Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 28 (2009), 223–232

Strongly Extreme Points in Musielak-Orlicz Spaces with the Orlicz Norm

Ping Wang, FeiFei Yu and Yunan Cui

Abstract. In this paper, criteria of strongly extreme points in Musielak–Orlicz spaces endowed with the Orlicz norm are given.

Keywords. Musielak–Orlicz function spaces, strongly extreme points, Orlicz norm, Local Δ-condition.

Mathematics Subject Classification (2000). 46B20, 46E30

1. Introduction

It is well known that both strongly extreme points and extreme points are important concepts in Banach Geometry Theory [1, 3, 15]. There are a lot of discussions about the criteria for strongly extreme points and extreme points (see [2–12]). The criteria for strongly extreme points and extreme points in the classical Orlicz spaces have been given in [5, 8, 12] already. However, because of the complication of Musielak–Orlicz spaces, at present there are only criteria for extreme points which were obtained by A. Kamińska [9] in 1981. But the criteria for strongly extreme points have not been discussed yet. In this paper, by virtue of the local Δ-condition which has been introduced [14], necessary and sufficient conditions for strongly extreme points in Musielak–Orlicz function spaces equipped with the Orlicz norm were given.

Let $(X, \|\cdot\|)$denote a Banach space, $B(X)$ and $S(X)$ denote the unit ball and the unit sphere of X, respectively. A point $x \in S(X)$ is said to be a strongly extreme point if for any $x_n, y_n \in X$ with $\|x_n\| \to 1, \|y_n\| \to 1$ and $\frac{x_n+y_n}{2} = x$, there holds $\|x_n - y_n\| \to 0 (n \to \infty)$. It is obvious that strongly extreme points are extreme points, but the converse is not true.

Ping Wang: Department of Mathematics, Harbin University of Science and Technology, Harbin 150080, P. R. China; w-ping8981@163.com
Yunan Cui: Department of Mathematics, Harbin University of Science and Technology, Harbin 150080, P. R. China; cuiya@hrbust.edu.cn
FeiFei Yu: Department of Mathematics, TianJin University of Science and Technology, Tianjin 200080, P. R. China;yuff@tjus.edu.cn

Let$(T, \sum, \mu)$be a nonatomic and complete measure space with $\mu(T) < \infty$. By M we denote a *Musielak–Orlicz function*, i.e., $M : T \times R \to [0, +\infty]$ satisfies the conditions:

(1) $M(\cdot, u)$ is a $\sum$-measurable function for any $u \in R$;
(2) $M(t, 0) = 0$, $\lim_{u \longrightarrow \infty} M(t, u) = \infty$, and there exists $u_t > 0$ such that $M(t, u_t) < \infty$ for a.e. $t \in T$.
(3) $M(\cdot, u)$ is even, convex and left-side continuous with respect to $u \in R$.

A point u_0 is said to be a point of strict convexity for $M(t, \cdot)$ if $M(t, u_0) < \frac{1}{2}M(t, u_0 - \epsilon) + \frac{1}{2}M(t, u_0 + \epsilon)$ holds for any $\epsilon > 0$.

We say that M satisfy the $\Delta[A]$-condition ($M \in \Delta[A]$ for short) if there are a positive constant $K > 0$, a set $C \in \sum$ with $\mu(C) = 0$ and a nonnegative function $\delta \in L(T, \sum, \mu)$ such that the inequality $M(t, 2u) \leq KM(t, u) + \delta(t)$ holds for all $t \in A \backslash C$ and $u \in R$. M is said to satisfy the Δ-condition if $A = T$. In addition, we denote

$$e(t) = \sup\{u \geq 0 : M(t, u) = 0\}$$
$$B(t) = \sup\{u \geq 0 : M(t, u) < \infty\},$$

and we may assume that $u = 0$ is not a point of strict convexity of $M(t, \cdot)$ if $e(t) > 0$. Let x denote the $\sum$-measurable real function on $(T, \sum, \mu)$. The convex modular of x with respect to M is $\rho_M(x) = \int_T M(t, x(t))\, dt$. The linear space $\{x : \rho_M\big(\frac{x}{\lambda}\big) < \infty \text{ for some } \lambda > 0\}$ equipped with the Amemiya–Orlicz norm $\|x\|^0 = \inf\{k > 0 : \frac{1}{k}\big(1 + \rho_M(kx)\big)\}$ is a Banach space, and we call it Musielak–Orlicz function space endowed with the Luxemburg norm, denoted by $L_M(T)$ (see [13]).

2. General results

In this section, we present some results relating the criteria for strongly extreme points in Musielak–Orlicz spaces endowed with the Orlicz norm. First, we look at the following lemma:

Lemma 2.1. *If $M \in \Delta$ and $x_n(t) \to 0$ for μ-a.e. $t \in T$, where $\mu(T) < \infty$, then $\|x_n\|^0 \to 0$ if and only if $\rho_M(x_n) \to 0$.*

Proof. We only need to prove that $\rho_M(x_n) \to 0$ implies that $\|x_n\|^0 \to 0$ as $n \to \infty$. Using $M \in \triangle$, for any $\varepsilon > 0$ there exists $K > 0$ and $\delta \in L^1(T, \Sigma, u)$ such that the inequality $M(t, \frac{u}{\varepsilon}) \leq KM(t, u) + \delta(t)$ holds for μ-a.e. $t \in T$ and all $u \in R$. By the Fegorov theorem, there exists $e_0 \in T$ with $\mu(e_0) < \eta$ such that $M(t, \frac{x_n(t)}{\varepsilon})$ convergent to 0 uniformly in $T \setminus e_0$. Hence, $\int_{T\setminus e_0} M(t, \frac{x_n(t)}{\varepsilon})d\mu < \frac{1}{3}$, when n large enough. By $\rho_M(x_n)\to 0$, we have that $\rho_M(x_n) < \frac{1}{3K}$ when n large enough. Therefore,

$$\begin{aligned}\int_T M\left(t,\frac{x_n(t)}{\varepsilon}\right)d\mu &= \int_{T\setminus e_0} M\left(t,\frac{x_n(t)}{\varepsilon}\right)d\mu + \int_{e_0} M\left(t,\frac{x_n(t)}{\varepsilon}\right)d\mu \\ &< \frac{1}{3} + \int_{e_0}\big(KM(t,x_n(t)) + \delta(t)\big)d\mu \\ &< \frac{1}{3} + K\frac{1}{3K} + \frac{1}{3} = 1.\end{aligned}$$

This implies that $\|x_n\|^0 < \varepsilon$ when n large enough, i.e., $\|x_n\|^0 \to 0$ as $n\to\infty$. □

Theorem 2.2. *$x \in S(L_M)$ is a strongly extreme point if and only if:*

1. *$K(x) \neq \phi$, where $K(x) = \{k : k^{-1}[1+\rho_M(kx)] = \|x\|_M^0\}$;*
2. *for $k \in K(x), k|x(t)|$ is a point of strict convexity for $M(t,\cdot)(a.e.\, t\in T)$;*
3. *if $T' \subset T$, $\xi_M(kx|_{T'}) < 1$ implies $M \in \Delta(T')$, where $\xi_M(u) = \inf\{\lambda > 0 : \rho_M(\frac{u}{\lambda}) < \infty\}$.*

Proof. Without loss of generality, we assume $x(t) \geq 0$ (a.e. $t\in T$).

`Necessity.` If (1) is not necessary, then $1 = \|x\|^0 = \int_T x(t)B(t)\,dt$. Take $T_1, T_2 \subset T$ such that $T_1\cup T_2 = \operatorname{supp} x$, $T_1\cap T_2 = \emptyset$ and $\int_{T_1} x(t)\widetilde{B}(t)\,dt = \int_{T_2} x(t)\widetilde{B}(t)\,dt$. Let

$$y(t) = \begin{cases}2x(t), & t\in T_1\\ 0, & t\in T\setminus T_1\,,\end{cases} \qquad z(t) = \begin{cases}2x(t), & t\in T_2\\ 0, & t\in T\setminus T_2\,.\end{cases}$$

Thus $y\neq z, y+z = 2x, \|y\|^0 = \int_{T_1} 2x(t)\widetilde{B}(t)\,dt = \int_T x(t)\widetilde{B}(t)\,dt = 1$, in the same way we have $\|z\|^0 = 1$, this contradicts the fact that x is an extreme point.

If (2) fails, then $\mu\{t\in T : kx(t) \in (a_t,b_t)\} > 0$, where (a_t,b_t) is some affine interval of $M(t,\cdot)$ and there exists $\epsilon_0 > 0$ small enough such that $\mu A > 0$, where $A = \{t\in T : kx(t)\in(a_t+\epsilon_0, b_t-\epsilon_0)\}$. We can suppose that $M(t,u) = \alpha(t)u+\beta(t)$ for $u\in(a_t,b_t)$. Choose $B, C\subset A, B\cap C = \phi$, satisfying $\int_B \alpha(t)\,dt = \int_C \alpha(t)\,dt$, and define

$$y(t) = \begin{cases}x(t)+\frac{\varepsilon}{k}, & t\in B\\ x(t)-\frac{\varepsilon}{k}, & t\in C\\ x(t), & t\in T\setminus(B\cup C),\end{cases} \qquad z(t) = \begin{cases}x(t)+\frac{\varepsilon}{k}, & t\in C\\ x(t)-\frac{\varepsilon}{k}, & t\in B\\ x(t), & t\in T\setminus(B\cup C).\end{cases}$$

Then $y \neq z, y+z=2x$, and

$$\begin{aligned}\|y\|^0 &\le \frac{1}{k}(1+\rho_M(ky)) \\ &= \frac{1}{k}\left\{1+\rho_M(kx|_{T\setminus(B\cup C)})+\int_B[\alpha(t)(kx(t)+\epsilon)+\beta(t)]\,dt\right. \\ &\quad\left.+\int_B[\alpha(t)(kx(t)-\epsilon)+\beta(t)]\,dt\right\} \\ &= \frac{1}{k}\left\{1+\rho_M(kx|_{T\setminus(B\cup C)})+\int_B M(t,kx(t))\,dt+\int_B M(t,kx(t))\,dt\right\} \\ &= \frac{1}{k}(1+\rho_M(kx))=1.\end{aligned}$$

In the same way we get $\|z\|^0 \le 1$, this shows that x is not an extreme point.

If the condition (3) fails, then there exists $A\subset T, 0<s<1$, such that $\xi_M(kx|_A)<1-s<1$ and $M\notin\Delta(A)$. There are $A_n\subset A$ with $\mu A_n\to 0$ as $n\to\infty$, and a sequence $\{u_n\}$ in $S(L_M^0)$ such that $u_n=u_n\chi_{A_n}$ and $\rho_M(u_n)\to 0$ (see [9]). Let

$$x(t)=\begin{cases}x(t), & t\in T\setminus A_n\\ x(t)+\frac{s}{k}u_n(t), & t\in A_n,\end{cases}\qquad y_n(t)=\begin{cases}x(t), & t\in T\setminus A_n\\ x(t)-\frac{s}{k}u_n(t), & t\in A_n.\end{cases}$$

Then $x_n+y_n=2x$ and $\|x_n-y_n\|^0=\frac{2s}{k}>0$, but

$$\begin{aligned}\|x_n\|^0 &\le \frac{1}{k}\left[1+\rho_M(kx|_{T\setminus A_n})+\int_{A_n}M(t,kx(t)+su_n(t))\,dt\right] \\ &\le \frac{1}{k}\left[1+\rho_M(kx|_{T\setminus A_n})+(1-s)\int_{A_n}M\left(t,\frac{kx(t)}{1-s}\right)dt\right. \\ &\quad\left.+s\int_{A_n}M(t,u_n(t))\,dt\right] \\ &\le \|x\|^0+\frac{1-s}{k}\rho_M\left(\frac{x}{1-s}|_{A_n}\right)+\frac{s}{k}\rho_M(u_n)\to 1.\end{aligned}$$

Thus $\lim_{n\to\infty}\|x_n\|^0\le 1$, Similarity we get $\lim_{n\to\infty}\|y_n\|^0\le 1$. Combining these facts with $x_n+y_n=2x$, we have $\lim_{n\to\infty}\|x_n\|^0=1, \lim_{n\to\infty}\|y_n\|^0=1$, this contradicts the fact that x is a strongly extreme point.

`Sufficiency`. Put $\|x_n\|^0\to 1, \|y_n\|^0\to 1, x_n+y_n=2x$, we want to prove $\|x_n-y_n\|^0\to 0 (n\to\infty)$.

First, we will show that $\|x_n\|^0=\frac{1}{k_n}[1+\rho_M(k_nx_n)], \|y_n\|^0=\frac{1}{h_n}[1+\rho_M(h_ny_n)]$ and $\overline{k}=\sup_n\{k_n,h_n\}<\infty$ can be assumed. Otherwise, $k_n\to\infty$ or $K(x_n)=\phi$, then we can assume that $k_n \to \infty$ satisfies $\|x_n\|^0 > \frac{1}{k_n}[1+\rho_M(k_n x_n)] - \frac{1}{n}$. Since $\frac{\|x_n+x\|}{2} \to 1$, we have

$$\begin{aligned} 0 &\leftarrow \|x_n\|^0 + \|x\|^0 - \|x_n+x\|^0 \\ &> \frac{1}{k_n}[1+\rho_M(k_n x_n)] - \frac{1}{n} + \frac{1}{k}[1+\rho_M(kx)] - \frac{k_n+k}{k_n k}\left[1+\rho_M\left(\frac{k_n}{k_n+k}(x_n+x)\right)\right] \\ &> -\frac{1}{n}. \end{aligned}$$

Hence,

$$\frac{k_n+k}{2k_n k}\left[1+\rho_M\left(\frac{2k_n k}{k_n+k}\cdot\frac{x_n+x}{2}\right)\right] - \frac{\|x_n+x\|^0}{2} \to 0.$$

If $\frac{\|x_n+x\|^0}{2} = \frac{1}{\omega_n}[1+\rho_M(\omega_n \cdot \frac{x_n+x}{2})]$, then $\lim_{n\to\infty}\omega_n = \lim_{n\to\infty}\frac{2kk_n}{k+k_n} = 2k$, i.e., $\{\omega_n\}$ is bounded. Let $x_n^{'} = \frac{x_n+x}{2}$, $y_n^{'} = \frac{y_n+x}{2}$, we also get $x_n^{'}+y_n^{'} = 2x$, $\|x_n^{'}\|^0 \to 1$, $\|y_n^{'}\|^0 \to 1$ and $\|x_n^{'} - y_n^{'}\|^0 \to 0$ if and only if $\|x_n - y_n\|^0 \to 0 (n\to\infty)$.

`Step1:` We will show that $k_n x_n - kx \xrightarrow{\mu} 0$ in measure and $k_n \to k$ as $n \to \infty$. Since

$$\begin{aligned} 0 &\leftarrow \|x_n\|^0 + \|y_n\|^0 - \|2x\|^0 \\ &\geq \frac{1}{k_n}[1+\rho_M(k_n x_n)] + \frac{1}{h_n}[1+\rho_M(h_n y_n)] - \frac{k_n+h_n}{k_n h_n}\left[1+\rho_M\left(\frac{k_n h_n}{k_n+h_n}(x_n+y_n)\right)\right] \\ &= \frac{k_n+h_n}{k_n h_n}\left[\frac{h_n}{k_n+h_n}\rho_M(k_n x_n) + \frac{k_n}{k_n+h_n}\rho_M(h_n y_n) - \rho_M\left(\frac{k_n h_n}{k_n+h_n}(x_n+y_n)\right)\right] \\ &\geq 0, \end{aligned}$$

so $\frac{k_n+h_n}{k_n h_n}\left[1+\rho_M(\frac{k_n h_n}{k_n+h_n}(x_n+y_n))\right] - 2\|x\|^0 \to 0$, i.e.,

$$\frac{k_n+h_n}{2k_n h_n}\left[1+\rho_M\left(\frac{2k_n h_n}{k_n+h_n}\left(\frac{x_n+y_n}{2}\right)\right)\right] - \|x\|^0 \to 0.$$

Combining the last conditions with the fact that $K(x)$ is singleton, we get $\lim_{n\to\infty}\frac{2k_n h_n}{h_n+k_n} = k$. If $k_n x_n - kx \xrightarrow{\mu} 0$ fails , there exist $\epsilon > 0$, $\sigma > 0$ such that $\mu A_n > \sigma$, where $A_n = \{t \in T : |k_n x_n(t) - h_n y_n(t)| > \epsilon\}$. Since $\overline{k} - 1 \geq k_n - 1 = \rho_M(k_n x_n)$, $\overline{k} - 1 \geq \rho_M(h_n y_n)$, there exists large enough $d > 0$, such that $\mu B_n < \frac{\sigma}{5}$, where $B_n = \{t \in T : |k_n x_n(t)| > d$ or $|h_n y_n(t)| > d\}$. Since $kx(t)$ is a point of strict convexity of $M(t,\cdot)$ and $0 < \frac{1}{1+\bar{k}} \leq \frac{k_n}{k_n+h_n}$, $\frac{h_n}{k_n+h_n} < \frac{\bar{k}}{1+\bar{k}} < \infty$, there holds $M(t,u) < \frac{k_n}{k_n+h_n}M(t,v) + \frac{h_n}{k_n+h_n}M(t,w)$ as $u \in [kx(t) - \frac{\epsilon}{2(1+\bar{k})}, kx(t) + \frac{\epsilon}{2(1+\bar{k})}]$, $u = \frac{k_n}{k_n+h_n}v + \frac{h_n}{k_n+h_n}w$ and $|v-w| \geq \epsilon$. Thus, there exists $\delta(t) \in (0,1)$ satisfying

$$M(t,u) \leq [1-\delta(t)]\left[\frac{k_n}{k_n+h_n}M(t,v) + \frac{h_n}{k_n+h_n}M(t,w)\right],$$

where $u \in [kx(t) - \frac{\epsilon}{2(1+\overline{k})}, kx(t) + \frac{\epsilon}{2(1+\overline{k}})]$, $\epsilon \le |v - w| \le d$. Because of $\lim_{n\to\infty} \frac{2k_n}{h_n+k_n} = k$, we get $\mu C_n < \frac{\sigma}{5}$ for n large enough, where $C_n = \big\{t \in T : \frac{k_n h_n}{k_n+h_n}(x_n(t) + y_n(t)) = \frac{2k_n h_n}{k_n+h_n} x(t) \notin \big[kx(t) - \frac{\epsilon}{2(1+\overline{k})}, kx(t) + \frac{\epsilon}{2(1+\overline{k})}\big]\big\}$. Since $\delta(t) > 0$, there exists $\delta_0 > 0$ small enough such that $\mu D < \frac{\sigma}{5}$, where $D = \{t \in T : \delta(t) < \delta_0\}$. Since $M(t, kx(t) + \frac{\epsilon}{2(1+\overline{k})}) > 0$, there is $\theta > 0$ small enough such that $\mu E < \frac{\sigma}{5}$, where $E = \big\{t \in T : M\big(t, kx(t) + \frac{\epsilon}{2(1+\overline{k})}\big) < \theta\big\}$. Defining $\Omega_n = A_n \setminus (B_n \cup C_n \cup D \cup E)$, we have $\mu\Omega_n > \frac{\sigma}{5}$. If $t \in \Omega_n$, then $\epsilon \le |k_n x_n(t) - h_n y_n(t)| \le 2d$,

$$\frac{k_n h_n}{k_n + h_n}(x_n(t) + y_n(t)) \in \left[x(t) - \frac{\epsilon}{2}, x(t) + \frac{\epsilon}{2}\right], \quad \delta(t) \ge \delta_0$$

and $M(t, kx(t) + \frac{\epsilon}{2(1+\overline{k})}) \ge \theta$, because $\frac{k_n}{k_n+h_n} < \frac{\overline{k}}{1+\overline{k}}$ implies $\frac{k_n+h_n}{k_n h_n} > \frac{1}{\overline{k}}$ and

$$\begin{aligned}
0 \leftarrow & \frac{k_n + h_n}{k_n h_n} \int_T \left[\frac{h_n}{k_n + h_n} M(t, k_n x_n(t)) + \frac{k_n}{k_n + h_n} M(t, h_n y_n(t))\right] dt \\
& - \frac{k_n + h_n}{k_n h_n} \int_T M\left(t, \frac{k_n h_n}{k_n + h_n}(x_n(t) + y_n(t))\right) dt \\
\ge & \frac{k_n + h_n}{k_n h_n} \int_{\Omega_n} \delta_0 \left[\frac{h_n}{k_n + h_n} M(t, k_n x_n(t)) + \frac{k_n}{k_n + h_n} M(t, h_n y_n(t))\right] dt \\
\ge & \frac{\delta_0}{\overline{k}} \int_{\Omega_n} M\left(t, kx(t) + \frac{\epsilon}{2t(1 + \overline{k})}\right) dt \\
\ge & \frac{\delta_0}{\overline{k}} \cdot \theta \cdot \frac{\sigma}{5} > 0.
\end{aligned}$$

This contradiction shows that $k_n x_n - kx \xrightarrow{\mu} 0$.

Since $\frac{k_n h_n}{k_n+h_n}(x_n + y_n)$ is a convex combination between $k_n x_n$ and $h_n y_n$, we get that $k_n x_n - \frac{k_n h_n}{k_n+h_n}(x_n + y_n) = k_n x_n - \frac{2k_n h_n}{k_n+h_n} x \xrightarrow{\mu} 0$ holds. Since

$$\lim_{n\to\infty} \rho_M \left(\frac{k_n h_n}{k_n + h_n}(x_n + y_n)\right) \le \lim_{n\to\infty} \left(\frac{2k_n h_n}{k_n + h_n} - 1\right) = k - 1 = \rho_M(x),$$

we have

$$\lim_{\mu e\to 0} \sup_n \int_e M\left(t, \frac{k_n h_n}{k_n + h_n}(x_n(t) + y_n(t))\right) dt = \lim_{\mu e\to 0} \sup_n \int_e M(t, kx(t))\, dt = 0\,.$$

Combining this fact with (1), we get

$$\lim_{\mu e\to 0} \sup_n \int_e \left[\frac{h_n}{k_n + h_n} M(t, k_n x_n(t)) + \frac{k_n}{k_n + h_n} M(t, h_n y_n(t))\right] dt = 0.$$

Thus

$$\lim_{\mu e\to 0}\sup_n \int_e [M(t,k_nx_n(t))+M(t,h_ny_n(t))]\,dt=0.$$

Using the condition $k_nx_n - kx \xrightarrow{\mu} 0$, $k_n = 1+\rho_M(k_nx_n)\to 1+\rho_M(kx)=k$ $(n\to\infty)$, i.e., $k_n\to k$ $(n\to\infty)$ and in the same way $h_n\to k$ $(n\to\infty)$.

`Step 2:` We will show that $\|k_nx_n-kx\|^0\to 0$.

If $\mu T_0=\mu\{t\in T: x(t)=0\}>0$, by the condition (3) we know that $M\in\Delta(T_0)$. By the Riesz lemma, we know that $k_nx_n(t)\to kx(t)$ (a.e. $t\in T$) (choose a subsequence if necessary), combining this fact with

$$\lim_{\mu e\to 0}\sup_n \int_e [M(t,k_nx_n(t))]\,dt=0,$$

we know that $\rho_M(k_nx_n|_{T_0})\to 0$, then by the Lemma 1 we know that $\|(k_nx_n-kx)|_{T_0}\|^0=\|(k_nx_n)|_{T_0}\|^0\to 0$ $(n\to\infty)$. So in the following discussion, we can assume $x(t)>0$ (a.e. $t\in T$).

If $\|k_nx_n-kx\|^0\to 0$ $(n\to\infty)$ fails, there exists $\epsilon_0>0$ satisfying $\|k_nx_n-kx\|^0\geq 8\overline{k}^2\epsilon_0$. For any m, choose $\eta_m>0$ such that

$$\int_e [M(t,k_nx_n(t))+M(t,h_ny_n(t))]\,dt<\frac{1}{2^m},$$

where $e\subset T, \mu e<\eta_m$. Choose $G_m\subset T, \mu G_m<\eta_m$ such that $\frac{|k_nx_n(t)|}{kx(t)}\to 1$ and $\frac{|h_ny_n(t)|}{kx(t)}\to 1$ holds uniformly on $T\setminus G_m$. Let n_m be large enough such that $|k_{n_m}x_{n_m}(t)|<(1+\epsilon_0)kx(t), |h_{n_m}y_{n_m}(t)|<(1+\epsilon_0)kx(t)$ if $t\in T\setminus G_m$. Put

$$\Omega=\left\{t\in T:\exists m \text{ such that } \frac{|k_{n_m}x_{n_m}(t)|}{kx(t)}\geq 1+\epsilon_0 \text{ or } \frac{|k_{n_m}x_{n_m}(t)|}{kx(t)}\geq 1+\epsilon_0\right\}.$$

Then we have $\Omega\subset\bigcup_m G_m$. Since

$$\begin{aligned}\int_\Omega M(t,(1+\epsilon_0)kx(t))\,dt &= \int_{\Omega\cap\bigcup_m G_m} M(t,(1+\epsilon_0)kx(t))\,dt\\ &\leq \int_{T_0} M(t,(1+\epsilon_0)kx(t))\,dt\\ &\quad+\sum_m\int_{\Omega\cap G_m} M(t,(1+\epsilon_0)kx(t))\,dt\\ &\leq \sum_m\int_{G_m}[M(t,k_{n_m}x_{n_m}(t))+M(t,h_{n_m}y_{n_m}(t))]\,dt\\ &<\sum_m\frac{2}{2^m}=2,\end{aligned}$$

by condition (3) we know $M \in \Delta(\Omega)$. Take m large enough satisfying the condition $\|\frac{k_{n_m}h_{n_m}}{k_{n_m}+h_{n_m}}(x_{n_m}+y_{n_m}-kx)\|^0 < \epsilon_0$. Denote

$$H_m = \big\{t \in T : \frac{|k_{n_m}x_{n_m}(t)|}{kx(t)} < 1+\epsilon_0, \frac{|h_{n_m}y_{n_m}(t)|}{kx(t)} < 1+\epsilon_0\big\}$$

and put

$$\begin{aligned} H_m^{'} &= \{t \in H_m : kx(t) \le k_{n_m}x_{n_m}(t)\} \\ H_m^{''} &= \{t \in H_m : k_{n_m}x_{n_m}(t) \le kx(t) \le h_{n_m}y_{n_m}(t)\} \\ H_m^{'''} &= \{t \in H_m : kx(t) > \max\{k_{n_m}x_{n_m}(t), h_{n_m}y_{n_m}(t)\}\}. \end{aligned}$$

If $t \in H_m^{'}$, $k_{n_m}x_{n_m}(t) < (1+\epsilon_0)kx(t)$, then $\mathrm{k}_{n_m}x_{n_m}(t) - kx(t) < \epsilon_0 kx(t)$. Hence, $\|(k_{n_m}x_{n_m} - kx)|_{H_m^{'}}\|^0 < \epsilon_0 k \le k^{-2}\epsilon_0$. If $t \in H_m^{''}$, we also can get $\|(h_{n_m}y_{n_m} - kx)|_{H_m^{''}}\|^0 < \epsilon_0 k$. Combining this fact with

$$\frac{|k_{n_m}x_{n_m}(t) - \frac{k_{n_m}h_{n_m}}{k_{n_m}+h_{n_m}}(x_{n_m}(t)+y_{n_m}(t))|}{|h_{n_m}y_{n_m}(t) - \frac{k_{n_m}h_{n_m}}{k_{n_m}+h_{n_m}}(x_{n_m}(t)+y_{n_m}(t))|} < \frac{\frac{\overline{k}}{1+\overline{k}}}{\frac{1}{1+\overline{k}}} = \overline{k},$$

we know that

$$\begin{aligned} &\|(k_{n_m}x_{n_m} - kx)|_{H_m^{''}}\|^0 \\ &\le \left\|\left(k_{n_m}h_{n_m} - \frac{k_{n_m}h_{n_m}}{k_{n_m}+h_{n_m}}(x_{n_m}+y_{n_m})\right)\Big|_{H_m^{''}}\right\|^0 \\ &\quad + \left\|\left(\frac{k_{n_m}h_{n_m}}{k_{n_m}+h_{n_m}}(x_{n_m}+y_{n_m}) - kx\right)\Big|_{H_m^{''}}\right\|^0 \\ &\le \overline{k}\left[\|(h_{n_m}y_{n_m} - kx)|_{H_m^{''}}\|^0 + \left\|\left(\frac{k_{n_m}h_{n_m}}{k_{n_m}+h_{n_m}}(x_{n_m}+y_{n_m}) - kx\right)\Big|_{H_m^{''}}\right\|^0\right] + \epsilon_0 \\ &\le \overline{k}(\epsilon_0 k + \epsilon_0) + \epsilon_0 < 3k^{-2}\epsilon_0. \end{aligned}$$

If $t \in H_m^{'''}$, $k_{n_m}x_{n_m}(t) > h_{n_m}y_{n_m}(t)$, then

$$\|(k_{n_m}x_{n_m} - kx)|_{H_m^{'''}}\|^0 < \left\|\left(\frac{k_{n_m}h_{n_m}}{k_{n_m}+h_{n_m}}(x_{n_m}+y_{n_m}) - kx\right)\Big|_{H_m^{'''}}\right\|^0 < \epsilon_0.$$

If $t \in H_m^{'''}$,$k_{n_m}x_{n_m}(t) < h_{n_m}y_{n_m}(t)$, then

$$\begin{aligned} \|(k_{n_m}x_{n_m} - kx)|_{H_m^{'''}}\|^0 &< \left\|\left(\frac{k_{n_m}h_{n_m}}{k_{n_m}+h_{n_m}}(x_{n_m}+y_{n_m}) - kx\right)\Big|_{H_m^{'''}}\right\|^0 \\ &\quad + \left\|\left(k_{n_m}x_{n_m} - \frac{k_{n_m}h_{n_m}}{k_{n_m}+h_{n_m}}(x_{n_m}+y_{n_m})\right)\Big|_{H_m^{'''}}\right\|^0 \\ &< (\overline{k}+1)\epsilon_0 \\ &< 2k^{-2}\epsilon_0. \end{aligned}$$

Thus we get $\|(k_{n_m}x_{n_m} - kx)|_{H_m}\|^0 < 6k^{-2}\epsilon_0$. It is clear that $T \setminus H_m \subset \Omega$ and $\frac{k_{n_m}x_{n_m}-kx}{2}|_{T\setminus H_m} \in L^0_M(\Omega)$. Since $\frac{k_{n_m}x_{n_m}(t)-kx(t)}{2} \to 0$ (a.e. $t \in T$), we have the following inequality:

$$\int_{T\setminus H_m} M\left(t, \frac{k_{n_m}x_{n_m}(t) - kx(t)}{2}\right)dt \leq \frac{1}{2}\int_{G_m}\big[M(t, k_{n_m}x_{n_m}(t)) + M(t, kx(t))\big]dt$$
$$< \frac{1}{2^m} \to 0 \quad (k \to \infty).$$

Thus $\|(k_{n_m}x_{n_m} - kx)|_{T\setminus H_m}\|^0 < k^{-2}\epsilon_0$ when m is large enough. Hence, we have $\|k_{n_m}x_{n_m} - kx\|^0 < 7k^{-2}\epsilon_0$. A contradiction to the assumption that $\|k_{n_m}x_{n_m} - kx\|^0 \geq 8k^{-2}\epsilon_0$, which completes the proof of the theorem. □

Acknowledgment. In case of Prof. Cui Yunan this paper was supported by NSF of CHINA, Grant number: 10571037 and overseas foundation of HeiLongJiang Province, Grant number: 1152hq07. Prof. Wang Ping was supported by HeiLongJiang Province, Grant number: A0204. The authors would like to give their great thanks to the referee for his many valuable suggestions and comments.

References

[1] Chen, S., *Geometry of Orlicz Spaces.* Dissertationes Math. 356. Warsaw 1996.

[2] Chen, S. and Wu, C., Extreme points and rotundity of Musielak–Orlicz spaces (in Chinese). *Northeast. Math. J.* 2 (1986), 138 – 149.

[3] Cui, Y., Hudzik, H. and Płuciennik, R., Extreme points and strongly extreme point equipped with the Orlicz norm. *Z. Anal. Anwendungen* 22 (2003)(4), 789 – 817.

[4] Cui, Y., Li, Y. and Wang, T., Packing constants and strongly extreme points in Orlicz spaces (in Chinese). *Adv. Math.* 15 (1986)(2), 217 – 219.

[5] Cui, Y. and Wang, T., Strongly extreme points in Orlicz spaces (in Chinese). *J. Math. (Wuhan)* 7 (1987), 335 – 340.

[6] Hudzik, H., Kurc, W. and Wisła, M., Strongly extreme points in Orlicz function spaces. *J. Math. Anal. Appl.* 189 (1995), 651 – 670.

[7] Hudzik, H. and Mastyło, M., Strongly extreme points in Kothe–Bochner spaces. *Rocky Mountain J. Math.* 23 (1993), 899 – 909.

[8] Hudzik, H. and Wisła, M., On extreme points of Orlicz spaces with Orlicz norm. *Collect. Math.* 44 (1993), 135 – 146.

[9] Kamińska, A., Rotundity of Musielak–Orlicz sequence spaces. *Bull. Acad. Polon. Sci. Sér. Sci. Math.* 29 (1981), 137 – 144.

[10] Kamińska, A., Strict convexity of Musielak–Orlicz sequence spaces with Orlicz norm. *J. Funct. Anal.* 50 (1983), 285 – 305.

[11] Kamińska, A., On some convexity properties of Musielak–Orlicz spaces. *Suppl. Rend. Circ. Mat. Palermo* 2 (1984), 63 – 72.

[12] Lao, B. and Zhu, X., Extreme points in Orlicz spaces (in Chinese). *J. Zhongshan Univ.* 2 (1983), 97 – 103.

[13] Musielak, J. and Orlicz, W., *On modular spaces.* Studia Math. 18 (1959), 49 – 65.

[14] Teng, Y. and Wang, T., Complex locally uniform rotundity of Musielak–Orlicz spaces. *Sci. China Ser. A* 43 (2000), 113 – 127.

[15] Yu, X., *Geometry Theory of Banach Spaces* (in Chinese). Huadong: Huadong Normal University Press 1986.

Received May 25, 2004; revised March 2, 2007