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Asymptotic Behavior of Approximate Solutions to Evolution Equations in Banach Spaces

Sergiu Aizicovici, Simeon Reich and Alexander J. Zaslavski

Abstract. We study evolution equations in Banach spaces governed by a class of mappings associated with continuous descent methods for the minimization of convex functions. In our previous work we showed that for most of these mappings (in the sense of Baire category) the corresponding solutions converged. In the present paper we show that this remains true even for approximate solutions.

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1. Introduction

Discrete and continuous descent methods are two important topics in optimization theory and in the study of dynamical systems. See, for example, [1–6,8–13]. Given a continuous convex function f on a Banach space X, we associate with f a complete metric space, say $\mathcal{A}$, of vector fields $V : X \to X$ such that $f^0(x, Vx) \leq 0$ for all $x \in X$. Here $f^0(x, u)$ is the right-hand derivative of f at x in the direction of $u \in X$ (see (1) below). To each such vector field there correspond two gradient-like iterative processes. In the papers [10, 11], it is shown that for most of these vector fields, both iterative processes generate sequences $\{x_n\}_{n=1}^{\infty}$ such that the sequences $\{f(x_n)\}_{n=1}^{\infty}$ tend to $\inf(f)$ as $n \to \infty$. Here by "most" we mean an everywhere dense G_δ subset of the space of vector fields $\mathcal{A}$ (cf., for example, [7, 10, 14]). In the paper [2], we studied the convergence of the trajectories of an analogous continuous dynamical system governed by such vector fields to the point where the function f attains

S. Aizicovici: Department of Mathematics, Ohio University, Athens, OH 45701-2979, USA; aizicovi@math.ohiou.edu
S. Reich: Department of Mathematics, The Technion–Israel Institute of Technology, 32000 Haifa, Israel; sreich@tx.technion.ac.il
A. J. Zaslavski: Department of Mathematics, The Technion–Israel Institute of Technology, 32000 Haifa, Israel; ajzasl@tx.technion.ac.il

its infimum. The first attempt to examine such continuous descent methods was made in [13]. However, it is assumed there that the convex function f is Lipschitz on all bounded subsets of X. No such assumption was made in [2]. In the more recent paper [3], we show that convergent continuous descent methods are stable under small perturbations (see Theorem 1.3 below). We remark in passing that continuous descent methods for the minimization of Lipschitz (not necessarily convex) functions are studied in [1].

It should be mentioned that in [3] we obtain stability results for small perturbations such that the perturbed vector fields still belong to the space $\mathcal{A}$. This implies that the function f decreases along all the trajectories of the corresponding dynamical system. In the present paper we improve upon the stability results of [3] by allowing any small perturbations (see Theorem 1.4 below).

More precisely, let $(X, \|\cdot\|)$ be a real Banach space and let $f : X \to R^1$ be a convex continuous function which satisfies the following conditions:

C(i) $\lim_{\|x\|\to\infty} f(x) = \infty$;

C(ii) there is $\bar{x} \in X$ such that $f(\bar{x}) \leq f(x)$ for all $x \in X$;

C(iii) if $\{x_n\}_{n=1}^{\infty} \subset X$ and $\lim_{n\to\infty} f(x_n) = f(\bar{x})$, then $\lim_{n\to\infty} \|x_n - \bar{x}\| = 0$.

By C(iii), the point $\bar{x}$, where the minimum of f is attained, is unique. For each $x \in X$, let

$$f^0(x, u) = \lim_{t\to 0^+} \frac{f(x + tu) - f(x)}{t}, \quad u \in X. \tag{1}$$

Let $(X^*, \|\cdot\|_*)$ be the dual space of $(X, \|\cdot\|)$. For each $x \in X$ and $r > 0$, set

$$B(x, r) = \{z \in X : \|z - x\| \leq r\} \text{ and } B(r) = B(0, r). \tag{2}$$

For each mapping $A : X \to X$ and each $r > 0$, put

$$\mathrm{Lip}(A, r) = \sup\left\{\frac{\|Ax - Ay\|}{\|x - y\|} : x, y \in B(r) \text{ and } x \neq y\right\}. \tag{3}$$

Denote by $\mathcal{A}_l$ the set of all mappings $V : X \to X$ such that $\mathrm{Lip}(V, r) < \infty$ for all $r > 0$ (this means that the restriction of V to any bounded subset of X is Lipschitz) and $f^0(x, Vx) \leq 0$ for all $x \in X$. For the set $\mathcal{A}_l$ we consider the uniformity determined by the base

$$E_s(n, \epsilon) = \left\{(V_1, V_2) \in \mathcal{A}_l \times \mathcal{A}_l : \begin{array}{c} \mathrm{Lip}(V_1 - V_2, n) \leq \epsilon \text{ and} \\ \|V_1x - V_2x\| \leq \epsilon \ \forall x \in B(n) \end{array}\right\}. \tag{4}$$

Clearly, this uniform space $\mathcal{A}_l$ is metrizable and complete. The topology induced by this uniformity in $\mathcal{A}_l$ will be called the strong topology. We also equip the space $\mathcal{A}_l$ with the uniformity determined by the base

$$E_w(n, \epsilon) = \{(V_1, V_2) \in \mathcal{A}_l \times \mathcal{A}_l : \|V_1x - V_2x\| \leq \epsilon \ \forall x \in B(n)\}, \tag{5}$$

where $n, \epsilon > 0$. The topology induced by this uniformity will be called the weak topology.

The following existence result was proved in Section 3 of [2].

Proposition 1.1. *Let $x_0 \in X$ and $V \in \mathcal{A}_l$. Then there exists a unique continuously differentiable mapping $x : [0,\infty) \to X$ such that*

$$x'(t) = Vx(t),\ t \in [0,\infty), \quad x(0) = x_0.$$

We now recall the main result of [2].

Theorem 1.2. *There exists a set $\mathcal{F} \subset \mathcal{A}_l$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}_l$ such that for each $V \in \mathcal{F}$ the following property holds:*

For each $\epsilon > 0$ and each $n > 0$, there exist $T_{\epsilon,n} > 0$ and a neighborhood $\mathcal{U}$ of V in $\mathcal{A}_l$ with the weak topology such that for each $W \in \mathcal{U}$ and each differentiable mapping $y : [0,\infty) \to X$ satisfying

$$|f(y(0))| \le n \qquad \text{and} \qquad y'(t) = Wy(t) \ \text{ for all } t \ge 0,$$

the inequality $\|y(t) - \bar{x}\|| \le \epsilon$ holds for all $t \ge T_{\epsilon,n}$.

Denote by $\mathcal{F}_*$ the set of all $V \in \mathcal{A}_l$ which have the following property:

(P1) For each $\epsilon > 0$ and each $n > 0$, there exists $T_{\epsilon,n} > 0$ such that for each differentiable mapping $y : [0,\infty) \to X$ satisfying $|f(y(0))| \le n$ and $y'(t) = Vy(t)$ for all $t \ge 0$, the inequality $\|y(t) - \bar{x}\| \le \epsilon$ holds for some $t \in [0, T_{\epsilon,n}]$.

By Theorem 1.2, $\mathcal{F}_*$ contains a subset which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}_l$. Denote by $\mathcal{A}$ the set of all mappings $V : X \to X$ which are bounded on bounded subsets of X and satisfy $f^0(x, Vx) \le 0$ for all $x \in X$. Clearly, $\mathcal{A}_l \subset \mathcal{A}$. For the set $\mathcal{A}$ we consider the uniformity determined by the base

$$G_w(n,\epsilon) = \big\{(V_1, V_2) \in \mathcal{A} \times \mathcal{A} :\ \|V_1x - V_2x\| \le \epsilon \ \forall x \in B(n)\big\},$$

where $n, \epsilon > 0$. Clearly, the space $\mathcal{A}$ with this uniformity is metrizable. In [3] we established the following result.

Theorem 1.3. *Let $V \in \mathcal{F}_*$ and $n, \epsilon > 0$. Then there exist $T_{\epsilon,n} > 0$ and a neighborhood $\mathcal{U}$ of V in $\mathcal{A}$ such that for each $W \in \mathcal{U}$, each $T \ge T_{\epsilon,n}$ and each $x \in W^{1,1}(0,T;X)$ satisfying*

$$|f(x(0))| \le n, \quad x'(t) = Wx(t),\ t \in [0,T] \ \text{(a.e.)},$$

the inequality $\|x(t) - \bar{x}\| \le \epsilon$ holds for all $t \in [T_{\epsilon,n}, T]$.

The following theorem is the main result of the present paper.

Theorem 1.4. *Let $V \in \mathcal{F}_*$ and $n, \epsilon > 0$. Then there exist $\delta > 0$ and $\tau > 0$ such that for each $T \geq \tau$ and each $x \in W^{1,1}(0,T;X)$ satisfying*

$$|f(x(0))| \leq n, \quad \|x'(t) - Vx(t)\| \leq \delta, \ t \in [0,T] \ a.e.,$$

the inequality $\|x(t) - \bar{x}\| \leq \epsilon$ holds for all $t \in [\tau, T]$.

Our paper is organized as follows. An auxiliary result, Proposition 2.1, is presented in Section 2. Section 3 contains a basic lemma. Our main result, Theorem 1.4, is proved in Section 4.

2. An auxiliary result

Let $x \in W^{1,1}(0,T;X)$, i.e., $x(t) = x_0 + \int_0^t u(s)ds$, $t \in [0,T]$, where $T > 0$, $x_0 \in X$ and $u \in L^1(0;T;X)$. Then $x : [0,T] \to X$ is absolutely continuous and $x'(t) = u(t)$ for a.e. $t \in [0,T]$.

Recall that the function $f : X \to R^1$ is assumed to be convex and continuous, and therefore it is, in fact, locally Lipschitz. It follows that its restriction to the set $\{x(t) : \ t \in [0,T]\}$ is Lipschitz. Indeed, since this set is compact, the restriction of f to it is Lipschitz. Hence the function $(f \circ x)(t) := f(x(t))$, $t \in [0,T]$, is absolutely continuous. It follows that for almost every $t \in [0,T]$, both the derivatives $x'(t)$ and $(f \circ x)'(t)$ exist:

$$x'(t) = \lim_{h \to 0} h^{-1}[x(t+h) - x(t)]$$
$$(f \circ x)'(t) = \lim_{h \to 0} h^{-1}[f(x(t+h)) - f(x(t))].$$

We now recall Proposition 3.1 in [13].

Proposition 2.1. *Assume that $t \in [0,T]$ and that both the derivatives $x'(t)$ and $(f \circ x)'(t)$ exist. Then*

$$(f \circ x)'(t) = \lim_{h \to 0} h^{-1}[f(x(t) + hx'(t)) - f(x(t))].$$

3. A basic lemma

The following lemma plays a key role in the proof of Theorem 1.4.

Lemma 3.1. *Let $V \in \mathcal{A}_l$ and let n, ϵ, l be positive numbers. Then there exists $\delta > 0$ such that for each continuously differentiable mapping $y : [0, \infty) \to X$ and each $x \in W^{1,1}(0,l;X)$ satisfying*

$$|f(y(0))| \leq n \tag{6}$$
$$y'(t) = Vy(t), \quad t \in [0, \infty), \tag{7}$$

and

$$y(0) = x(0), \quad \|x'(t) - Vx(t)\| \le \delta, \ \textit{a.e. on } [0,l], \tag{8}$$

the following inequality holds:

$$\|x(t) - y(t)\| \le \epsilon \quad \textit{for all } t \in [0,l]. \tag{9}$$

Proof. We may assume that $\epsilon < \frac{1}{2}$. By C(i), there is $n_1 > n$ such that

$$\text{if } z \in X, \ f(z) \le n+2, \quad \text{then } \|z\| \le n_1. \tag{10}$$

Since $V \in \mathcal{A}_l$, there exists a constant $L > 1$ such that

$$\|Vz_1 - Vz_2\| \le L\|z_1 - z_2\| \quad \text{for all } z_1, z_2 \in B(n_1 + 1). \tag{11}$$

Choose a positive number δ such that

$$\delta(l+1)e^{Ll} < \frac{\epsilon}{2}. \tag{12}$$

Assume that a continuously differentiable mapping $y : [0,\infty) \to X$ satisfies (6), (7), and that $x \in W^{1,1}(0,l;X)$ satisfies (8). By Proposition 2.1, the inclusion $V \in \mathcal{A}_l$ and (7), the function $f(y(\cdot))$ is decreasing on $[0,\infty)$. Therefore in view of (6),

$$f(y(t)) \le f(y(0)) \le n \quad \text{for all } t \in [0,\infty). \tag{13}$$

When combined with (10), inequality (13) implies that

$$\|y(t)\| \le n_1 \quad \text{for all } t \in [0,\infty). \tag{14}$$

Set

$$\Omega = \{s \in [0,l] : \ \|x(t)\| \le n_1 + 1 \quad \text{for all } t \in [0,s]\}. \tag{15}$$

It follows from (8) and (14) that $\|x(0)\| = \|y(0\| \le n_1$. We conclude that $\Omega \neq \emptyset$. Set

$$\tau = \sup \Omega. \tag{16}$$

It is easy to see that $\tau \le l$. Since the function $\|x(\cdot)\|$ is continuous, it follows from (15) and (16) that $\tau \in \Omega$. Hence

$$\|x(t)\| \le n_1 + 1 \quad \text{for all } t \in [0,\tau]. \tag{17}$$

Let $s \in [0, \tau]$. It follows from (7) and (8) that

$$\begin{aligned}\|x(s) - y(s)\| &= \left\|x(0) + \int_0^s x'(t)\,dt - (y(0) + \int_0^s y'(t)\,dt)\right\| \\ &= \left\|\int_0^s (x'(t) - y'(t))\,dt\right\| \\ &= \left\|\int_0^s (x'(t) - Vy(t))\,dt\right\| \\ &\le \int_0^s \|x'(t) - Vy(t)\|\,dt \\ &\le \int_0^s \|x'(t) - Vx(t)\|\,dt + \int_0^s \|Vx(t) - Vy(t)\|\,dt \\ &\le \delta s + \int_0^s \|Vx(t) - Vy(t)\|\,dt,\end{aligned}$$

so that

$$\|x(s) - y(s)\| \le \delta s + \int_0^s \|Vx(t) - Vy(t)\|\,dt. \tag{18}$$

By (17), (14) and (11), $\int_0^s \|Vx(t) - Vy(t)\|\,dt \le \int_0^s L\|x(t) - y(t)\|\,dt$. When combined with (18), this inequality implies that

$$\|x(s) - y(s)\| \le \delta s + L\int_0^s \|x(t) - y(t)\|\,dt \le \delta l + L\int_0^s \|x(t) - y(t)\|\,dt.$$

Since this inequality holds for all $s \in [0, \tau]$, it follows from the Gronwall inequality that for any $s \in [0, \tau]$, $\|x(s) - y(s)\| \le \delta l \exp(\int_0^s L\,dt) = \delta l \exp(Ls) \le \delta l e^{Ll}$. Combined with (12), this implies that

$$\|x(s) - y(s)\| < \frac{\epsilon}{2} \quad \text{for all } s \in [0, \tau]. \tag{19}$$

Since $\epsilon < \frac{1}{2}$, it follows from (19) and (14) that $\|x(\tau)\| < \|y(\tau)\| + \frac{1}{2} \le n_1 + \frac{1}{2}$. If $\tau < l$, then there is $\tau_1 \in (\tau, l)$ such that $\|x(t)\| \le n_1 + 1$ for all $t \in [\tau, \tau_1]$ whence $\tau_1 \in \Omega$, a contradiction. Therefore $\tau = l$ and by (19), $\|x(s) - y(s)\| < \frac{\epsilon}{2}$ for all $s \in [0, l]$. Thus (9) holds and Lemma 3.1 is proved. □

4. Proof of Theorem 1.4

There is $r_0 > 0$ such that

$$|f(z) - f(\bar{x})| < 1 \quad \text{for all } z \in X \text{ satisfying } \|z - \bar{x}\| \le r_0. \tag{20}$$

We may assume without loss of generality that

$$n > |f(\bar{x})| + 4 \quad \text{and} \quad \epsilon < \frac{r_0}{4}. \tag{21}$$

By Theorem 1.3, there exists $\bar{T} > 0$ such that the following property holds:

(P2) For each continuously differentiable mapping $y : [0, \infty) \to X$ satisfying $|f(y(0))| \le n$, $y'(t) = Vy(t)$ for all $t \in [0, \infty)$, the inequality $\|y(t) - \bar{x}\| \le \frac{\epsilon}{4}$ holds for all $t \in [\bar{T}, \infty)$.

By Lemma 3.1, there exists $\delta > 0$ such that the following property holds:

(P3) For each continuously differentiable mapping $y : [0, \infty) \to X$ satisfying $|f(y(0))| \le n$, $y'(t) = Vy(t)$, $t \in [0, \infty)$, each $s \in \{2\bar{T}, 8\bar{T}\}$, and each $x \in W^{1,1}(0, s; X)$ satisfying $y(0) = x(0)$, $\|x'(t) - Vx(t)\| \le \delta$, a.e. on $[0, s]$, the inequality $\|x(t) - y(t)\| \le \frac{\epsilon}{4}$ holds for all $t \in [0, s]$.

Now assume that $T \ge 8\bar{T}$, and that $x \in W^{1,1}(0, T; X)$ satisfies

$$|f(x(0))| \le n \tag{22}$$

and

$$\|x'(t) - Vx(t)\| \le \delta, \quad \text{a.e. on } [0, T]. \tag{23}$$

By Proposition 1.1, there is a continuously differentiable mapping $y : [0, \infty) \to X$ such that

$$y(0) = x(0), \quad y'(t) = Vy(t), \ t \in [0, \infty). \tag{24}$$

By (P3) (with $s = 8\bar{T}$) and (22)–(24),

$$\|x(t) - y(t)\| \le \frac{\epsilon}{4}, \quad t \in [0, 8\bar{T}]. \tag{25}$$

By (P2), (22) and (24),

$$\|y(t) - \bar{x}\| \le \frac{\epsilon}{4}, \quad t \in [\bar{T}, \infty). \tag{26}$$

Inequalities (25) and (26) imply that

$$\|x(t) - \bar{x}\| \le \frac{\epsilon}{2}, \quad t \in [\bar{T}, 8\bar{T}]. \tag{27}$$

We will now show that

$$\|x(t) - \bar{x}\| \le \epsilon \quad \text{for all } t \in [\bar{T}, T]. \tag{28}$$

Assume the contrary. Then there is $t \in [\bar{T}, T]$ such that

$$\|x(t) - \bar{x}\| > \epsilon. \tag{29}$$

In view of (27),

$$t > 8\bar{T}. \tag{30}$$

By (27), (29) and (30), there is a number s such that $s \in (8\bar{T}, T)$, $\|x(s) - \bar{x}\| = \epsilon$ and

$$\|x(t) - \bar{x}\| < \epsilon \quad \text{for all } t \in [\bar{T}, s). \tag{31}$$

Define

$$x_1(t) = x(t + s - 2\bar{T}), \quad t \in [0, 2\bar{T}]. \tag{32}$$

By Proposition 1.1, there exists a continuously differentiable mapping $y_1 : [0, \infty) \to X$ such that

$$y_1(0) = x_1(0), \quad y_1'(t) = V y_1(t),\ t \in [0, \infty). \tag{33}$$

In view of (31),

$$s - 2\bar{T} > 6\bar{T}. \tag{34}$$

By (32), (34), (31) and (21), $\|x_1(0) - \bar{x}\| = \|x(s - 2\bar{T}) - \bar{x}\| < \epsilon < \frac{r_0}{4}$. Together with (20) and (21), this implies that

$$|f(x_1(0)) - f(\bar{x})| < 1 \quad \text{and} \quad |f(x_1(0))| < |f(\bar{x})| + 1 < n. \tag{35}$$

By (P2), (35) and (33),

$$\|y_1(t) - \bar{x}\| \le \frac{\epsilon}{4} \quad \text{for all } t \ge \bar{T}. \tag{36}$$

In view of (32) and (23), for a.e. $t \in [0, 2\bar{T}]$, $x_1'(t) = x'(t + s - 2\bar{T})$, $Vx'(t + s - 2\bar{T}) = V x_1(t)$, and

$$\|x_1(t) - V x_1(t)\| \le \delta. \tag{37}$$

By (37), (33), (35), (32) and (P3) (with $s = 2\bar{T}$),

$$\|x_1(t) - y_1(t)\| \le \frac{\epsilon}{4}, \quad t \in [0, 2\bar{T}]. \tag{38}$$

By (32), (38) and (36),

$$\|x(s) - \bar{x}\| = \|x_1(2\bar{T}) - \bar{x}\| \le \|x_1(2\bar{T}) - y_1(2\bar{T})\| + \|y_1(2\bar{T}) - \bar{x}\| \le \frac{\epsilon}{2}.$$

This contradicts (31). The contradiction we have reached proves (28). Hence the conclusion of Theorem 1.4 holds with $\tau = \bar{T}$. □

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References

[1] Aizicovici, S., Reich, S. and Zaslavski, A. J., Convergence theorems for continuous descent methods. *J. Evol. Equ.* 4 (2004), 139 – 156.

[2] Aizicovici, S., Reich, S. and Zaslavski, A. J., Most continuous descent methods converge. *Arch. Math. (Basel)* 85 (2005), 268 – 277.

[3] Aizicovici, S., Reich, S. and Zaslavski, A. J., Stability of convergent continuous descent methods. *Electron. J. Diff. Equ.* 32 (2007), 1 – 6.

[4] Choudhary, R., A stability theorem for convergence of a Lyapunov function along trajectories of nonexpansive semigroups. *Electron. J. Diff. Equ.* 120 (2006), 1 – 10.

[5] Choudhary, R., Generic convergence of a convex Lyapunov function along trajectories of nonexpansive semigroups in Hilbert space. *J. Nonlin. Convex Anal.* 7 (2006), 245 – 268.

[6] Curry, H. B., The method of steepest descent for nonlinear minimization problems. *Quart. Appl. Math.* 2 (1944), 258 – 261.

[7] De Blasi, F. S. and Myjak, J., Sur la convergence des approximations successives pour les contractions non linéaires dans un espace de Banach (in French). *C. R. Acad. Sci. Paris Ser. A–B* 283 (1976), 185 – 187.

[8] Hiriart-Urruty, J.-B. and Lemaréchal, C., *Convex Analysis and Minimization Algorithms.* Berlin: Springer 1993.

[9] Neuberger, J. W., *Sobolev Gradients and Differential Equations.* Lect. Notes Math. 1670. Berlin: Springer 1997.

[10] Reich, S. and Zaslavski, A. J., Generic convergence of descent methods in Banach spaces. *Math. Oper. Res.* 25 (2000), 231 – 242.

[11] Reich, S. and Zaslavski, A. J., The set of divergent descent methods in a Banach space is σ-porous. *SIAM J. Optim.* 11 (2001), 1003 – 1018.

[12] Reich, S. and Zaslavski, A. J., Porosity of the set of divergent descent methods. *Nonlin. Anal.* 47 (2001), 3247 – 3258.

[13] Reich, S. and Zaslavski, A. J., Two convergence results for continuous descent methods. *Electron. J. Diff. Equ.* 24 (2003), 1 – 11.

[14] Vidossich, G., Most of the successive approximations do converge. *J. Math. Anal. Appl.* 45 (1974), 127 – 131.

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