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Linear q-Difference Equations

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Abstract. We prove that a linear q-difference equation of order n has a fundamental set of n-linearly independent solutions. A q-type Wronskian is derived for the n-th order case extending the results of Swarttouw–Meijer (1994) in the regular case. Fundamental systems of solutions are constructed for the n-th order linear q-difference equation with constant coefficients. A basic analog of the method of variation of parameters is established.

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1. Introduction and basic definitions

In the following, q is a positive number, $0 < q < 1$, and I is an open interval containing zero. Now we state the basic definitions used in this article, cf. [4, 9]. Then we introduce a brief account about the q-calculus established in [3]. Let $n \in \mathbb{N}$. The q-shifted factorial $(a;q)_n$ of $a \in \mathbb{C}$ is defined by

$$(a;q)_0 := 1 \quad \text{and, for } n > 0, \quad (a;q)_n := \prod_{k=1}^{n} \left(1 - aq^{k-1}\right).$$

The multiple q-shifted factorial for complex numbers $a_1, \ldots, a_k$ is defined by

$$(a_1, a_2, \ldots, a_k; q)_n := \prod_{j=1}^{k} (a_j; q)_n.$$

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The limit $\lim_{n\to\infty}(a;q)_n$ exists and is denoted by $(a;q)_\infty$. The third type of the q-Bessel functions of Jackson of order ν is defined to be, see [10],

$$J_\nu^{(3)}(x;q) = x^\nu \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} \frac{x^{2n}}{(q;q)_n (q^{\nu+1};q)_n}.$$

We denote this function by $J_\nu(x;q)$ instead of $J_\nu^{(3)}(x;q)$ for simplicity. In some literature this function is called the Hahn–Exton q-Bessel function, see [12, 14]. The functions $\cos_q x$ and $\sin_q x$ are defined for $x \in \mathbb{C}$, $|x|(1-q) < 1$, by

$$\cos_q x := \sum_{n=0}^{\infty} (-1)^n \frac{\big(x(1-q)\big)^{2n}}{(q;q)_{2n}}$$
$$\sin_q x := \sum_{n=0}^{\infty} (-1)^n \frac{\big(x(1-q)\big)^{2n+1}}{(q;q)_{2n+1}}.$$

The functions $\cos(x;q)$ and $\sin(x;q)$ are defined in $\mathbb{C}$ by

$$\cos(x;q) := \sum_{n=0}^{\infty} (-1)^n \frac{q^{n^2}(x(1-q))^{2n}}{(q;q)_{2n}}$$
$$\sin(x;q) := \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}(x(1-q))^{2n+1}}{(q;q)_{2n+1}},$$

and they are q-analogs of the sine and cosine functions, [4, 9]. See also [1, 2], [5]–[7] for a study of the zeros and completeness of q-trigonometric and q-Bessel systems.

Let $\mu \in \mathbb{R}$ be fixed . A set $A \subseteq \mathbb{R}$ is called a μ-geometric set if for $x \in A$, $\mu x \in A$. Now, we define the q-difference operator of Heine. Let f be a function defined on a q-geometric set $A \subseteq \mathbb{R}$. The q-difference operator is defined by the formula

$$D_q f(x) := \frac{f(x) - f(qx)}{x - qx}, \quad x \in A \setminus \{0\}.$$

If $0 \in A$, we say that f has q-derivative at zero if the limit

$$\lim_{n\to\infty} \frac{f(xq^n) - f(0)}{xq^n}, \quad x \in A$$

exists and does not depend on x. In this case, we shall denote this limit by $D_q f(0)$. In some literature the q-derivative at zero is defined to be $f'(0)$ if it exists, cf. [12, 14], but the above definition is more suitable for our approach. The non-symmetric Leibniz' rule

$$D_q(fg)(x) = g(x)D_q f(x) + f(qx)D_q g(x) \tag{1.1}$$

holds. Relation (1.1) can be symmetrized using the relation $f(qx) = f(x) - x(1-q)D_q f(x)$, giving the additional term $-x(1-q)D_q f(x) D_q g(x)$. The q-integration of F. H. Jackson [11] is defined for a function f defined on a q-geometric set A to be

$$\int_a^b f(t)\,d_q t = \int_0^b f(t)\,d_q t - \int_0^a f(t)\,d_q t, \quad a, b \in A,$$

where

$$\int_0^x f(t)\,d_q t = \sum_{n=0}^{\infty} x q^n (1-q) f(xq^n), \quad x \in A, \tag{1.2}$$

provided that the series converge.

Theorem 1.1 ([3]). *The q-integral* (1.2) *exists only if* $\lim_{k\to\infty} xq^k f(xq^k) = 0$.

Consider the non-homogeneous q-difference equation of order n

$$a_0(x) D_q^n y(x) + a_1(x) D_q^{n-1} y(x) + \cdots + a_n(x) y(x) = b(x), \quad x \in I, \tag{1.3}$$

for which a_i, $0 \leqslant i \leqslant n$, and b are continuous at zero functions defined on I and $a_0(x) \neq 0$ for all $x \in I$. Equation (1.3) together with the initial conditions

$$D_q^{i-1} y(0) = b_i, \quad b_i \in \mathbb{C},\ i = 1, \ldots, n, \tag{1.4}$$

form a q-type Cauchy problem. By a solution of problem (1.3)–(1.4), we mean a continuous at zero function which satisfies (1.3) subject to the initial conditions (1.4). According to [3], there exists a unique solution of (1.3)–(1.4) in a subinterval J of I, $J = [-h, h]$, $h > 0$. In the next section, we shall study the n-th order homogeneous linear equation

$$a_0(x) D_q^n y(x) + a_1(x) D_q^{n-1} y(x) + \cdots + a_n(x) y(x) = 0, \quad x \in I. \tag{1.5}$$

A fundamental set of solutions for (1.5) when the coefficients are constants is derived in §2. In §3, a q-type Wronskian for the solutions of (1.5) is introduced and it is proved that it satisfies a first order q-difference equation and its solution is given. This extends the results of Swarttouw–Meijer [15] in the regular case. As applications, a formula for a solution of (1.3) in terms of a fundamental set of solutions of (1.5) will be given in §4 by using a q-analog of the method of variation of parameters.

2. Linear homogeneous q-difference equations

Let M denote the set of solutions of (1.5) valid in a subset $J \subseteq I$ which contains zero. Then it is easy to see that M is a linear space over $\mathbb{C}$. Also from the existence and uniqueness of the solutions, cf. [3], if $\phi \in M$ and $D_q^i \phi(0) = 0$, $0 \leqslant i \leqslant n-1$, then $\phi(x) \equiv 0$ on J. Moreover, cf. [3], $\{D_q^i \phi\}_{i=0}^{n-1}$, $0 \leqslant i \leqslant n-1$, are continuous at zero for any $\phi \in M$. A set of n solutions of (1.5) is said to be a fundamental set (f.s.) for (1.5) valid in J or a f.s. of M if it is linearly independent in J. Moreover, as in differential equations, if b_{ij}, $1 \le i, j \le n$, are numbers, and, for each j, ϕ_j is the unique solution of (1.5) which satisfies the initial conditions

$$D_q^{i-1}\phi_j(0) = b_{ij}, \quad 1 \le i \le n,$$

then $\{\phi_j\}_{j=1}^n$ is a f.s. of (1.5) if and only if $\det(b_{ij}) \neq 0$. Hence M is a linear space of dimension n. In the following we are concerned with constructing a f.s. for (1.5) when it has constant coefficients, a_r, $0 \leqslant r \leqslant n$. Set $L := a_0 D_q^n + a_1 D_q^{n-1} + \cdots + a_n$. Then, (1.5) can be written as

$$Ly(x) = a_0 D_q^n y(x) + a_1 D_q^{n-1} y(x) + \cdots + a_n y(x) = 0. \tag{2.1}$$

The characteristic polynomial $P(\lambda)$ of (2.1) is defined by

$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n\,, \quad \lambda \in \mathbb{C}.$$

Let λ_i, $1 \leqslant i \leqslant k$, denote the distinct roots of $P(\lambda)$ and m_i denotes the multiplicity of λ_i, so that $\sum_{i=1}^k m_i = n$. Corresponding to each λ_i we define an m_i-dimensional subspace M_i by

$$M_i = \big\{v \in M : \ \big(D_q - \lambda_i\big)^{m_i} v = 0\big\}.$$

The construction of a f.s. of (2.1) depends on the fact that, cf. [13],

$$M = M_1 \oplus \cdots \oplus M_k. \tag{2.2}$$

Lemma 2.1. *Let $(X, \mathbb{K})$ be a vector space, and let T be a linear operator on X. For any $\lambda \in \mathbb{K}$, if there exist $y_0, y_1, \ldots, y_{m-1}$ in X such that*

$$\begin{aligned} Ty_0 &= \lambda y_0, && y_0 \neq 0 \\ Ty_i &= \lambda y_i + y_{i-1}, && 1 \leqslant i \leqslant m-1, \end{aligned}$$

then $y_1, \ldots, y_{m-1}$ are linearly independent.

Proof. By induction on i, $0 \leqslant i \leqslant m-1$. □

Lemma 2.2. *If $\lambda_i \neq 0$, then the initial value problem*

$$\begin{aligned} D_q\phi_{0,i} &= \lambda_i\phi_{0,i}, && \phi_{0,i}(0) = 1, r \\ D_q\phi_{r,i} &= \lambda_i\phi_{r,i} + \phi_{r-1,i}, && \phi_{r,i}(0) = 0, \ \ r = 1, \ldots, m_i - 1, \end{aligned}$$

has the solution

$$\phi_{r,i}(x) = \begin{cases} e_q(\lambda_i x) := \sum_{k=0}^\infty \frac{(\lambda_i x(1-q))^k}{(q;q)_k}, & r = 0 \\ \frac{1}{\lambda_i^r} \sum_{k=r}^\infty \frac{k(k-1)\cdots(k-r+1)}{r!} \frac{(\lambda_i x(1-q))^k}{(q;q)_k}, & r = 1, 2, \ldots, m_i - 1, \end{cases} \tag{2.3}$$

which is valid for $|x| < \frac{1}{\lambda_i(1-q)}$. If $\lambda_i = 0$, then

$$\phi_{r,i}(x) = \frac{x^r(1-q)^r}{(q;q)_r}, \quad r = 0, 1, \ldots, m_i - 1. \tag{2.4}$$

Proof. The proof follows by direct computations. □

One can see that $(D_q - \lambda_i)^{m_i}\phi_{r,i} = 0$, $r = 0, 1, \dots, m_i - 1$. Thus, $\phi_{r,i} \in M_i$, for $r = 0, 1, \dots, m_i - 1$. Therefore, these functions form a basis for M_i since they are linearly independent by Lemma 2.1. This fact and (2.2) above imply the following theorem.

Theorem 2.3. *The set* $\{\phi_{i,r}\}_{r=0}^{m_i-1}$ *of* (2.3) *when* $\lambda_i \neq 0$ *or of* (2.4) *when* $\lambda_i = 0$ *is a linearly independent set of solutions of* (2.1). *Moreover,* $\bigcup_{i=1}^{k}\{\phi_{i,r}\}_{r=0}^{m_i-1}$ *is a fundamental set of solutions of* (2.1).

Example 2.4. The q-difference equation

$$D_q^3 y(x) - 4D_q^2 y(x) + 5D_q y(x) - 2y(x) = 0,$$

has the functions $e_q(2x)$, $e_q(x)$ and $\sum_{k=1}^{\infty} k\frac{(x(1-q))^k}{(q;q)_k}$ as a f.s..

3. A q-type Wronskian

This section contains a q-analog of the Wronskian of linear differential equations, we prove that the q-analog satisfies a first order q-difference equation and we derive its solution. We also derive a q-type Liouville's formula for the q-Wronskian.

Definition 3.1. Let y_i, $1 \le i \le n$, be functions defined on a q-geometric set A. The *q-Wronskian* of the functions y_i which will be denoted by $W_q(y_1, \dots, y_n)(x)$ is defined to be

$$W_q(y_1, \dots, y_n)(x) := \begin{vmatrix} y_1(x) & \cdots & y_n(x) \\ D_q y_1(x) & \cdots & D_q y_n(x) \\ \vdots & \ddots & \vdots \\ D_q^{n-1} y_1(x) & \cdots & D_q^{n-1} y_n(x) \end{vmatrix},$$

provided that the derivatives exist in I. For convenience we write $W_q(x)$ instead of $W_q(y_1, \dots, y_n)(x)$.

Lemma 3.2. *Let* $y_1, \dots, y_n$ *be functions defined on a* q*-geometric set* A*. Then for any* $x \in A$, $x \neq 0$,

$$D_q W_q(y_1, y_2, \dots, y_n)(x) = \begin{vmatrix} y_1(qx) & y_2(qx) & \cdots & y_n(qx) \\ (D_q y_1)(qx) & (D_q y_2)(qx) & \cdots & (D_q y_n)(qx) \\ \vdots & \vdots & \ddots & \vdots \\ (D_q^{n-2} y_1)(qx) & (D_q^{n-2} y_2)(qx) & \cdots & (D_q^{n-2} y_n)(qx) \\ D_q^n y_1(x) & D_q^n y_2(x) & \cdots & D_q^n y_n(x) \end{vmatrix}. \quad (3.1)$$

Proof. We prove the lemma by induction on n. The lemma is trivial when $n = 1$. Assume that (3.1) holds at $k \in \mathbb{N}$, $k \geqslant 1$, then expanding $W_q(y_1, y_2, \ldots, y_{k+1})$ in terms of the first row we obtain

$$W_q(y_1, y_2, \ldots, y_{k+1})(x) = \sum_{j=1}^{k+1} (-1)^{j+1} y_j(x) W_q^{(j)}(x),$$

where

$$W_q^{(j)} := \begin{cases} W_q\big(D_q y_2, \ldots, D_q y_{k+1}\big), & j = 1 \\ W_q\big(D_q y_1, \ldots, D_q y_{j-1}, D_q y_{j+1}, \ldots, D_q y_{k+1}\big), & 1 \leqslant j \leqslant k+1 \\ W_q\big(D_q y_1, \ldots, D_q y_k\big), & j = k+1. \end{cases}$$

Consequently, from (1.1),

$$\begin{aligned} &D_q W_q\big(y_1, y_2, \ldots, y_{k+1}\big)(x) \\ &\quad = \sum_{j=1}^{k+1} (-1)^{j+1} D_q y_j(x) W_q^{(j)}(x) + \sum_{j=1}^{k+1} (-1)^{j+1} y_j(qx) D_q W_q^{(j)}(x). \end{aligned}$$

Now

$$\sum_{j=1}^{k+1} (-1)^{j+1} D_q y_j(x) W_q^{(j)}(x) = \begin{vmatrix} D_q y_1(x) & \cdots & D_q y_{k+1}(x) \\ D_q y_1(x) & \cdots & D_q y_{k+1}(x) \\ D_q^2 y_1(x) & \cdots & D_q^2 y_{k+1}(x) \\ \vdots & \ddots & \vdots \\ D_q^{k-1} y_1(x) & \cdots & D_q^{k-1} y_{k+1}(x) \\ D_q^k y_1(x) & \cdots & D_q^k y_{k+1}(x) \end{vmatrix} = 0,$$

and from the induction hypothesis,

$$\begin{aligned} &\sum_{j=1}^{k+1} (-1)^{j+1} y_j(qx) D_q W_q^{(j)}(x) = \sum_{j=1}^{k+1} (-1)^{j+1} y_j(qx) \times \\ &\begin{vmatrix} (D_q y_1)(qx) & \cdots & (D_q y_{j-1})(qx) & (D_q y_{j+1})(qx) & \cdots & (D_q y_{k+1})(qx) \\ (D_q^2 y_1)(qx) & \cdots & (D_q^2 y_{j-1})(qx) & (D_q^2 y_{j+1})(qx) & \cdots & (D_q^2 y_{k+1})(qx) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (D_q^{k-1} y_1)(qx) & \cdots & (D_q^{k-1} y_{j-1})(qx) & D_q^{k-1} y_{j+1}(qx) & \cdots & D_q^{k-1} y_{k+1}(x)(qx) \\ D_q^{k+1} y_1(x) & \cdots & D_q^{k+1} y_{j-1}(x) & D_q^{k+1} y_{j+1}(x) & \cdots & D_q^{k+1} y_{k+1}(x) \end{vmatrix}, \end{aligned} \tag{3.2}$$

where when $i = 1$ the determinant of (3.2) starts with $D_q y_2(qx)$ and when $j = k+1$, the determinant ends with $D_q^{k+1} y_k(x)$. Thus

$$\sum_{j=1}^{k+1} (-1)^{j+1} y_j(qx) D_q W_q^{(j)}(x) = \begin{vmatrix} y_1(qx) & \cdots & y_{k+1}(qx) \\ (D_q y_1)(qx) & \cdots & (D_q y_{k+1})(qx) \\ (D_q^2 y_1)(qx) & \cdots & (D_q^2 y_{k+1})(qx) \\ \vdots & \ddots & \vdots \\ (D_q^{k-1} y_1)(qx) & \cdots & (D_q^{k-1} y_{k+1})(qx) \\ D_q^{k+1} y_1(x) & \cdots & D_q^{k+1} y_{k+1}(x) \end{vmatrix},$$

proving (3.1) for $n = k+1$ and hence all $k \in \mathbb{N}$. $\square$

Theorem 3.3. *If $y_1, y_2, \ldots, y_n$ are solutions of (1.5) in $J \subseteq I$, then their q-Wronskian satisfies the first order q-difference equation*

$$\begin{aligned} D_q W_q(x) &= -R(x) W_q(x), \quad x \in J \setminus \{0\} \\ R(x) &= \sum_{k=0}^{n-1} (x-qx)^k \frac{a_{k+1}(x)}{a_0(x)}. \end{aligned} \tag{3.3}$$

Proof. From the definition of the operator D_q, we have

$$(D_q^m y)(qx) = D_q^m y(x) - x(1-q) D_q^{m+1} y(x), \quad m \in \mathbb{N}.$$

Substituting in (3.1) yields

$$D_q W_q\big(y_1, \ldots, y_n\big)(x) = \begin{vmatrix} y_1(x) - x(1-q)D_q y_1(x) & \cdots & y_n(x) - x(1-q)D_q y_n(x) \\ D_q y_1(x) - x(1-q)D_q^2 y_1(x) & \cdots & D_q y_n(x) - x(1-q)D_q^2 y_n(x) \\ \vdots & \ddots & \vdots \\ D_q^{n-2} y_1(x) - x(1-q)D_q^{n-1} y_1(x) & \cdots & D_q^{n-2} y_n(x) - x(1-q)D_q^{n-1} y_n(x) \\ D_q^n y_1(x) & \cdots & D_q^n y_n(x) \end{vmatrix}.$$

We shall prove by induction on n that

$$D_q W_q\big(y_1, \ldots, y_n\big)(x) = \sum_{k=1}^{n} (-1)^{k-1} (x-qx)^{k-1} \begin{vmatrix} y_1(x) & \cdots & y_n(x) \\ D_q y_1(x) & \cdots & D_q y_n(x) \\ \vdots & \ddots & \vdots \\ D_q^{n-k-1} y_1(x) & \cdots & D_q^{n-k-1} y_n(x) \\ D_q^{n-k+1} y_1(x) & \cdots & D_q^{n-k+1} y_n(x) \\ \vdots & \ddots & \vdots \\ D_q^n y_1(x) & \cdots & D_q^n y_n(x) \end{vmatrix}. \tag{3.4}$$

If (3.4) holds at $n = m$, then

$$D_q W_q(y_1, y_2, \ldots, y_{m+1})(x) = \sum_{j=1}^{m+1} (-1)^{j+1} \big(y_j(x) - x(1-q) D_q y_j(x)\big) A_{1j},$$

where

$$A_{1j} = D_q W_q\big(D_q y_1, \ldots, D_q y_{j-1}, D_q y_{j+1}, \ldots, D_q y_{m+1}\big), \quad j = 1, 2, \ldots, m.$$

Hence from the previous hypothesis we obtain

$$\begin{aligned}
&D_qW_q(y_1,y_2,\dots,y_{m+1})(x)\\
&\quad=\sum_{j=1}^{m+1}(-1)^{j+1}\big(y_j(x)-x(1-q)D_qy_j(x)\big)\sum_{k=1}^{m}(-1)^{k-1}\big(x(1-q)\big)^{k-1}B_{jk}\\
&\quad=\sum_{k=1}^{m}(-1)^{k-1}\big(x(1-q)\big)^{k-1}\sum_{j=1}^{m+1}(-1)^{j+1}y_j(x)B_{jk}\\
&\qquad+\sum_{k=1}^{m}(-1)^{k}\big(x(1-q)\big)^{k}\sum_{j=1}^{m+1}(-1)^{j+1}D_qy_j(x)B_{jk},
\end{aligned}$$

with

$$B_{jk}:=\begin{vmatrix}
D_qy_1(x) & \cdots & D_qy_{j-1}(x) & D_qy_j(x) & \cdots & D_qy_{m+1}(x)\\
D_q^2y_1(x) & \cdots & D_q^2y_{j-1}(x)(x) & D_q^2y_j(x) & \cdots & D_q^2y_{m+1}(x)\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
D_q^{m-k}y_1(x) & \cdots & D_q^{m-k}y_{j-1}(x) & D_q^{m-k}y_j(x) & \cdots & D_q^{m-k}y_{m+1}(x)\\
D_q^{m-k+2}y_1(x) & \cdots & D_q^{m-k+2}y_{j-1}(x) & D_q^{m+k+1}y_{j+1}(x) & \cdots & D_q^{m+k+1}y_{m+1}(x)\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
D_q^{m+1}y_1(x) & \cdots & D_q^{m+1}y_{j-1}(x) & D_q^{m+1}y_{j+1}(x) & \cdots & D_q^{m+1}y_{m+1}(x)
\end{vmatrix},$$

$k=1,2,\dots,m$, where when $j=1$ the determinant B_{1k} start with $D_qy_2(x)$ and when $j=m+1$, the determinant $B_{(m+1)k}$ ends with $D_q^{m+1}y_m(x)$. From the properties of the determinants we conclude that

$$\sum_{j=1}^{m+1}(-1)^{j+1}y_j(x)B_{jk}=\begin{vmatrix}
y_1(x) & \cdots & y_{m+1}(x)\\
D_qy_1(x) & \cdots & D_qy_{m+1}(x)\\
\vdots & \ddots & \vdots\\
D_q^{m-k}y_1(x) & \cdots & D_q^{m-k}y_{m+1}(x)\\
D_q^{m-k+2}y_1(x) & \cdots & D_q^{m-k+2}y_{m+1}(x)\\
\vdots & \ddots & \vdots\\
D_q^{m+1}y_1(x) & \cdots & D_q^{m+1}y_{m+1}(x)
\end{vmatrix} \tag{3.5}$$

$$\sum_{j=1}^{m+1}(-1)^{j+1}D_qy_j(x)B_{jk}=0,\quad \text{for } k=1,2,\dots,m-1, \tag{3.6}$$

and

$$\sum_{j=1}^{m+1}(-1)^{j+1}D_qy_j(x)B_{jm}=\begin{vmatrix}
D_qy_1(x) & \cdots & D_qy_{m+1}(x)\\
D_q^2y_1(x) & \cdots & D_q^2y_{m+1}(x)\\
\vdots & \ddots & \vdots\\
D_q^{m+1}y_1(x) & \cdots & D_q^{m+1}y_{m+1}(x)
\end{vmatrix}. \tag{3.7}$$

Combining equations (3.5)–(3.7) with (3), we obtain (3.4) when $n = m+1$. One can easily see that (3.4) holds at $n = 1$. Consequently it holds for all $n \in \mathbb{N}$. From (1.5), we have

$$D_q^n y_j(x) = -\sum_{i=1}^{i=n} \frac{a_i(x)}{a_0(x)} D_q^{n-i} y_j(x), \quad j = 1, 2, \ldots, n.$$

Then (3.4) is nothing but

$$D_q W_q(x) = -\left[\sum_{k=0}^{k=n-1} (x - qx)^k \frac{a_{k+1}(x)}{a_0(x)}\right] W_q(x) = -R(x) W_q(x).$$

This completes the proof of the theorem. □

The next theorem gives a q-type Liouville's formula for the q-Wronskian.

Theorem 3.4. *Let $x(1-q)R(x) \neq -1$ for all $x \in J$. Then the q-Wronskian of any set of solutions $\{\phi_i\}_{i=1}^n$ of equation (1.5) is given by*

$$\begin{aligned} W_q(x) &= W_q(\phi_1, \ldots, \phi_n)(x) \\ &= \frac{1}{\prod_{k=0}^{\infty}(1 + x(1-q)q^k R(xq^k))} W_q(0), \quad x \in J. \end{aligned} \tag{3.8}$$

Proof. Equation (3.3) is

$$\frac{W_q(x) - W_q(qx)}{x - qx} = -R(x) W_q(x),\ x \neq 0,$$

i.e., $W_q(x) - W_q(qx) = -x(1-q)R(x)W_q(x)$. Hence, under the assumption $1 + x(1-q)R(x) \neq 0$, we obtain $W_q(x) = \frac{W_q(qx)}{1+x(1-q)R(x)}$. Therefore,

$$W_q(x) = \frac{W_q(xq^{m+1})}{\prod_{k=0}^{m}\big(1 + x(1-q)q^k R(xq^k)\big)}, \quad \text{for all}\, m \in \mathbb{N}\, \text{and}\, x \in I.$$

Since all functions $\frac{a_j}{a_0}$ are continuous at zero, then $\sum_{k=0}^{\infty} q^k |R(xq^k)|$ is convergent. Consequently, $\prod_{k=0}^{\infty}\big(1 + x(1-q)q^k R(xq^k)\big)$ converges for every $x \in I$. Thus, using the continuity of $W_q(x)$ at zero, (3.8) follows. □

Corollary 3.5. *Let $\{\phi_i\}_{i=1}^n$ be a set of solutions of* (1.5) *in some subinterval J of I which contains zero. Then $W_q(x)$ is either never zero or identically zero in I. The first case occurs when $\{\phi_i\}_{i=1}^n$ is a fundamental set of* (1.5) *and the second when it is not.*

Proof. A set of solutions $\{\phi_i\}_{i=1}^n$ forms a f.s. of (1.5) if and only if

$$W_q(0) = \det\left(D_q^{i-1}\phi_j(0)\right)_{i,j=1}^n \neq 0,$$

cf. [3]. This proves the corollary since from Theorem 3.4, $W_q(x) \neq 0$ for all $x \in J$ if and only if $W_q(0) \neq 0$. □

Example 3.6. In this example we calculate the q-Wronskian of

$$\frac{-1}{q} D_{q^{-1}} D_q y(x) + y(x) = 0, \quad x \in \mathbb{R}. \tag{3.9}$$

The solutions of (3.9) subject to the initial conditions

$$y(0) = 0,\ D_q y(0) = 1 \qquad \text{and} \qquad y(0) = 1,\ D_q y(0) = 0,$$

are $\sin(x;q)$, $\cos(x;q)$, $x \in \mathbb{R}$, respectively. Since (3.9) can be written as

$$D_q^2 y(x) + qx(1-q) D_q y(x) - q y(x) = 0.$$

Then $a_0(x) \equiv 1$, $a_1(x) = qx(1-q)$ and $a_2(x) = -q$. Thus $R(x) \equiv 0$ on $\mathbb{R}$ and $W_q(x) \equiv W_q(0)$. But

$$\begin{aligned} W_q(0) &= W_q\big(\cos(\cdot;q), \sin(\cdot;q)\big)(0) \\ &= \big(\cos(x;q)\cos(\sqrt{q}x;q) + \sqrt{q}\sin(x;q)\sin(\sqrt{q}x;q)\big)\big|_{x=0} \\ &= 1. \end{aligned}$$

Then, $W_q(x) \equiv 1$ for all $x \in \mathbb{R}$.

Example 3.7. We calculate the q-Wronskian of the solutions of the q-difference equations

$$-D_q^2 y(x) + y(x) = 0, \quad x \in \mathbb{R}. \tag{3.10}$$

The functions $\sin_q x$, $\cos_q x$, $|x|(1-q) < 1$, are solutions of (3.10) subject to the initial conditions

$$y(0) = 0,\ D_q y(0) = 1 \qquad \text{and} \qquad y(0) = 1,\ D_q y(0) = 0,$$

respectively. Here $R(x) = x(1-q)$. So, $x(1-q)R(x) \neq -1$ for all x in $\mathbb{R}$. Hence,

$$W_q(x) = \frac{W_q(0)}{\prod_{n=0}^{\infty}(1 + q^{2n}\{x(1-q)\}^2)}, \quad |x|(1-q) < 1.$$

But

$$W_q(0) = W_q\big(\cos_q, \sin_q\big)(0) = \big({\cos_q}^2 x + {\sin_q}^2 x\big)\big|_{x=0} = 1.$$

Therefore, $W_q(x) \equiv \left(\prod_{n=0}^{\infty}(1 + q^{2n}\{x(1-q)\}^2)\right)^{-1}$, $|x|(1-q) < 1$.

Remarks. 1. Theorem 3.3 might be satisfied for less restrictive conditions. But a general treatment needs a separate consideration. The q-Wronskian of (1.5) satisfies the first order q-difference equation (3.3) whatever the conditions which the functions a_j, $0 \le j \le n$ satisfy. But, in this case, the q-Wronskian can not be determined by using Theorem 3.4. An example of this case is the second order q-difference equation

$$qx^2(1-q)^2 D_q^2 y(x) + x(1-q)^2 D_q y(x) \\ + (x^2 q^{2-\nu} + (1-q^\nu)(1-q^{-\nu}))y(xq) = 0,$$

where $\nu > -1$, which has a f.s. $\{J_\nu(x;q^2), J_{-\nu}(xq^{-\nu};q^2)\}$ and it has been treated by R. F. Swarttouw and H. G. Meijer [15]. This class of problems may be considered as singular q-difference equations, while we are dealing with regular equations.

2. It is worthy to mention here that if equation (1.5) has the form

$$a_0(x) D_q^n y(x) + a_1(x)(D_q^{n-1})y(qx) + \cdots + a_n(x)y(qx) = 0, \quad x \in I, \tag{3.11}$$

then substituting with $D_q^n y(x) = -\sum_{j=1}^n \frac{a_j(x)}{a_0(x)}(D_q^{n-j}y)(qx)$ in (3.1) above we could derive a theory similar to that of the present section. In this case the associated q-Wronskian of solutions $z_1, \ldots, z_n$ of (3.11) will satisfy the simplified first order q-difference equation

$$D_q W_q(x) = -\frac{a_1(x)}{a_0(x)} W_q(qx).$$

Consequently

$$W_q(x) = \prod_{k=0}^{\infty} \left(1 - xq^k(1-q)\frac{a_1(xq^k)}{a_0(xq^k)}\right) W_q(0), \quad x \in J \setminus \{0\}.$$

Similar to differential equations if $a_1 \equiv 0$, then $W_q(x)$ is identically a constant. It should be noted that problems involving equation of the form (3.11) plays an important role in defining self adjoint eigenvalue problems, see e.g. [5, 8].

4. Applications

The theory introduced in the previous two sections can be used to obtain a general formula for the solutions of the inhomogeneous equation (1.3). Obviously, if ψ_1 and ψ_2 are two solutions of (1.3), then $\psi_1 - \psi_2$ is a solution of the corresponding homogeneous equation (1.5). Thus if ψ is a solution of (1.3) and $\{\phi_i\}_{i=1}^n$ is a f.s. for (1.5), then there are unique constants $\{c_i\}_{i=1}^n$ such that

$$\psi = c_1\phi_1 + \cdots + c_n\phi_n + \psi_0,$$

where ψ_0 is a particular solution of (1.3). Now, we introduce a q-analog of the method of variation of parameters to find a particular solution ψ_0 of (1.3). Here also the functions a_r and b are continuous at zero functions defined on I such that $a_0(x) \neq 0$ for all $x \in I$.

Theorem 4.1. *Let $\{\phi_i\}_{i=1}^n$ be a fundamental set of (1.5) in J. Then, any solution ψ of (1.3) is given by*

$$\psi(x) = \sum_{i=1}^{n} \left(c_i + \int_0^x \frac{W_{q,i}\left(\phi_1,\, \ldots,\, \phi_n\right)(qt)}{W_q\left(\phi_1,\, \ldots,\, \phi_n\right)(qt)} \cdot \frac{b(t)}{a_0(t)}\, d_qt \right) \phi_i(x), \tag{4.1}$$

where the c_i's are constants and $W_{q,r}(\phi_1, \ldots, \phi_n)(x)$ is the determinant obtained from $W_q(\phi_1, \ldots, \phi_n)(x)$ by replacing the r-th column by $(0, \ldots, 0, 1)$.

Proof. Let ψ be a solution of (1.3). If ψ_0 is a particular solution of (1.3), then for some constants $c_1, c_2, \ldots, c_n$,

$$\psi = \psi_0 + c_1\phi_1 + \cdots + c_n\phi_n,$$

where $c_1, \ldots, c_n$ are constants. Assume that ψ_0 has the form

$$\psi_0(x) = u_1(x)\phi_1(x) + \cdots + u_n(x)\phi_n(x),$$

where $u_1, \ldots, u_n$ are functions satisfying the system

$$\begin{aligned} D_qu_1(x)\phi_1(qx) + \cdots + D_qu_n(x)\phi_n(qx) &= 0 \\ D_qu_1(x)D_{q,qx}\phi_1(qx) + \cdots + D_qu_n(x)D_{q,qx}\phi_n(qx) &= 0 \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \qquad\quad & \quad \vdots \\ D_qu_1(x)D^{n-2}_{q,qx}\phi_1(qx) + \cdots + D_qu_n(x)D^{n-2}_{q,qx}\phi_n(qx) &= 0 \\ D_qu_1(x)D^{n-1}_{q,qx}\phi_1(qx) + \cdots + D_qu_n(x)D^{n-1}_{q,qx}\phi_n(qx) &= \frac{b(x)}{a_0(x)}. \end{aligned} \tag{4.2}$$

System (4.2) is an inhomogeneous linear system of equations in the n unknowns $\{D_qu_i\}_{i=1}^n$. The determinant of the coefficients is $W_q(\phi_1, \ldots, \phi_n)(qx) \neq 0$ since ϕ_r, $1 \leqslant r \leqslant n$, is a f.s. for (1.5). Hence (4.2) can be solved for the D_qu_r and

$$D_qu_r(x) = \frac{W_{q,r}(\phi_1, \ldots, \phi_n)(qx)}{W_q(\phi_1, \ldots, \phi_n)(qx)} \cdot \frac{b(x)}{a_0(x)}, \quad r = 1, \ldots, n.$$

Since $\frac{b}{a_0}$ is continuous at zero, then from Theorem 1.1 D_qu_r is q-integrable on $[0, x]$, for all $x \in J$. Thus, a suitable choice for $u_r(x)$ is

$$u_r(x) = \int_0^x \frac{W_{q,r}(\phi_1, \ldots, \phi_n)(qt)}{W_q(\phi_1, \ldots, \phi_n)(qt)} \cdot \frac{b(t)}{a_0(t)}\, d_qt,$$

and then ψ_0 has the form

$$\psi_0(x) = \sum_{i=1}^{n} \phi_i(x) \int_0^x \frac{W_{q,i}\left(\phi_1, \ldots, \phi_n\right)(qt)}{W_q\left(\phi_1, \ldots, \phi_n\right)(qt)} \cdot \frac{b(t)}{a_0(t)}\, d_q t,$$

proving formula (4.1). □

Example 4.2. Consider the equation

$$-\frac{1}{q} D_{q^{-1}} D_q y(x) + y(x) = b(x), \tag{4.3}$$

where $b(\cdot)$ is a continuous function defined in $\mathbb{R}$. The corresponding homogeneous equation is

$$D_q^2 y(x) - q y(qx) = 0. \tag{4.4}$$

A fundamental set of solutions of (4.4) is $\{\sin(x;q), \cos(x;q)\}$. Substituting in (4.1) and using $W_q(\sin(\cdot;q), \cos(\cdot;q))(x) \equiv -1$, every solution of (4.3) has the form

$$\begin{aligned}\psi(x) &= c_1 \sin(x;q) + c_2 \cos(x;q) \\ &\quad - q \int_0^x \big(\sin(x;q)\cos(qt;q) - \cos(x;q)\sin(qt;q)\big) b(qt)\, d_q t\,,\end{aligned}$$

where $x \in \mathbb{R}$, c_1 and c_2 are arbitrary constants.

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