from MacCamy's model of heat conduction in materials with memory (see [10])

$$\partial_t \theta(t,x) = \int_0^t a(t-s)\sigma'(\partial_x\theta(s,x))\partial_{xx}\theta(s,x)\,\mathrm{d}s + h(t,x), \quad 0 < x < 1.$$

For more about linear and quasilinear nonautonomous evolution equations without delay see [4–9]. Some results on abstract quasilinear integrodifferential equations can be found in [11, 12].

This paper is organized as follows. The second section is devoted to the general setting and local existence while the third section deals with existence of global solutions and long-time behavior. In the last section we apply the results to the equation describing heat conduction.

2. Local existence

Let us assume that X, Y are reflexive Banach spaces with Y continuously and densely embedded in X. Denote by $B(Y, X)$ the space of all bounded linear operators from Y to X. Let W be an open bounded subset of Y contained in the ball $B(R, 0)$ of radius R centered at zero. The appropriate space for history functions is

$$\tilde{C}(X) := \left\{ f : (-\infty, 0] \to X \,\middle|\, \exists t < 0 : \begin{array}{l} f \text{ continuous on } [t, 0] \text{ and} \\ f \equiv 0 \text{ on } (-\infty, t) \end{array} \right\}$$

equipped with sup-norm. Denote $I = [0, T]$. $\mathrm{Bdd}(I, Y)$ means the space of all functions $f : I \to Y$ bounded in Y-norm. We say that a family of operators $(A(t))_{t\geq 0}$ on X is stable if $-A(t)$ generates a strongly continuous semigroup $\left(e^{-sA(t)}\right)_{s\geq 0}$ on X and the stability condition

$$\|e^{-s_1A(t_1)}e^{-s_2A(t_2)}\cdots e^{-s_nA(t_n)}\|_{X\to X} \leq Me^{\beta(s_1+s_2+\cdots+s_n)}$$

holds for all $0 \leq t_1 \leq \cdots \leq t_n$, $s_i \geq 0$. We write $A \in \mathrm{sta}(X, M, \beta)$.

We say that u is a *mild solution* for (1) on $[0, T]$ if there exists a countable set $N \subset [0, T]$ such that $u \in C([0, T] \setminus N, Y) \cap \mathrm{Lip}([0, T], X) \cap C^1([0, T] \setminus N, X)$ and (1) holds for all $t \in [0, T] \setminus N$.

Let us introduce our assumptions:

(i) $A : I \times W \to B(Y,X)$ and $A(\cdot, v(\cdot)) \in \mathrm{sta}(X, M, \beta)$ for every $v \in \mathrm{Bdd}(I,W) \cap \mathrm{Lip}(I,X)$ with M, β depending only on $\mathrm{Lip}_X(v)$;

(ii) $\|A(t,w)\|_{Y\to X} \le \lambda_A$ and

$$\|A(t,w) - A(t,w')\|_{Y\to X} \le \mu_A \|w - w'\|_X, \quad t \in I,\ w, w' \in W;$$

(iii) $t \mapsto A(t,\cdot)$ is a mapping of bounded variation from I to $C(W, B(Y,X))$;

(iv) $f : I \times \tilde{C}(W) \to Y$, $\|f(t,w)\|_Y \le \lambda_f$,

$$\|f(t,w) - f(t,w')\|_X \le \mu_f \|w - w'\|_\infty, \quad t \in [0,T],\ w, w' \in \tilde{C}(W),$$

and $t \mapsto f(t, v_t)$ is continuous with values in Y, whenever $v \in \mathrm{Bdd}(I,W) \cap \mathrm{Lip}(I,X)$;

(v) for all $y_0 \in Y$ and every $\epsilon > 0$ there exists $y \in Y$ with $\|y - y_0\|_Y < \epsilon$ and $\|A(t,w)y\|_Y \le M$ for all $t \in I$, $w \in W$.

The main result of this section is the following local existence theorem.

Theorem 2.1. *Let* (i)–(v) *hold. If* $u_0 \in W$, *then* (1) *has a unique mild solution* u *on* $[0,T']$ *for some* $0 < T' \le T$.

We will follow the approach of Kato. Let us introduce the following notation:

$$A_v(t) := A(t, v(t)) \quad \text{and} \quad f_v(t) := f(t, v_t).$$

We first show that the linear problem

$$\dot{u}(t) = A_v(t)u(t) + f_v(t), \quad u(0) = u_0, \tag{3}$$

has a solution u for every v. We denote $\Phi : v \mapsto u$ and show that $\Phi : E \to E$ is a contraction in X-norm for an appropriate set E.

Let $\rho > 0$ be such that the closed ball with center u_0 and radius ρ is contained in W. Consider a set E of functions $v \in \mathrm{Bdd}(I',Y) \cap \mathrm{Lip}(I',X)$ satisfying

$$\|v(t) - u_0\|_Y \le \rho, \quad \|v(t) - v(s)\|_X \le L|t-s|,$$

where $I' := [0,T']$, $T' \le T$. The values of T' and L will be specified later.

Let $B : I \to B(Y,X)$. By an *evolution operator* for B we mean a family of operators $(U(t,s))_{0\le s\le t\le T}$ satisfying $U(t,s) = U(t,r)U(r,s)$, $\|U(t,s)\| \le M_1 e^{\beta_1(t-s)}$ for $0 \le s \le r \le t \le T$, $U(t,s)Y \subset Y$, $\|U(t,s)\|_{Y\to Y} \le M_2 e^{\beta_2(t-s)}$, and for every $y \in Y$ the mappings $t \mapsto U(t,s)y$ and $s \mapsto U(t,s)y$ are continuous in X-norm, continuous with countably many exceptions in Y-norm and differentiable in X-norm in all t and s with countably many exceptions and the derivatives satisfy $\partial_s U(t,s)y = -U(t,s)B(s)y$ and $\partial_t U(t,s)y = B(t)U(t,s)y$.

Proposition 2.2. *Let $v \in E$ and $u_0 \in Y$. Then there exists an evolution operator U_v for A_v and $\|U_v(t,s)\|_X \le C$, $\|U_v(t,s)\|_Y \le C'$ for $0 \le s \le t \le T$, where C, C' are independent of v. A unique solution of* (3) *is given by*

$$u(t) := U_v(t,0)u_0 + \int_0^t U_v(t,s) f_v(s)\, ds. \tag{4}$$

Proof. The following estimate shows that A_v is of bounded variation. For $0 \le t_0 < t_1 < \cdots < t_n \le T$ we have

$$\begin{aligned}
&\sum_{k=1}^{n} \|A(t_k, v(t_k)) - A(t_{k-1}, v(t_{k-1}))\|_{Y\to X} \\
&\le \sum_{k=1}^{n} \|A(t_k, v(t_k)) - A(t_k, v(t_{k-1}))\|_{Y\to X} \\
&\quad + \sum_{k=1}^{n} \|A(t_k, v(t_{k-1})) - A(t_{k-1}, v(t_{k-1}))\|_{Y\to X} \\
&\le \sum_{k=1}^{n} \mu_A \|v(t_k) - v(t_{k-1})\|_X + \sum_{k=1}^{n} \sup_{w\in W} \|A(t_k, w) - A(t_{k-1}, w)\|_{Y\to X} \\
&\le \mu_A \cdot LT + \nu_A.
\end{aligned} \tag{5}$$

Hence, the assumptions of [2, Theorem 1.2] hold and the existence of U_v follows. If u is given by (4), then it is easy to show that (3) is satisfied for all but countably many t's. □

The rest of the proof of Theorem 2.1 is the same as in [3]. The only difference is that the derivative of $U(t,s)$ does not exist in every t and s but this fact does not influence the proof. So, the proofs of the following propositions can be found in [3].

Proposition 2.3. *Let $u_0 \in W$. Then constants ρ, L, and T' can be chosen such that $\Phi : E \to E$.*

Proposition 2.4. *If $T' > 0$ is sufficiently small, then the mapping Φ is a contraction for the supremum norm of $\mathcal{X} := C(I', X)$.*

3. Global existence

In this section we prove global existence on bounded and unbounded intervals. If $T < +\infty$, then we obtain the same result as in [3] for Lipschitz continuous A.

Theorem 3.1. *Let* $T < +\infty$, (i)–(iv) *and*

$$\|U_v(t,s)\|_{X\to X} \le M_1 e^{\beta_1(t-s)}, \quad \|U_v(t,s)\|_{Y\to Y} \le M_2 e^{\beta_2(t-s)}, \quad \beta_1, \beta_2 < 0, \quad (6)$$

hold for all $v \in \mathrm{Bdd}(I,W) \cap \mathrm{Lip}(I,X)$ *and* $0 \le s \le t \le T$. *Let*

$$M_1 \cdot \mu_f + \beta_1 < 0 \quad (7)$$

and one of the following inequalities hold:

$$\|f(t,v_t)\|_Y \le -\frac{\beta_2}{M_2} \sup_{t\in I} \|v(t)\|_Y \quad (8)$$

or

$$\|f(t,v_t)\|_Y \le M \sup_{t\in I} \|v(t)\|_Y + \varepsilon \quad \text{for some } M < -\frac{\beta_2}{M_2},\ \varepsilon > 0. \quad (9)$$

If $u_0 \in W$ *and* ε *are small enough, then* (1) *has a unique mild solution* u *on* I.

The proof is also the same as in [3] since we have solutions of the linear problem (3) for every v since A_v is of bounded variation by (5).

If $T = +\infty$, then we cannot expect A_v to be of bounded variation if v is only Lipschitz continuous. However, if v is of bounded variation, we can modify (5) in the following way:

$$\begin{aligned}
&\left\| \sum_{k=1}^{n} A(t_k, v(t_k)) - A(t_{k-1}, v(t_{k-1})) \right\|_{Y\to X} \\
&\le \sum_{k=1}^{n} \|A(t_k, v(t_k)) - A(t_k, v(t_{k-1}))\|_{Y\to X} \\
&\quad + \sum_{k=1}^{n} \|A(t_k, v(t_{k-1})) - A(t_{k-1}, v(t_{k-1}))\|_{Y\to X} \\
&\le \sum_{k=1}^{n} \mu_A \|v(t_k) - v(t_{k-1})\|_X + \sum_{k=1}^{n} \sup_{w\in W} \|A(t_k, w) - A(t_{k-1}, w)\|_{Y\to X} \\
&\le \mu_A \operatorname{Var} v + \operatorname{Var} A.
\end{aligned}$$

Hence, $t \mapsto A(t, v(t))$ is of bounded variation on $\mathbb{R}_+$ and according to [2, Theorem 1.2] we have a solution u of (3) for every v that is bounded in Y-norm and Lipschitz continuous and of bounded variation in X-norm. We will prove two theorems. The first one yields bounded solutions and the second one yields exponentially bounded solutions, both in Y-norm.

Theorem 3.2. *Let* (ii) *and* (iii) *hold and let* (i), (iv) *and* (6) *hold for all* $v \in Z := L^1(\mathbb{R}_+, W) \cap \mathrm{Lip}(\mathbb{R}_+, X) \cap \mathrm{BV}(\mathbb{R}_+, X)$ *and* $0 \le s \le t$. *Moreover, let* (7) *hold and assume that there exist* $\varepsilon > 0$ *and* $M < -\frac{\beta_2}{M_2}$ *such that*

$$\|f_v\|_{L^1(\mathbb{R}_+,Y)} \le M\|v\|_{L^1(\mathbb{R}_+,Y)} + \varepsilon, \quad \|f_v(t)\|_Y \le M \sup_{t\ge 0} \|v\|_Y + \varepsilon$$

hold for all $v \in Z$. If $u_0 \in W$ and ε are small enough, then (1) *has a unique mild solution $u \in \mathrm{Bdd}(\mathbb{R}_+, W) \cap \mathrm{BV}(\mathbb{R}_+, X)$ on $\mathbb{R}_+$.*

Let L and V be fixed constants defined below by (11) and (12). Let $\rho_1 \le 1$ be arbitrary and $\rho \le 1$, $B(0,\rho) \subset W$ is a fixed constant that will be specified below (in (15)). Denote by E the set of all $v \in L^1(\mathbb{R}_+, W) \cap \mathrm{Lip}(\mathbb{R}_+, X) \cap \mathrm{BV}(\mathbb{R}_+, X)$ such that

$$\|v(t)\|_Y \le \rho, \ \ \|v\|_{L^1(\mathbb{R}_+,Y)} \le \rho_1, \ \ \|v(t) - v(s)\|_X \le L|t-s|, \ \ \mathrm{Var}\, v \le V \quad (10)$$

for all t, $s \in I$. As in the previous section we define the mapping Φ by $\Phi(v) := u$ where u is the solution of (3). We show that $\Phi : E \to E$.

Proposition 3.3. *The mapping $\Phi : v \mapsto u$ maps E into E.*

Proof. Since $u = \Phi(v)$ is given by (4), we can estimate

$$\begin{aligned}
\|u(t)\|_Y &\le \|U_v(t,0)u_0\|_Y + \left\| \int_0^t U_v(t,s) f_v(s)\, \mathrm{d}s \right\|_Y \\
&\le M_2 e^{\beta_2 t} \|u_0\|_Y + \int_0^t M_2 e^{\beta_2 (t-s)} (M\rho + \varepsilon)\, \mathrm{d}s \\
&\le M_2 \|u_0\|_Y + M_2 \frac{1}{-\beta_2} \big(1 - e^{\beta_2 t}\big)(M\rho + \varepsilon) \\
&\le \rho
\end{aligned}$$

if $MM_2 < -\beta_2$ and ε, $\|u_0\|_Y$ are small enough. To estimate L^1-norm of u we write

$$\begin{aligned}
\|u(t)\|_{L^1(\mathbb{R}_+,Y)} &\le \|U_v(t,0)\|_{L^1(\mathbb{R}_+,B(Y))} \|u_0\|_Y + \left\| \int_0^t U_v(t,s) f_v(s)\, \mathrm{d}s \right\|_{L^1(\mathbb{R}_+,Y)} \\
&\le \frac{M_2}{-\beta_2} \big(1 - e^{\beta_2 t}\big) \|u_0\|_Y + \frac{M_2}{-\beta_2} \big(1 - e^{\beta_2 t}\big)(M\rho_1 + \varepsilon) \\
&\le \rho_1
\end{aligned}$$

provided $MM_2 < -\beta_2$ and ε, $\|u_0\|_Y$ are small enough. The derivative of u is estimated by

$$\|\partial_t u(t)\|_{\tilde{X}} \le \lambda_A \|u(t)\|_Y + \|f_v(t)\|_Y \le \lambda_A \rho + M\rho + \varepsilon < \lambda_A - \frac{\beta_2}{M_2} + 1 =: L \quad (11)$$

for all t with countably many exceptions (we assume ρ, $\varepsilon < 1$). Hence, u is L-Lipschitz continuous. And, finally,

$$\mathrm{Var}\, u \le \int_0^{+\infty} \|\partial_t u\| \le \int_0^{+\infty} \lambda_A \|u(t)\|_Y + \|f_v(t)\|_Y \mathrm{d}t \le \lambda_A \rho_1 + M\rho_1 + \varepsilon < V \quad (12)$$

if $\varepsilon < 1$, where $V := \lambda_A - \frac{\beta_2}{M_2} + 1$. Hence, Φ maps E into E. □

Proposition 3.4. *Φ is a contraction in $C(I,X)$, i.e.,*

$$\sup_{t\geq 0}\|\Phi(v)(t)-\Phi(v')(t)\|\leq \alpha\cdot\sup_{t\geq 0}\|v(t)-v'(t)\|$$

holds for some $\alpha<1$.

Proof. Let us estimate

$$\begin{aligned}\|\Phi(v)(t)-\Phi(v')(t)\|_X \leq{}& \|(U_v(t,0)-U_{v'}(t,0))u_0\|_X\\ &+\left\|\int_0^t (U_v(t,s)-U_{v'}(t,s))f_v(s)\,\mathrm{d}s\right\|_X\\ &+\left\|\int_0^t U_{v'}(t,s)(f_v(s)-f_{v'}(s))\,\mathrm{d}s\right\|_X.\end{aligned}\tag{13}$$

The identity

$$\begin{aligned}U_v(t,s)y-U_{v'}(t,s)y &= \int_s^t \frac{d}{dr}U_v(t,r)U_{v'}(r,s)\,\mathrm{d}r\\ &= \int_s^t U_v(t,r)(A_v(r)-A_{v'}(r))U_{v'}(r,s)y\,\mathrm{d}r\end{aligned}$$

yields

$$\begin{aligned}&\|(U_v(t,0)-U_{v'}(t,0))u_0\|_X\\ &\leq \int_0^t \|U_v(t,r)\|_X\mu_A\|v(r)-v'(r)\|_X\|U_{v'}(r,0)u_0\|_Y\,\mathrm{d}r\\ &\leq \mu_A\int_0^t M_1e^{\beta_1(t-r)}M_2e^{\beta_2 r}\,\mathrm{d}r\cdot\sup\|v(r)-v'(r)\|_X\|u_0\|_Y\\ &\leq M_1M_2\mu_A te^{\beta t}\|u_0\|_Y\sup\|v(r)-v'(r)\|_X,\end{aligned}\tag{14}$$

where $\beta=\max(\beta_1,\beta_2)$. If $\|u_0\|_Y$ is small enough, then $M_1M_2\mu_A te^{\beta t}\|u_0\|_Y<\delta$ for all $t\geq 0$. The second term on the right-hand side of (13) is estimated by

$$\begin{aligned}&\int_0^t\int_s^t \|U_v(t,r)\|_X\mu_A\|v(r)-v'(r)\|_X\|U_v(r,s)\|_Y\|f_v(s)\|_Y\,\mathrm{d}r\,\mathrm{d}s\\ &\leq \mu_A(M\rho+\varepsilon)\int_0^t (t-s)e^{\beta(t-s)}\,\mathrm{d}s\,\sup\|v(r)-v'(r)\|_X\\ &\leq \delta\sup\|v(r)-v'(r)\|_X\end{aligned}\tag{15}$$

provided ρ, ε are small enough. The last term in (13) is less than

$$\frac{M_1}{-\beta_1}\left(1-e^{\beta_1 t}\right)\cdot\mu_f\sup\|v(t)-v'(t)\|_X\leq q\cdot\sup\|v(t)-v'(t)\|_X,$$

where $q<1$ by (7). If $MM_2<-\beta_2$ and ρ, ε, $\|u_0\|_Y$ are small enough such that $q+2\delta<1$, then Φ is a contraction. □

Since Φ is a contraction in $C_b(\mathbb{R}_+, X)$-norm, there is a fixed point u of Φ. This u is a solution of (1) on $\mathbb{R}_+$ and it satisfies (10) except the L^1 estimate (the estimate of Y-norm follows from the reflexivity of X and Y. In fact, the sequence $u_n(t)$ from the Banach contraction theorem is bounded for every t, so there exists a weakly* convergent subsequence, this subsequence is weakly convergent and the norm of the limit $u(t)$ is bounded by ρ by the weak semicontinuity of the norm).

Theorem 3.5. *Let* (ii) *and* (iii) *hold and let* (i), (iv) *and* (6) *hold for all* $v \in Z := L^1(\mathbb{R}_+, W) \cap \mathrm{Lip}(\mathbb{R}_+, X) \cap \mathrm{BV}(\mathbb{R}_+, X)$ *and* $t \in \mathbb{R}_+$. *Moreover, let* (7) *hold and assume that there exist* $\beta_3 \in (\beta, 0)$ (*where* $\beta = \max\{\beta_1, \beta_2\}$) *and* $\rho_0 > 0$ *such that for every* $0 < \rho < \rho_0$ *it holds that*

$$v \in Z,\ \|v(t)\| \leq \rho e^{\beta_3 t} \ \Rightarrow\ \|f_v(t)\| \leq M e^{\beta_3 t}, \quad M < \frac{\beta_3 - \beta_2}{M_2}\rho. \tag{16}$$

If $u_0 \in W$ *is small enough, then* (1) *has a unique mild solution* $u \in \mathrm{BV}(\mathbb{R}_+, X)$ *satisfying* $\|u(t)\|_Y \leq K e^{\beta_3 t}$ *for some* K *and all* $t \geq 0$.

Let ρ, L, and V be small enough (exact values will be specified later). Denote by E the set of all $v \in \mathrm{Bdd}(\mathbb{R}_+, W) \cap \mathrm{Lip}(\mathbb{R}_+, X) \cap \mathrm{BV}(\mathbb{R}_+, X)$ such that

$$\|v(t)\|_Y \leq \rho e^{\beta_3 t}, \quad \|v(t) - v(s)\|_X \leq L|t - s| \quad \text{and} \quad \mathrm{Var}\, v \leq V \tag{17}$$

for all t, $s \in I$. As in the previous section we define the mapping Φ by $u = \Phi(v)$ where u is the solution of (3). We show that $\Phi : E \to E$.

Proposition 3.6. *The mapping* $\Phi : v \mapsto u$ *maps* E *into* E.

Proof. Let us estimate

$$\begin{aligned}
\|u(t)\|_Y &\leq \|U_v(t, 0)u_0\|_Y + \left\| \int_0^t U_v(t, s) f_v(s)\,\mathrm{d}s \right\|_Y \\
&\leq M_2 e^{\beta_2 t} \|u_0\|_Y + \int_0^t M_2 e^{\beta_2 (t-s)} (M e^{\beta_3 s})\,\mathrm{d}s \\
&\leq M_2 e^{\beta_3 t} \|u_0\|_Y + M M_2 e^{\beta_2 t} \frac{1}{\beta_3 - \beta_2} (e^{(\beta_3 - \beta_2)t} - 1) \\
&\leq \rho e^{\beta_3 t}
\end{aligned}$$

if $M < \frac{1}{M_2}(\beta_3 - \beta_2)\rho$ and $\|u_0\|_Y$ is small enough. Then:

$$\|\partial_t u(t)\|_{\tilde{X}} \leq \lambda_A \|u(t)\|_Y + \|f_v(t)\|_Y \leq \lambda_A \rho + \lambda_f \leq \lambda_A + \lambda_f =: L$$

for all t with countably many exceptions. Hence, u is L-Lipschitz continuous. And, finally,

$$\operatorname{Var} u \leq \int_0^{+\infty} \|\partial_t u\| \leq \int_0^{+\infty} \lambda_A \|u(t)\|_Y + \|f_v(t)\|_Y \mathrm{d}t \leq \int_0^{+\infty} \lambda_A \rho e^{\beta_3 t} + M e^{\beta_3 t} \mathrm{d}t \leq V.$$

where $V := \frac{\frac{\beta_3-\beta_2}{M_2}+\lambda_A\rho}{-\beta_3}$. Hence, Φ maps E into E. □

Proposition 3.7. *Φ is a contraction in $C(I,X)$, i.e.,*

$$\sup_{t\geq 0} \|\Phi(v)(t) - \Phi(v')(t)\| \leq \alpha \cdot \sup_{t\geq 0} \|v(t) - v'(t)\|$$

holds for some $\alpha < 1$.

Proof. The proof is the same as the proof of Proposition 3.4 when we replace (14) by

$$\begin{aligned}
&\int_0^t \int_s^t \|U_v(t,r)\|_X \mu_A \|v(r) - v'(r)\|_X \|U_v(r,s)\|_Y \|f_v(s)\|_Y \,\mathrm{d}r\,\mathrm{d}s \\
&\leq \mu_A M M_1 M_2 \int_0^t \int_s^t e^{\beta_1(t-r)} e^{\beta_2(r-s)} e^{\beta_3 s} \,\mathrm{d}r\,\mathrm{d}s \sup \|v(r) - v'(r)\|_X \\
&\leq \delta \sup \|v(r) - v'(r)\|_X
\end{aligned}$$

provided M is small enough (i.e., if ρ is small enough). □

Since Φ is a contraction in $C_b(\mathbb{R}_+, X)$-norm, there is a fixed point u of Φ. This u is a solution of (1) on $\mathbb{R}_+$ and it satisfies (17) (the estimate of Y-norm follows from reflexivity of X and Y).

We finally formulate three corollaries for f being of the following type:

$$f(t, v_t) := \int_0^t k(t-s) v(s) \,\mathrm{d}s + g(t), \tag{18}$$

where $k \in L^1(I)$ and $g \in C(I, W)$. The first one is for bounded intevals.

Corollary 3.8. *Let $T < +\infty$,* (ii) *and* (iii) *hold. Let* (i) *and* (6) *hold for all $v \in \operatorname{Bdd}([0,T], W) \cap \operatorname{Lip}([0,T], X)$ and $0 \leq s \leq t \leq T$. Let f be given by* (18) *and the following inequalities hold:*

$$\|k\|_1 < \frac{-\beta_2}{M_2} \text{ and } \|g\|_Y \text{ small enough} \qquad \text{or} \qquad \|k\|_1 = \frac{-\beta_2}{M_2} \text{ and } g \equiv 0 \tag{19}$$

$$\|k\|_1 < \frac{-\beta_1}{M_1}. \tag{20}$$

If $\|u_0\|_Y$ is small enough, then (1) *has a unique mild solution u on $[0,T]$.*

Proof. This corollary follows from Theorem 3.1. In fact, estimate (8) resp. (9) follows immediately from (19) and estimate (7) from (20). Assumption (iv) of Theorem 3.1 is satisfied with $\lambda_f = \|k\|_1 \cdot R + \|g\|$, $\mu_f = \|k\|_1$ and continuity of f_v follows from the estimate

$$\begin{aligned}\|f_v(t) - f_v(s)\| &\le \int_0^t |k(t-\sigma) - k(s-\sigma)| \cdot \|v(\sigma)\|_Y \,\mathrm{d}\sigma \\ &\quad + \int_s^t |k(t-\sigma)| \cdot \|v(\sigma)\|_Y \,\mathrm{d}\sigma + \|g(t) - g(s)\|_Y \\ &\le \delta \sup \|v(\sigma)\|_Y + \delta \sup \|v(\sigma)\|_Y + \delta,\end{aligned}$$

where $\delta \to 0$ if $|t - s| \to 0$. The other assumptions of Theorem 3.1 are satisfied obviously. □

The second corollary yields a bounded solution on $\mathbb{R}_+$. It follows from Theorem 3.2 by the same arguments as in the previous proof.

Corollary 3.9. *Let* (ii) *and* (iii) *hold and let* (i) *and* (6) *hold for all* $v \in L^1(\mathbb{R}_+, W) \cap \mathrm{Lip}(\mathbb{R}_+, X) \cap \mathrm{BV}(\mathbb{R}_+, X)$ *and* $0 \le s \le t \le T$. *Let* f *be given by* (18) *and*

$$\|k\|_1 < \frac{-\beta_2}{M_2} \qquad \text{and} \qquad \|k\|_1 < \frac{-\beta_1}{M_1} \tag{21}$$

are satisfied. If $\|u_0\|_Y$, $\|g\|_{L^1(\mathbb{R}_+,Y)}$ *and* $\sup_{t\in\mathbb{R}_+} \|g(t)\|_Y$ *are small enough, then* (1) *has a unique solution* $u \in L^1(\mathbb{R}_+, W) \cap \mathrm{Lip}(\mathbb{R}_+, X) \cap \mathrm{BV}(\mathbb{R}_+, X)$.

Let k and g satisfy

$$|k(t)| \le K_1 e^{\beta_4 t} \quad \text{and} \quad |g(t)| \le K_2 e^{\beta_4 t}. \tag{22}$$

Then we have for $\|v(t)\|_Y \le \rho e^{\beta_3 t}$, $\beta_4 < \beta_3 < 0$,

$$\|f(t, v_t)\|_Y \le \int_0^t K_1 \rho e^{\beta_4 t} e^{(\beta_3-\beta_4)s} \,\mathrm{d}s + K_2 e^{\beta_4 t} \le \left(K_1 \rho \frac{1}{\beta_3 - \beta_4} + K_2\right) e^{\beta_3 t} \tag{23}$$

and

$$\|f(t, v) - f(t, w)\|_X \le \|k\|_1 \|v - w\|_\infty \le \frac{K_1}{-\beta_4} \|v - w\|_\infty. \tag{24}$$

We have the following corollary.

Corollary 3.10. *Let* (ii) *and* (iii) *hold and let* (i) *and*

$$\|U_v(t, s)\|_{X\to X} \le M_1 e^{\beta_1 t} \quad \text{and} \quad \|U_v(t, s)\|_{Y\to Y} \le M_2 e^{\beta_2 t} \tag{25}$$

hold with β_1, $\beta_2 < 0$ *for all* $v \in \mathrm{Bdd}(\mathbb{R}_+, W) \cap \mathrm{Lip}(\mathbb{R}_+, X) \cap \mathrm{BV}(\mathbb{R}_+, X)$ *with* $\|v(t)\|_Y \le R e^{\beta_3 t}$ *and* $0 \le s \le t \le T$. *Let* f *be given by* (18) *and* (22) *hold. Assume*

$$M_1 K_1 < \beta_1 \beta_4 \quad \text{and} \quad M_2 K_1 < (\beta_3 - \beta_2) \cdot (\beta_3 - \beta_4) \tag{26}$$

hold for some $0 > \beta_3 > \beta_4$ *If* $u_0 \in W$ *and* K_2 *are small enough, then* (1) *has a unique mild solution* $u \in L^1(\mathbb{R}_+, W) \cap \mathrm{BV}(\mathbb{R}_+, X)$ *on* $\mathbb{R}_+$ *satisfying* $\|u(t)\|_Y \leq Ce^{\beta_3 t}$ *for some* $C > 0$.

Proof. We show that the assumptions of Theorem 3.5 are satisfied. From (23) and (24) we obtain assumption (iv) with $\mu_f = \frac{K_1}{-\beta_4}$ and the assumption (16) with $M = K_1 \frac{\rho}{\beta_3 - \beta_4} + K_2$. This is less than $\frac{\rho(\beta_3-\beta_2)}{M_2}$ by the second inequality in (26) provided K_2 is small enough. Assumption (7) follows immediately from the first inequality in (26). So, Corollary 3.10 follows from Theorem 3.5. □

4. Application to heat conduction

Consider the following modified Mac Camy's model for heat conduction in materials with memory, where the temperature θ in position x and time t is given by

$$\partial_t \theta(t,x) = \int_0^t a(t-s)\partial_2\sigma(s, \partial_x\theta(s,x))\partial_{xx}\theta(s,x)\,\mathrm{d}s + h(t,x), \quad 0 < x < 1, \tag{27}$$

with prescribed boundary $\theta(t,0) = \theta(t,1) = 0$ and an initial value $\theta(0,x) = \theta_0(x)$ (see [10] for details). If a is positive, non-increasing, log-convex and $a \to 0$ for $t \to +\infty$, then there exists a completely positive function $c \in L^1_{loc}$ satisfying $1 * a = c * c$ and a creep function r (i.e., positive, non-decreasing and concave) such that $c * r = t$ (see Prüss [12, Section 4]). Assume $r'' \in L^1([0,T])$, $h \in C([0,T], H_0^3)$ and $\sigma \in C^3$ with $0 < c_1 < \sigma' < c_2$. Then (27) can be rewritten as (details are given in [1])

$$\dot{u}(t) = A(t, u(t))u(t) + \int_0^t k(t-s)u(s)\,\mathrm{d}s + g(t), \quad u(0) = u_0, \tag{28}$$

where u takes values in a product of two function spaces:

$$\begin{pmatrix}\theta(t,\cdot)\\ \eta(t,\cdot)\end{pmatrix}, \quad \text{where } \eta(t,\cdot) = \int_0^t c(t-s)\partial_2\sigma(s,\partial_x\theta(s,\cdot))\partial_{xx}\theta(s,\cdot)\,\mathrm{d}s.$$

We have

$$A(t,w) = \begin{pmatrix} -r'(0) & I \\ \partial_2\sigma(t,\partial_x w_1)\Delta & -r'(0)\end{pmatrix} \quad \text{for } w = \begin{pmatrix} w_1 \\ w_2\end{pmatrix}$$

and

$$k(t) = -r''(t), \quad g(t) = \begin{pmatrix} r(0)h(t) + \int_0^t r'(t-s)h(s)\,\mathrm{d}s \\ 0 \end{pmatrix}, \quad u_0 = \begin{pmatrix}\theta_0 \\ 0\end{pmatrix}$$

provided $r(0)$ and $r'(0)$ are finite.

As in [3] we will work on the spaces

$$X := H_0^2(0,1) \times H_0^1(0,1), \quad Y := H_0^3(0,1) \times H_0^2(0,1) = D(A(t)),$$

where

$$H_0^2(0,1) = \{f \in H^2(0,1) :\ f(0) = f(1) = 0\}$$
$$H_0^3(0,1) = \{f \in H^3(0,1) :\ f(0) = f(1) = f''(0) = f''(1) = 0\}.$$

Define $\tilde{A}_v := A_v + r'(0)I$. In [2] we have shown that $\tilde{A}_v$ is stable on $H_0^1(0,1) \times L^2(0,1)$ if $t \mapsto \partial_2\sigma(t, \partial_x v(t,x))$ is in $\mathrm{BV}(I, L^\infty((0,1)))$. We will proceed in a similar way. According to [2, Corollary 1.4.] we need $t \mapsto \tilde{A}_v(t)$ to be of bounded variation and to find a family of scalar products $(\cdot,\cdot)_t$ on X such that $t \mapsto \|\cdot\|_t$ is of bounded variation and $\tilde{A}_v(t)$ generates a semigroup of contractions on $(X, \|\cdot\|_t)$ for every fixed t. So, we define a family of scalar products $(\cdot,\cdot)_t$ on X by

$$\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right)_t := \int_0^1 u_1'' w_1''\, \mathrm{d}x + \int_0^1 u_2' w_2' \frac{1}{\sigma_2(t, v_1(t,x))}\, \mathrm{d}x.$$

It is easy to show that $\tilde{A}_v(t)$ is skew-adjoint with respect to $(\cdot,\cdot)_t$, hence it generates a semigroup of contractions. The mapping $t \mapsto \|\cdot\|_t$ is of bounded variation if

$$t \mapsto \partial_2\sigma(t, \partial_x v_1(t,x)) \in \mathrm{BV}(I, L^\infty((0,1))) \tag{29}$$

(the proof is the same as in [2]). We show that (29) holds if

$$\begin{aligned} &\sigma_2 = \partial_2\sigma \in \mathrm{BV}(I, L^\infty(0,1)) \cap L^\infty(I, \mathrm{Lip}(0,1)) \\ &\text{and } \partial_x v_1 \in \mathrm{BV}(I, L^\infty(0,1)). \end{aligned} \tag{30}$$

In fact,

$$\begin{aligned} &\sum |\sigma_2(t_k, \partial_x v_1(t_k,x)) - \sigma_2(t_{k-1}, \partial_x v_1(t_{k-1},x))| \\ &\le \sum |\sigma_2(t_k, \partial_x v_1(t_k,x)) - \sigma_2(t_{k-1}, \partial_x v_1(t_k,x))| \\ &\quad + |\sigma_2(t_{k-1}, \partial_x v_1(t_k,x)) - \sigma_2(t_{k-1}, \partial_x v_1(t_{k-1},x))| \\ &\le \sum \sup_w |\sigma_2(t_k,w) - \sigma_2(t_{k-1},w)| + L_s|\partial_x v_1(t_k,x) - \partial_x v_1(t_{k-1},x)| \\ &\le \mathrm{Var}\,\sigma_2 + L_{\sigma_2}\,\mathrm{Var}(\partial_x v_1). \end{aligned} \tag{31}$$

It is easy to show that $t \mapsto \tilde{A}(t, v(t))$ is of bounded variation if (30) holds. Then, [2, Corollary 1.4] yields $\tilde{A}_v \in \mathrm{sta}(X, M, 0)$ for $v \in \mathrm{BV}(I, X)$. Hence, $A_v \in \mathrm{sta}(X, M, -r'(0))$ and assumption (i) holds. We have also proven the assumption (iii).

Assumptions (ii), (iv) and (v) are easy to verify, so Theorem 2.1 yields local existence and Corollary 3.8 yields existence on bounded time intervals provided the data are sufficiently small. In fact, in (19) and (20) we have $\|k\|_1 = r'(0) - r'(T)$ if we restrict k to a bounded interval $[0,T]$, $\beta_1 = \beta_2 = -r'(0)$ and M_1, M_2 depend on the variation of $t \mapsto \|\cdot\|_t$. In particular, according to [2, Section 2] and estimates (31) we have

$$\operatorname{Var} \|\cdot\|_t \le C(\operatorname{Var}\sigma_2 + L_{\sigma_2} \operatorname{Var}(\partial_x v_1))$$

and according to [2, Lemma 1.1]

$$M_1 \le e^{2C((\operatorname{Var}\sigma_2 + L_{\sigma_2}\operatorname{Var}(\partial_x v_1)))}. \tag{32}$$

The estimate

$$M_2 \le e^{3C((\operatorname{Var}\sigma_2 + L_{\sigma_2}\operatorname{Var}(\partial_x v_1)))}$$

then follows from (32) and [2, Lemma 1.1] if we take $\|y\|_{Y,t} := \|A(t)y\|_t$. Since the second term in the exponent can be made arbitrarily small, M_1, M_2 are close to 1 if $\operatorname{Var}\sigma_2 = 0$. In this case, T can be taken arbitrarily large and we have existence on every bounded interval $[0,T]$ (this case was investigated in [3]). If $\operatorname{Var}\sigma_2 \ne 0$, then

$$\begin{aligned} M_1\|k\|_1 + \beta_1 &< M_2\|k\|_1 + \beta_2 \\ &= (r'(0) - r'(T))e^{3C(\operatorname{Var}\sigma_2 + L_{\sigma_2}\operatorname{Var}(\partial_x v_1))} - r'(0) \\ &\le r'(0)\left(e^{3C(\operatorname{Var}\sigma_2 + \epsilon)}\frac{r'(0) - r'(T)}{r'(0)} - 1\right). \end{aligned} \tag{33}$$

Hence, (19) and (20) hold if the right-hand side in (33) is less than zero, e.g., if

$$e^{3C\operatorname{Var}\sigma_2}\frac{r'(0) - r'(T)}{r'(0)} < 1 \tag{34}$$

and ε is small enough. Hence, (34) gives an estimate for T for which the mild solution exists on $[0,T]$ if the data are sufficiently small. If $r'(\infty) > 0$, then (34) holds with $T = +\infty$ if $\operatorname{Var}\sigma_2$ is small enough. In this case, we can apply Corollary 3.9 (resp. Corollary 3.10 if k and g tend to zero exponentially) and obtain a bounded (resp. exponentially bounded) solution on $\mathbb{R}_+$. However, $r'(\infty) > 0$ never holds if the equation (28) arose as a reformulation of equation (27), as is noticed in [3].

Acknowledgement. This work is a part of the research project MSM 0021620839 financed by MSMT and it was also partly supported by GACR 201/06/P171.

References

[1] Bárta, T., *Integrodifferential equations in Banach spaces.* Doctoral Thesis. Prague: Charles Univ. 2008.

[2] Bárta, T., A generation theorem for hyperbolic equations with coefficients of bounded variation in time. *Riv. Mat. Univ. Parma(7)* 9 (2008), 17 – 30.

[3] Bárta, T., Delayed quasilinear evolution equations with applications in heat flow (submitted).

[4] Beyer, H., *Beyond Partial Differential Equations. On Linear and Quasi-Linear Abstract Hyperbolic Evolution Equations.* Lect. Notes Math. 1898. Berlin: Springer 2007.

[5] Kato, T., Linear evolution equations of "hyperbolic" type. *J. Fac. Sci. Univ. Tokyo, Sec. I* 17 (1970), 241 – 258.

[6] Kato, T., Linear evolution equations of "hyperbolic" type. II. *J. Math. Soc. Japan* 25 (1973), 648 – 666.

[7] Kato, T., Quasi-linear equations of evolution with applications to partial differential equations, In: *Spectral theory and differential equations.* (Proc. Sympos., Dundee 1974; ed.: Everitt, W. N.). Lect. Notes Math. 448. Berlin: Springer 1975, pp. 25 – 70.

[8] Kato, T., *Abstract Differential Equations and Nonlinear Mixed Problems* (Scuola Normale Superiore Pisa). Fermi Lectures. Rome: Accademia Nazionale dei Lincei 1985.

[9] Kato, T., Abstract evolution equations, linear and quasilinear, revisited. In: *Functional Analysis and Related Topics* (Kyoto 1991; ed.: Komatsu, H.). Lect. Notes Math. 1540. Berlin: Springer 1993, pp. 103 – 125.

[10] MacCamy, R. C., An integro-differential equation with application in heat flow. *Quart. J. Appl. Math.* 35 (1977), 1 – 19.

[11] Oka, H., Abstract quasilinear Volterra integrodifferential equations. *Nonlin. Anal.* 28 (1997), 1019 – 1045.

[12] Prüss, J., *Evolutionary Integral Equations and Applications.* Monographs Math. 87. Basel: Birghäuser 1993.

Received May 12, 2008