Regularity of Minimizers
of some Variational Integrals with Discontinuity

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Abstract. We prove regularity properties in the vector valued case for minimizers of variational integrals of the form

$$\mathcal{A}(u) = \int_{\Omega} A(x, u, Du) \, dx$$

where the integrand $A(x, u, Du)$ is not necessarily continuous respect to the variable $x$, grows polynomially like $|\xi|^p$, $p \geq 2$.

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1. Introduction

In this note we consider the regularity problem of minimizers of the variational integral

$$\mathcal{A}(u) = \int_{\Omega} A(x, u, Du) \, dx$$

where $\Omega$ is a bounded domain of $\mathbb{R}^m$, $u : \Omega \to \mathbb{R}^n$ is a mapping in a suitable Sobolev space, $Du = (D\alpha u^i)$ ($\alpha = 1, \ldots, m$, $i = 1, \ldots, n$). The nonnegative integrand function $A : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$ is in the class $VMO$ with respect to the variable $x$, continuous in $u$ and of class $C^2$ with respect to $Du$. It is also assumed that for some $p \geq 2$ there exist two constants $\lambda_1$ and $\Lambda_1$ such that

$$\lambda_1(1 + |\xi|^p) \leq A(x, u, \xi) \leq \Lambda_1(1 + |\xi|^p), \quad \forall (x, u, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}.$$
A minimizer for the functional \( A \) is a function \( u \in W^{1,p}(\Omega, \mathbb{R}^n) \) such that, for every \( \varphi \in W^{1,p}_0(\Omega, \mathbb{R}^n) \),
\[
A(u; \text{supp} \varphi) \leq A(u + \varphi; \text{supp} \varphi).
\]

For the case that \( A(x, u, \xi) \) is continuous in \( x \), many sharp regularity results for minimizers of \( A \) have been already known (see, e.g., [7, 8, 10, 12]). On the other hand, when \( A(\cdot, u, \xi) \) is assumed only to be \( L^\infty \), we can not expect the regularity of minimizers in general, as a famous example due to De Giorgi contained in [5] asserts. So, it seems to be natural to consider the regularity problems for \( A(x, u, \xi) \) with “mild” discontinuity with respect to \( x \). In 1996 Huang in [13] investigates regularity results for the elliptic system
\[
-D_a(a_{ij}^{\alpha\beta}(x)D_{\beta}u^j) = g_i(x) - \text{div} f_i(x), \quad i, j = 1, \ldots, n; \quad \alpha, \beta = 1, \ldots, m
\]
assuming that \( a_{ij}^{\alpha\beta} \) belong to the Sarason class \( VMO \) of vanishing mean oscillation functions. Then he generalizes Acquistapace’s [1] and Campanato’s results [7, p. 88, Theorem 3.2]. Campanato showed regularity properties under the assumption that the coefficients \( a_{ij}^{\alpha\beta} \) are in \( C^\alpha(\Omega) \). Acquistapace refined the results by Campanato, considering the coefficients in the class so-called “small multipliers of \( BMO \)”.

In a recent study made by Daněček and Viszus [4], it is considered the following functional:
\[
\int_{\Omega} \left\{ A_{ij}^{\alpha\beta}(x)D_{\alpha}u^iD_{\beta}u^j + g(x, u, Du) \right\} dx,
\]
where \( A_{ij}^{\alpha\beta} \) are in general discontinuous, more precisely belong to the vanishing mean oscillation class \( (VMO \text{ class}) \) and satisfy a strong ellipticity condition while the lower order term \( g \) is a Carathéodory function satisfying the following growth condition:
\[
|g(x, u, z)| \leq f(x) + H|z|^\kappa,
\]
where \( f \geq 0 \), a.e. in \( \Omega \), \( f \in L^p(\Omega) \), \( 2 < p \leq \infty \), \( H \geq 0 \), \( 0 \leq \kappa < 2 \).

We also recall the paper by Di Gironimo, Esposito and Sgambati [6] where is treated the Morrey regularity for minimizers of the functional
\[
\int_{\Omega} A_{ij}^{\alpha\beta}(x, u)D_{\alpha}u^iD_{\beta}u^j dx,
\]
where \( (A_{ij}^{\alpha\beta}(x, u)) \) are elliptic and of the \( VMO \) class in the variable \( x \).

In [17] the authors extend the results of [4] and [6] to the case that the functional is given by
\[
\int_{\Omega} \left\{ A_{ij}^{\alpha\beta}(x, u)D_{\alpha}u^iD_{\beta}u^j + g(x, u, Du) \right\} dx.
\]
In the note [18], it is studied the Morrey regularity for minimizer of the more general functionals
\[ A(u) = \int_{\Omega} A(x, u, Du) \, dx, \]
where \( A(x, u, \xi) \) is a nonnegative function defined on \( \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \) which is of class \( \text{VMO} \) as a function of \( x \), continuous in \( u \) and of class \( C^2 \) with respect to \( \xi \). We point out that it is assumed that for some positive constants \( \mu_0 \leq \mu_1 \),
\[ \mu_0 |\xi|^2 \leq A(x, u, \xi) \leq \mu_1 |\xi|^2 \quad \forall (x, u, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}. \]

We point out that in the above mentioned papers concerning functionals given by integrals with \( \text{VMO} \) class integrands, we have considered quadratic growth functionals. The super quadratic cases with continuous coefficients are treated in [2] and [11].

In the present note we investigate the partial regularity of the minima of \( A \), defined by (1.1) under \( p \)-growth hypothesis of the integrand function \( A \), \( p \geq 2 \). This study can be considered as an improving of [17] and [18] because of the growth condition is more general.

2. Definitions and preliminary tools

In the sequel we set
\[ Q(x, R) = \{ y \in \mathbb{R}^m : |y^\alpha - x^\alpha| < R, \ \alpha = 1, \ldots, m \} \]
a generic cube in \( \mathbb{R}^m \) having center \( x \) and side \( 2R \).

Let us now give some useful definitions, starting to the Morrey space \( L^{p,\lambda} \).

**Definition 2.1.** (see [16]). Let \( 1 \leq p < \infty \), \( 0 \leq \lambda < m \). A measurable function \( G \in L^p(\Omega, \mathbb{R}^n) \) belongs to the Morrey class \( L^{p,\lambda}(\Omega, \mathbb{R}^n) \) if
\[
\| G \|_{L^{p,\lambda}(\Omega)} = \sup_{0 < \rho < \text{diam} \Omega} \frac{1}{\rho^\lambda} \int_{\Omega \cap Q(x, \rho)} |G(y)|^p \, dy < +\infty,
\]
where \( Q(x, \rho) \) ranges in the class of the cubes of \( \mathbb{R}^m \).

**Definition 2.2.** Let \( H \in L^1(\Omega, \mathbb{R}^n) \). The integral average \( H_{x,R} \) is defined by
\[
H_{x,R} = \frac{1}{|\Omega \cap Q(x, R)|} \int_{\Omega \cap Q(x, R)} H(y) \, dy,
\]
where \( |\Omega \cap Q(x, R)| \) is the Lebesgue measure of \( \Omega \cap Q(x, R) \). In the case that we are not interested in specifying which the center is considered, we simply write \( H_R \).
Let us introduce the Bounded Mean Oscillation class.

**Definition 2.3** ([15]). Let \( H \in L^1_{\text{loc}}(\mathbb{R}^m) \). We say that \( H \) belongs to \( \text{BMO}(\mathbb{R}^m) \) if
\[
\| H \|_* \equiv \sup_{Q(x,R)} \frac{1}{|Q(x,R)|} \int_{Q(x,R)} |H(y) - H_{x,R}| dy < \infty.
\]

Let us now introduce the space of vanishing mean oscillation functions.

**Definition 2.4** ([19]). If \( H \in \text{BMO}(\mathbb{R}^m) \) and
\[
\eta(H; R) = \sup_{\rho \leq R} \frac{1}{|Q(x,\rho)|} \int_{Q(x,\rho)} |H(y) - H_{\rho}| dy,
\]
we define that \( H \in \text{VMO}(\Omega) \) if \( \lim_{R \to 0} \eta(H; R) = 0 \).

Throughout the present paper we consider \( p \geq 2 \) and \( u : \Omega \to \mathbb{R}^n \) a minimizer of the functional
\[
\mathcal{A}(u) = \int_{\Omega} A(x, u, Du) \, dx
\]
where the hypothesis on the integrand function \( A(x, u, \xi) \) are the following.

(A-1) For every \( (u, \xi) \in \mathbb{R}^n \times \mathbb{R}^{mn} \), \( A(\cdot, u, \xi) \in \text{VMO}(\Omega) \) and the mean oscillation of \( A(\cdot, u, \xi) \) vanishes uniformly with respect to \( u, \xi \) in the following sense: there exist a positive number \( \rho_0 \) and a function \( \sigma(z, \rho) : \mathbb{R}^m \times [0, \rho_0) \to [0, \infty) \) with
\[
\lim_{R \to 0} \sup_{\rho < R} \int_{Q(0, \rho) \cap \Omega} \sigma(z, \rho) \, dz = 0,
\]
such that \( A(\cdot, u, \xi) \) satisfies, for every \( x \in \overline{\Omega} \) and \( y \in Q(x, \rho_0) \cap \Omega \),
\[
|A(y, u, \xi) - A_{x,\rho}(u, \xi)| \leq \sigma(x - y, \rho)(1 + |\xi|^2)^{\frac{p}{2}} \ \forall (u, \xi) \in \mathbb{R}^n \times \mathbb{R}^{mn},
\]
where \( A_{x,\rho}(u, \xi) = \int_{Q(x,\rho) \cap \Omega} A(y, u, \xi) \, dy \).

(A-2) For every \( x \in \Omega, \xi \in \mathbb{R}^{mn} \) and \( u, v \in \mathbb{R}^n \),
\[
|A(x, u, \xi) - A(x, v, \xi)| \leq (1 + |\xi|^2)^{\frac{p}{2}} \omega(|u - v|^2),
\]
where \( \omega \) is some monotone increasing concave function with \( \omega(0) = 0 \).

(A-3) For almost all \( x \in \Omega \) and all \( u \in \mathbb{R}^n \), \( A(x, u, \cdot) \in C^2(\mathbb{R}^{mn}) \).

(A-4) There exist positive constants \( \lambda_1, \Lambda_1 \) such that
\[
\lambda_1(1 + |\xi|^p) \leq A(x, u, \xi) \leq \Lambda_1(1 + |\xi|^p)
\]
and
\[
\lambda_1(1 + |\eta|^p) \leq \frac{\partial^2 A(x, u, \xi)}{\partial \xi_i \partial \xi_j} \eta_i \eta_j \leq \Lambda_1(1 + |\eta|^p)
\]
for all \( (x, u, \xi, \eta) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \times \mathbb{R}^{mn} \).
Let us state the main theorem of the paper concerning the partial regularity of the minimizers of the functionals $A$.

**Theorem 2.5.** Assume that $\Omega \subset \mathbb{R}^m$ is a bounded domain with sufficiently smooth boundary $\partial \Omega$ and that $p \geq 2$. Let $u \in H^{1,p}(\Omega, \mathbb{R}^n)$ a minimizer of the functional $A(u, \Omega) = \int_\Omega A(x,u, Du) \, dx$ in the class $X_g(\Omega) = \{ u \in H^{1,p}(\Omega) \; ; \; u - g \in H^{1,p}_0(\Omega) \}$ for a given boundary data $g \in H^{1,s}(\Omega)$ with $s > p$. Suppose that assumptions (A-1), (A-2), (A-3) and (A-4) are satisfied. Then, for some positive $\varepsilon$, for every $0 < \tau < \min \{ 2 + \varepsilon, m(1 - \frac{p}{s}) \}$ we have

$$Du \in L^{p,\tau}(\Omega_0, \mathbb{R}^{mn}),$$

where $\Omega_0$ is a relatively open subset of $\overline{\Omega}$ which satisfies

$$\overline{\Omega} \setminus \Omega_0 = \left\{ x \in \Omega : \liminf_{R \to 0} \frac{1}{R^{m-p}} \int_{\Omega(x,R)} |Du(y)|^p \, dy > 0 \right\}.$$

Moreover, we have $H^{m-p-\delta}(\overline{\Omega} \setminus \Omega_0) = 0$ for some $\delta > 0$, where $H^r$ denotes the $r$-dimensional Hausdorff measure.

As a corollary of the above theorem we have the following partial Hölder regularity result.

**Corollary 2.6.** Let $g$, $u$ and $\Omega_0$ be as in Theorem 2.5. Assume that $p+2 \geq m$ and that $s > \max\{m,p\}$. Then, for some $\alpha \in (0,1)$, we have

$$u \in C^{0,\alpha}(\Omega_0, \mathbb{R}^n).$$

Moreover, as a corollary of the proof of Theorem 2.5, we have the following full-regularity result for the case that $A$ does not depend on $u$.

**Corollary 2.7.** Assume that $A$ and $g$ satisfy all assumptions of Theorem 2.5 and that $A$ does not depend on $u$. Let $u$ be a minimizer of $A$ in the class $X_g$ then

$$Du \in L^{p,\tau}(\Omega, \mathbb{R}^{mn}).$$

Moreover, if $p+2 \geq m$ and $s > \max\{m,p\}$, we have full-Hölder regularity of $u$, namely $u \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^n)$.

### 3. Preliminary lemmas and proof of the main results

Throughout the paper we use the following notation:

$$Q^+(x,R) = \{ y \in \mathbb{R}^m \; ; \; |y^\alpha - x^\alpha| < R, \; \alpha = 1, \ldots, m, \; y^m > 0 \}$$
for $x \in \mathbb{R}^m \cap \{ x \; ; \; x^m = 0 \}$, $R > 0$,

$$\Omega(x,R) = Q(x,R) \cap \Omega$$

$$\Gamma(x,R) = Q(x,R) \cap \partial \Omega.$$
When the center \( x \) is understood, we sometimes omit the center and write simply \( Q(R), \ Q^+(R) \) etc. For the sake of simplicity, we always assume that \( 0 < R < 1 \) in the following.

We can always reduce locally to the case of flat boundary, by means of a diffeomorphism which does not change properties of the functional assumed in the conditions (A-1)–(A-4). More precisely, we can choose a positive constant \( R_1 \) depending only on \( \partial \Omega \) which has the following properties:

1. A finite number of cubes \( \{Q(x, R_1)\} \) centered at \( x \in \partial \Omega \) cover the boundary, namely \( \partial \Omega \subset \bigcup_{k=1}^N Q(x_k, R_1), \ x_k \in \partial \Omega, \ k = 1, \ldots, N. \)

2. For every \( Q(x_k, 2R_1) \), by means of a suitable diffeomorphism, we can assume that \( x_k = 0 \) and that

\[
\Gamma(x_k, 2R_1) = Q(0, 2R_1) \cap \partial \Omega \subset \{ x \in \mathbb{R}^m ; x^m = 0 \}
\]

\[
Q(x_k, 2R_1) \cap \Omega = Q^+(0, 2R_1) = \{ x \in \mathbb{R}^m ; |x| < 2R_1, \ x^m > 0 \}.
\]

Let us define a so-called frozen functional. For some fixed point \( x_0 \in \Omega \) and \( R > 0 \) let us define \( A^0_\alpha(x, \Omega, \Omega(x_0, R)) \) by

\[
A^0_\alpha(x, \Omega, \Omega(x_0, R)) := \int_{\Omega(x_0, R)} A^0(Du) \, dx,
\]

where \( u_R = u_{x_0,R} = \int_{\Omega(x_0, R)} u(y) \, dy. \)

For weak solutions of the Euler-Lagrange equation of \( A^0 \), we have the following regularity results.

For interior points, we have the following (see [2, Theorem 3.1]).

**Lemma 3.1.** Let \( u \in H^{1,p}(\Omega, \mathbb{R}^n) \) \( p \geq 2 \), be a solution of the system

\[
D_\alpha a^\alpha_i(Du) = 0, \quad i = 1, \ldots, n, \quad in \ \Omega,
\]

in the sense that \( \int_\Omega a^\alpha_i(Du)D_\alpha \varphi^i dx = 0, \) for all \( \varphi \in C_0^\infty(\Omega, \mathbb{R}^n) \), under the conditions

1. \( a^\alpha_i(0) = 0; \)

2. there exist two constants \( \nu > 0 \) and \( M > 0 \) such that, for all \( x \in \Omega \) and for all \( \xi, \zeta \in \mathbb{R}^{mn}, \)

\[
\| A(\xi) \| \leq M \cdot (1 + \| \xi \|^2)^{\frac{p-2}{2}}
\]

\[
A^\alpha_\beta(x; \xi)^i_j \zeta^i_\alpha \zeta^j_\beta \geq \nu \cdot (1 + \| \xi \|^2)^{\frac{p-2}{2}} \| \zeta \|^2,
\]

where \( A = (A^\alpha_\beta) \) and \( A^\alpha_\beta(\xi) = \frac{\partial^\alpha_i(\xi)}{\partial \xi^j}. \)
Then, for all \( Q(\sigma) = Q(x_0, \sigma) \subset \subset \Omega \) and for all \( t \in (0, 1) \),

\[
\int_{Q(\sigma)} |Du|^p \, dx \leq ct^{\lambda_0} \int_{Q(\sigma)} |Du|^p \, dx, \quad \lambda_0 = \min\{2 + \varepsilon_0, m\},
\]

for some positive constants \( \varepsilon_0 \) and \( c \) which do not depend on \( t, \sigma \) and \( x^0 \).

In the neighborhood of the boundary, by the proof of [2, Theorem 7.1], we have the following.

**Lemma 3.2.** Let \( a_i^\sigma(\xi) \) and \( \lambda_0 \) be as in Lemma 3.1 and \( v \in H^{1,p}(Q^+(0, R)) \) be a solution of the problem

\[
\begin{aligned}
\int_{Q^+(0, R)} a_i^\sigma(Dv + Dg)D\alpha_i \varphi^i \, dx &= 0 \quad \forall \varphi \in C_0^\infty(Q^+(0, R)) \\
v &= 0 \quad \text{on } \Gamma(0, R),
\end{aligned}
\]

where \( g \) is a given function with \( Dg \in L^s(Q^+(0, R)) \) for some \( s > p \). Then, for every \( x_0 \in \Gamma(0, R) \) and \( \tau_0 \) with \( 0 < \tau_0 < \min\{\lambda_0, m(1 - \frac{2}{s})\} \), there exist a constant \( c > 0 \) such that

\[
\int_{Q^+(x_0, \tau)} |W(Dv)|^2 \, dx \leq c t^{\tau_0} \int_{Q^+(x_0, \sigma)} |W(Dv)|^2 \, dx + c\sigma^{\tau_0} \left( \int_{Q^+(x_0, \sigma)} |W(Dg)|^{\frac{2s}{s-2}} \, dx \right)^{\frac{s}{2}}.
\]

for any \( \sigma \in (0, R - |x_0|) \) and \( t \in (0, 1) \), where \( W(\xi) = (1 + |\xi|^2)^{\frac{s-2}{2}} \xi \).

**Outline of the proof.** Since (3.1) is exactly (7.6) of [2], we can proceed as in [2, pp. 148–150] and get the following estimates:

\[
\begin{aligned}
&\int_{Q^+(x_0, \sigma)} |W(Dv)|^2 \, dx \\
&\leq c_1 t^{\tau_0} \int_{Q^+(x_0, \sigma)} |W(Dv)|^2 \, dx + c_1 \int_{Q^+(x_0, \sigma)} (1 + |Dv| + |Dg|)^{p-2}|Dg|^2 \, dx \\
&\int_{Q^+(x_0, \sigma)} (1 + |Dv| + |Dg|)^{p-2}\, |Dg|^2 \, dx \\
&\leq c_2 \int_{Q^+(x_0, \sigma)} |W(Dg)|^2 \, dx + c_2 \int_{Q^+(x_0, \sigma)} |Dv|^{p-2}|Dg|^2 \, dx \\
&\int_{Q^+(x_0, \sigma)} |Dv|^{p-2}|Dg|^2 \, dx \\
&\leq \left( 1 - \frac{2}{p} \right) \delta \int_{Q^+(x_0, \sigma)} |W(Dv)|^2 \, dx + \frac{2}{p} \delta^{1-\frac{2}{s}} \int_{Q^+(x_0, \sigma)} |W(Dg)|^2 \, dx
\end{aligned}
\]
for any \( \delta > 0 \). These estimates are nothing else than (17)–(19) of [2]. Combining them, we get

\[
\int_{Q^+(x_0,\rho)} |W(Dv)|^2 \, dx \leq c_1 \left\{ t^\lambda + c_2 \left( 1 - \frac{2}{p} \right) \delta \right\} \int_{Q^+(x_0,\sigma)} |W(Dv)|^2 \, dx \\
+ c_1 c_2 \left( 1 + \frac{2}{p} \delta^{1-\frac{2}{p}} \right) \int_{Q^+(x_0,\sigma)} |W(Dg)|^2 \, dx \\
\leq c_1 \left\{ t^\lambda + c_1 c_2 \left( 1 - \frac{2}{p} \right) \delta \right\} \int_{Q^+(x_0,\sigma)} |W(Dv)|^2 \, dx \\
+ c_3(p, \delta) \sigma^{m(1-\frac{2}{p})} \left( \int_{Q^+(x_0,\sigma)} |W(Dg)|^\frac{2q}{p} \, dx \right)^{\frac{p}{q}}.
\]

Now, using “A useful lemma” of [8, p. 44], we get (3.2).

Moreover, we have the following \( L^q \)-estimate for \( u \).

**Lemma 3.3.** Assume that \( u \in H^{1,p}(Q^+(0, R)) \) satisfies

\[
\mathcal{A}(u, Q^+(0, R)) \leq \mathcal{A}(u + \varphi, Q^+(0, R)), \quad \varphi \in H_0^{1,p}(Q^+(0, R)),
\]

and that \( u = g \) on \( \Gamma(0, R) \) for some \( g \in H^{1,q_1}(Q^+(0, R)) \) with \( q_1 > p \). Then there exists an exponent \( q \in (p, q_1] \) such that \( u \in H^{1,q}(Q^+(0, r)) \) for any \( r < R \). Moreover, if \( x_0 \in Q^+(0, r) \cup \Gamma(0, r) \) and \( \rho < R - r \), we have the estimate

\[
\left( \int_{Q(x_0,\rho/2)\cap Q^+(0,R)} (1 + |Du|^2)^{\frac{q}{2}} \, dx \right)^{\frac{1}{q}} \leq \left( \int_{Q(x_0,\rho)\cap Q^+(0,R)} (1 + |Du|^2)^{\frac{q}{2}} \, dx \right)^{\frac{1}{q}} + c \left( \int_{Q(x_0,\rho)\cap Q^+(0,R)} (1 + |Dg|^2)^{\frac{q}{2}} \, dx \right)^{\frac{1}{q}}.
\]

(3.3)

In addition, if \( Q(x_0,\rho) \subset \subset Q^+(0,R) \), then we have

\[
\left( \int_{Q(x_0,\rho/2)} (1 + |Du|^2)^{\frac{q}{2}} \, dx \right)^{\frac{1}{q}} \leq c \left( \int_{Q(x_0,\rho)} (1 + |Du|^2)^{\frac{q}{2}} \, dx \right)^{\frac{1}{q}}.
\]

(3.4)

**Outline of the Proof.** For the case that \( Q(x_0,\rho) \subset \subset Q^+(0,R) \), we can proceed as in the proof of [9, Theorem 4.1] to get (3.4). For general case, mentioning the difference on the growth conditions, we can proceed as in the proof of [14, Lemma 1].

Mention that the above lemma is valid for minimizers of \( \mathcal{A}^0 \) also.

For bounded domain \( D \) with smooth boundary, covering \( \partial D \) with a finite number of cubes and using the above local estimates we get the following global \( L^q \)-estimates for a minimizer.
Corollary 3.4. Let $D \subset \mathbb{R}^m$ be an open set with smooth boundary $\partial D$, and let $v \in H^{1,p}(D)$ be a minimizer for the functional $A$ (or $A^0$) in the class

$$X_g := \{w \in H^{1,p}(D); w-g \in H^{1,p}_0(D)\}$$

for a given map $g \in H^{1,q_1}(D)$, $q_1 > p$. Then $Dv \in L^q(D)$ for some $q \in (p, q_1)$ and

$$\int_D \left(1 + |Dv|^2\right)^{\frac{q}{2}} dx \leq c \int_D \left(1 + |Dg|^2\right)^{\frac{q}{2}} dx.$$

We show the partial regularity of $u$ by comparing $u$ with $v$. For this purpose, we need the following lemma which can be shown as [11, Theorem 4.2, (4.8)].

Lemma 3.5. Let $v \in H^{1,p}(\Omega(x_0, r))$ is a minimizer for $A^0(w, \Omega(x_0, r))$ in the class $\{w \in H^{1,p}(\Omega(x_0, r)); w-u \in H^{1,p}_0(\Omega(x_0, r))\}$ for a given function $u \in H^{1,p}(\Omega(x_0, r))$. Then we have

$$\int_{\Omega(x_0, r)} |Du - Dv|^p dx \leq c \left\{A^0(u; \Omega(x_0, r)) - A^0(v; \Omega(x_0, r))\right\}.$$

Now, we can prove our main theorem.

Proof of Theorem 2.5. Assume that $Q(R) = Q(x_0, R) \subset \subset \Omega$. Let $v \in H^{1,p}(Q(R))$ be a minimizer of $A^0(\tilde{v}, Q(R))$ in the class $\{\tilde{v} \in H^{1,p}(Q(R)); u-\tilde{v} \in H^{1,p}_0(Q(R))\}$, and let $w = u - v$. First we will estimate $\int_{Q(R)} |Dw|^p dx$. By Lemma 3.5 we can see that

$$\int_{Q(R)} |Dw|^p dx = c \left\{A^0(u) - A^0(v)\right\}$$

$$\leq c \int_{Q(R)} |A_R(u_R, Du) - A(x, u_R, Du)| dx$$

$$+ c \int_{Q(R)} |A(x, u_R, Du) - A(x, u, Du)| dx$$

$$+ c \int_{Q(R)} |A(x, v, Dv) - A(x, u_R, Dv)| dx$$

$$+ c \int_{Q(R)} |A(x, u_R, Dv) - A_R(u_R, Dv)| dx.$$ 

Here we have used the minimality of $u$. So, using the assumptions on $A$, we get

$$\int_{Q(R)} |Dw|^p dx \leq \int_{Q(R)} \left\{\sigma(x, R) + \omega(|u - u_R|^2)\right\} \left(1 + |Du(x)|^2\right)^{\frac{q}{2}} dx$$

$$+ \int_{Q(R)} \left\{\sigma(x, R) + \omega(|v - u_R|^2)\right\} \left(1 + |Dv(x)|^2\right)^{\frac{q}{2}} dx. \quad (3.5)$$
Using Hölder’s inequality, Lemma 3.3, (3.4) and the boundedness of $\omega$ and $\sigma$, we have
\[
\int_{Q(R)} \{|\sigma(x, R) + \omega(|u - u_R|^2)| (1 + |Du(x)|^2)^{\frac{p}{q}} \, dx 
\leq C \left\{ \left( \int_{Q(R)} \sigma(x, R) \, dx \right)^{\frac{q-p}{q}} + \left( \int_{Q(R)} \omega(|u - u_R|^2) \, dx \right)^{\frac{q-p}{q}} \right\} \tag{3.6}
\times \int_{Q(2R)} (1 + |Du(x)|^2)^{\frac{p}{q}} \, dx.
\]

Using Corollary 3.4, and (3.4) we get similarly
\[
\int_{Q(R)} \{|\sigma(x, R) + \omega(|v - u_R|^2)| (1 + |Dv(x)|^2)^{\frac{p}{q}} \, dx 
\leq C \left\{ \left( \int_{Q(R)} \sigma(x, R) \, dx \right)^{\frac{q-p}{q}} + \left( \int_{Q(R)} \omega(|v - u_R|^2) \, dx \right)^{\frac{q-p}{q}} \right\} \tag{3.7}
\times \int_{Q(2R)} (1 + |Du(x)|^2)^{\frac{p}{q}} \, dx.
\]

By virtue of concavity of $\omega$, using Jensen’s inequality and Poincaré inequality, we have
\[
\int_{Q(R)} \omega(|u - u_R|^2) \, dx \leq \int_{Q(R)} \omega(|v - u_R|^2) \, dx \leq C \omega \left( R^{p-m} \int_{Q(R)} |Du|^p \, dx \right). \tag{3.8}
\]

Combining (3.5) – (3.8), we obtain
\[
\int_{Q(R)} |Dw|^p \, dx \leq C \left\{ \left( \int_{Q(R)} \sigma(x, R) \, dx \right)^{\frac{q-p}{q}} + \omega \left( R^{p-m} \int_{Q(R)} |Du|^p \, dx \right)^{\frac{q-p}{q}} \right\} \tag{3.9}
\times \int_{Q(2R)} (1 + |Du(x)|^2)^{\frac{p}{q}} \, dx.
\]

Now, from Lemma 3.1 and the above inequality, we get
\[
\int_{Q(R)} |Du|^p \, dx \leq \int_{Q(R)} (|Dv|^p + |Dw|^p) \, dx 
\leq C \left\{ \left( \frac{1}{R} \right)^\lambda + \left( \int_{Q(R)} \sigma(x, R) \, dx \right)^{\frac{q-p}{q}} \right. \tag{3.9}
+ \omega \left( R^{p-m} \int_{Q(R)} |Du|^2 \, dx \right)^{\frac{q-p}{q}} \right\} \int_{Q(2R)} (1 + |Du(x)|^p)^{\frac{p}{q}} \, dx.
\]
Let us consider the behavior of \( u \) near the boundary. Let \( Q(x_1, 2R_1) \) be a member of the covering \( \{Q(x_k, 2R_1)\} \) which is introduced at the beginning of this section. Then, \( u \) satisfies

\[
\begin{align*}
  \mathcal{A}(u, Q^+(x_1, 2R_1)) &\leq \mathcal{A}(u + \varphi, Q^+(x_1, 2R_1)) \quad \forall \varphi \in H_0^{1,p}(Q^+(x_1, 2R_1)) \\
  u &= g \quad \text{on} \quad \Gamma(x_1, 2R_1).
\end{align*}
\]

Fix a point \( x_0 \in \Gamma(x_1, R_1) \) and a positive number \( R < R_1 \) arbitrarily (here, mention that \( Q^+(x_0, R) \subset Q^+(x_1, 2R_1) \)). Let \( v \in H^{1,p}(Q^+(x_0, R)) \) be a minimizer of \( \mathcal{A}^0(v, Q^+(x_0, R)) \) in the class \( \{v \in H^{1,p}(Q^+(x_0, R)); u - v \in H_0^{1,p}(Q^+(x_0, R))\} \), and put \( w = u - v \). Then, using Lemma 3.5, we can proceed as in the interior case and get

\[
\int_{Q^+(R)} |Dw|^p \, dx \leq \int_{Q^+(R)} \{\sigma(x, R) + \omega(|u - u_R|^2)\} \left(1 + |Du(x)|^2\right)^{\frac{p}{2}} \, dx \\
+ \int_{Q^+(R)} \{\sigma(x, R) + \omega(|v - u_R|^2)\} \left(1 + |Dv(x)|^2\right)^{\frac{p}{2}} \, dx.
\]

Moreover, using (3.3) instead of (3.4) and proceeding as in the interior case, we have

\[
\int_{Q^+(R)} |Dw|^p \, dx \\
\leq C \left\{ \left( \int_{Q^+(R)} \sigma(x, R) \, dx \right)^{\frac{2-p}{2}} + \omega \left( R^{p-m} \int_{Q^+(R)} |Du|^p \, dx \right)^{\frac{2-p}{p}} \right\} \\
\times \left(1 + |Du(x)|^2\right)^{\frac{p}{2}} \, dx \\
+ CR^{m \frac{2-p}{q}} \left( \int_{Q^+(R)} (1 + |Dg|^2)^{\frac{p}{2}} \, dx \right)^{\frac{p}{q}}.
\]

Now, combining (3.2) and (3.10), we obtain

\[
\int_{Q^+(R)} |Du|^p \, dx \leq C \left\{ \left( \frac{r}{R} \right)^{\tau_0} + \left( \int_{Q^+(R)} \sigma(x, R) \, dx \right)^{\frac{2-p}{q}} \right\} \\
+ \omega \left( R^{p-m} \int_{Q^+(R)} |Du|^p \, dx \right)^{\frac{2-p}{p}} \\
\times \left(1 + |Du(x)|^2\right)^{\frac{p}{2}} \, dx \\
+ cr^{\tau_0} \left( \int_{Q^+(R)} (1 + |Dg|^2)^{\frac{p}{2}} \, dx \right)^{\frac{p}{q}} \\
+ CR^{m \frac{2-p}{q}} \left( \int_{Q^+(R)} (1 + |Dg|^2)^{\frac{p}{2}} \, dx \right)^{\frac{p}{q}}.
\]

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Since we are assuming that $Dg \in L^s$ for some $s > p$, and we can choose $q > p$ sufficiently near to $p$, without loss of generality we can assume that $s > q > p$. So, we can estimate the last term of (3.11) as follows:

$$R^{m \frac{s-p}{s}} \left( \int_{Q^+(2R)} (1 + |Dg|^2)^\frac{q}{2} \, dx \right)^{\frac{p}{q}} \leq C R^{m \left(1 - \frac{p}{s} \right)} \left( \int_{Q^+(2R)} \left(1 + |Dg|^2\right)^{\frac{p}{q}} \, dx \right)^{\frac{p}{q}}.$$

Here, we can assume that $R < 1$, so the above estimates hold even if $m \left(1 - \frac{p}{s} \right)$ can be replaced by the smaller constant $\tau_0$. Mentioning the above fact and combining the above estimate with (3.11), we get the following estimate:

$$\int_{Q^+(r)} |D_{u}|^p \, dx \leq C \left\{ \left( \frac{r}{R} \right)^{\tau_0} \left( \int_{Q^+(r)} \sigma(x, R) \, dx \right)^{\frac{q-p}{q}} + \omega \left( \int_{Q^+(r)} |D_{u}|^p \, dx \right)^{\frac{q-p}{q}} \right\} \times \int_{Q^+(2R)} \left(1 + |Du(x)|^2\right)^{\frac{p}{q}} \, dx + C(g)R^{\tau_0}.$$  \hspace{1cm} (3.12)

By the assumption (A-1), we have $\int_{Q(R)} \sigma(x, R) \, dx \to 0$ as $R \to 0$. So, using “a useful Lemma” on p. 44 of [8] for (3.9) and (3.12), and putting

$$\Phi(x, r) = \int_{Q(x, r)} \left(1 + |Du|^2\right)^{\frac{p}{q}} \, dx,$$

we can see that for any $\tau$ with $0 < \tau < \tau_0 (< \lambda_0)$ there exist positive constants $\delta$, $M$ and $R_0$ ($R_0 < \frac{R_1}{2}$) with the following properties:

**Interior Case.** If $r_1, r^{p-m}_1 \Phi(x, r_1) < \delta$ for some $r_1 \in (0, R_0)$ with $Q(x, r_1) \subset \subset \Omega$, then for $0 < \rho < r < r_1$ we have

$$\Phi(x, \rho) \leq M \left( \frac{\rho}{r} \right)^{\tau} \Phi(x, r).$$

**Boundary Case.** For $x \in \partial \Omega$, if $r_1, r^{p-m}_1 \Phi(x, r_1) < \delta$ for some $r_1 \in (0, R_0)$, then we have

$$\Phi(x, \rho) \leq M \left( \frac{\rho}{r} \right)^{\tau} \Phi(x, r) + M \rho^{\tau}.$$

Now, we can proceed as in Giusti’s book [12, pp. 318–319] to show partial Morrey-type regularity of $u$. Namely, there exist positive constants $\delta$ and $M$ with the following properties: for any $x \in \Omega$, if $r_0, r^{p-m}_0 \Phi(x, r_0) \leq \delta$ for some $r_0 > 0$, then $\rho^{-\tau} \Phi(x, \rho) \leq M$. So, we get the assertion. \qed

**Proof of Corollary 2.6.** When $p + 2 \geq m$ and $s > \max\{m, p\}$, we can take $\tau$ sufficiently near to $\min\{2 + \varepsilon, m(1 - \frac{q}{s})\}$ so that $\tau > m - p$. So, Corollary 2.6 is a direct consequence of Theorem 2.5 and Morrey’s theorem on the growth of the Dirichlet integral (see, for example, [8, p.43]). \qed
Proof of Corollary 2.7. When $A(x,u,\xi)$ does not depend on $u$, we can proceed as in the proof of Theorem 2.5 without the term with $\omega$ and get, instead of (3.9) and (3.11),

$$\int_{Q(x_0,r)} |Du|^p \, dx \leq C \left\{ \left( \frac{r}{R} \right)^{\lambda} + \left( \int_{Q(R)} \sigma(x,R) \, dx \right)^{\frac{q-p}{q}} \right\} \times \int_{Q(2R)} \left( 1 + |Du(x)|^2 \right)^{\frac{p}{2}} \, dx.$$

for $Q(2R) = Q(x_0, 2R) \subset\subset \Omega$ and

$$\int_{Q^+(x_0,r)} |Du|^p \, dx \leq C \left\{ \left( \frac{r}{R} \right)^{\lambda} + \left( \int_{Q^+(R)} \sigma(x,R) \, dx \right)^{\frac{q-p}{q}} \right\} \times \int_{Q^+(2R)} \left( 1 + |Du(x)|^2 \right)^{\frac{p}{2}} \, dx + C(g)R^\tau,$$

for $x_0 \in \partial\Omega$. So, we can proceed as in the last part of Theorem 2.5 without assuming that

$$r_1^{p-m} \Phi(x_1, r_1) = r_1^{p-m} \int_{\Omega(x_1 r_1)} \left( 1 + |Du|^2 \right)^{\frac{p}{2}} \, dx < \delta.$$

and see that $\rho^{-\tau} \Phi(x, \rho) \leq \tilde{M}$ for all $x \in \Omega$. Thus we get the assertions. \hfill \square

Remark 3.6. Without any restriction on the dimension of the domain, it is not possible to obtain a Hölder regularity result in all the domain $\Omega$ as showed by V. Šverak and X. Yan in a counterexample contained in [20].

References


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