Interpolation and Transmutation

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Abstract. We show that the existence of a transmutation between two self-adjoint operators $L_1$ and $L_2$ is equivalent to the existence of an interpolation operator in the spectral variable. This equivalence helps construct a transmutation operator between abstract self-adjoint operators.

Keywords. Sampling, interpolation, transmutation

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1. Introduction

We are concerned with the existence of a transmutation also known as a transformation operator between two given self-adjoint operators, $L_1$ and $L_2$ that act in the Hilbert spaces $H_1$ and $H_2$, respectively. Recall that a linear operator $W$ is said to be a transmutation operator if $H_2 \xrightarrow{W} H_1$ and

$$L_1 W = W L_2$$

(1)

holds on a dense subspace of the Hilbert space $H_2$. If the operator $W$ is invertible, then $L_1 = W L_2 W^{-1}$ and this helps reconstruct the operator $L_1$ from the knowledge of both $L_2$ and $W$. The concept of transmutation became an essential tool for the inverse spectral problem by the Gelfand-Levitan theory, see [9, 12]. Further concepts and applications of transmutations can be found in the books by Carroll, see [5, 6]. Observe that (1) can also be seen as the homogeneous part of an operator equation in $X$

$$L_1 X - X L_2 = Y,$$

(2)

where $Y$, $L_1$ and $L_2$ are given operators. When $L_1$ and $L_2$ are bounded operators, one can prove the existence and uniqueness of a solution $X$, see [2, 13],

$$X = \frac{1}{2\pi i} \int_{\Gamma} (L_1 - \lambda I)^{-1} Y (L_2 - \lambda I)^{-1} d\lambda$$

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and (2) has a unique solution if and only if (1) has the trivial solution. Ob-
serv that equation (1), in the simple case when $L_1$ and $L_2$ are finite matrices 
with disjoint spectra, has the trivial solution $W = 0$, see also the Sylvester-
Rosemblum theorem [2]. A simple way to see this classical result is to assume 
that if $v$ is an eigenvector for $L_2$, i.e., $L_2v = \lambda v$ where $\lambda \in \sigma_2$ and $\sigma_i$ 
denotes the spectrum of $L_i$, for $i = 1, 2$. Then (1) implies $L_1Wv = WL_2v = \lambda Wv$, 
and so either $Wv = 0$ or $\lambda \notin \sigma_1$. Since $\sigma_1 \cap \sigma_2 = \emptyset$, we must have $Wv = 0$ and the 
fact that $v$ is an arbitrary eigenvector implies that $W = 0$.

It is also known that if $L_1$ and $L_2$ are unbounded operators uniqueness may 
not hold, see also examples using the shift operator in [2]. Observe that in the 
case where operators have continuous spectra, the above simple argument fails 
because eigenfunctions are now distributions see [10]. Let us define the linear 
operator $\tau_{12}$ by

$$\tau_{12}(X) := L_1X - XL_2$$

and thus (2) becomes $\tau_{12}(X) = Y$. Then the existence and uniqueness of a 
solution $X$ to (2) is equivalent to the invertibility of the operator $\tau_{12}$. It turns out 
that the spectrum of $\tau_{12}$ always contains the direct sum $\sigma_1 - \sigma_2$, see [1], and thus 
if $\sigma_1 \cap \sigma_2 \neq \emptyset$, then it is not invertible. In other words, any nontrivial bounded 
operator solution $W$ for (1) belongs to the null space of the operator $\tau_{12}$.

In this note we show that equation (1) has non trivial unbounded solutions 
even if $\sigma_1 \cap \sigma_2 = \emptyset$, which means that (2) has no uniqueness in the class of 
unbounded operators. More precisely we show that a nontrivial solution $W$ 
for (1) exists if and only if a special interpolation operator between the spaces 
of the transforms does. When both operators are self-adjoint, the approach 
also allows for interpolation on the real line, and more precisely reconstructing 
values of a transform on $\sigma_1$ from its known values on $\sigma_2$. Most interesting cases 
will arise when the spectra are discrete and disjoint as the interpolation reduces 
to the well known idea of sampling, see [14, 16].

To motivate the approach, let us explain how to construct an explicit solu-
tion of (1) while $\sigma_1 \cap \sigma_2 = \emptyset$. Consider the unbounded self-adjoint differential 
operators

$$\begin{aligned}
L_1(f)(x) &:= -f''(x) + q_1(x)f(x), \ x \geq 0 \\
f'(0) - h_1f(0) &= 0 \\
L_2(f)(x) &:= -f''(x) + q_2(x)f(x), \ x \geq 0 \\
f'(0) - h_2f(0) &= 0
\end{aligned}$$

which act in the Hilbert space $H_2 = H_1 = L^2(0, \infty)$. For $i = 1, 2$, let us denote 
their eigenfunctionals by

$$L_i(y_i)(x, \lambda) = \lambda y_i(x, \lambda)$$

(4)
which we normalize by $y_i(0, \lambda) = 1$. By the Gelfand–Levitan theory, we can always construct $q_1$ and $q_2$ such that $\sigma_1$ and $\sigma_2$ are discrete and disjoint $\sigma_1 \cap \sigma_2 = \emptyset$, see [8]. On the other hand, we have the existence of transformation operators such that

$$y_i(x, \lambda) = \cos(x\sqrt{\lambda}) + \int_0^x K_i(x, t) \cos(t\sqrt{\lambda}) \, dt$$

$$\cos(x\sqrt{\lambda}) = y_i(x, \lambda) + \int_0^x H_i(x, t)y_i(t, \lambda) \, dt,$$

where $K_i$ and $H_i$ are continuous kernels. The next step is to compose the above mappings, as to eliminate $\cos(x\sqrt{\lambda})$ and write

$$y_2(x, \lambda) = y_1(x, \lambda) + \int_0^x (H_1(x, t) + K_2(x, t)) y_1(t, \lambda) \, dt$$

$$+ \int_0^x K_2(x, t) \int_0^t H_1(t, s)y_1(s, \lambda) \, ds \, dt$$

$$= y_1(x, \lambda) + \int_0^x K_{12}(x, t)y_1(t, \lambda) \, dt,$$

where $K_{12}$ is continuous in $(x, t)$, and so we can write

$$y_2(x, \lambda) = V(y_1)(x, \lambda).$$

The operator $V$ then is an unbounded operator solution to (1) since $L_2V = VL_1$ holds over the set $\{y_1(x, \lambda)\}_{\lambda \in \sigma_1}$ which is a complete set of functionals. To see the unboundedness of $V$ observe that if $\lambda_n \in \sigma_1$, then $y_1(x, \lambda_n) \in L^2(0, \infty)$ while $y_2(x, \lambda_n) = V(y_1)(x, \lambda_n) \notin L^2(0, \infty)$ since the spectra are disjoint. This adds a simple counter example to the Sylvester-Rosemblum theorem in the case the operators are unbounded.

2. Notation

We shall assume that $L_1$ and $L_2$ are both unbounded self-adjoint operators acting in the separable Hilbert spaces $H_1$ and $H_2$, respectively. For the sake of simplicity, we assume that their respective spectra $\sigma_1$ and $\sigma_2$ are simple. Then by the spectral theorem, [15, p. 31], for $i = 1, 2$, each operator $L_i$ generates an isomorphism or a transform $F_i$ such that

$$H_i \xrightarrow{F_i} L_{dp_i}^2$$

with

$$L_{dp_i}^2 := \left\{ F \text{ measurable: } \int_{-\infty}^\infty |F(\lambda)|^2 \, d\rho_i(\lambda) < \infty \right\}$$

$$F_i(L_i f)(\lambda) = \lambda F_i(f)(\lambda) \quad \text{and} \quad \|f\|_i^2 = \int_{-\infty}^\infty |F_i(f)(\lambda)|^2 \, d\rho_i(\lambda),$$
where \( \| \cdot \|_i \) is the norm in \( H_i, i = 1, 2 \). The function \( \rho_i \) is called the spectral function and defines a Lebesgue-Stieltjes measure \( d\rho_i \). Thus it is non-decreasing, has a jump discontinuity at an eigenvalue only, is increasing on the continuous spectrum and its support \( \text{supp} d\rho_i = \sigma_i \). The existence of a spectral function guarantees that the spectrum is simple otherwise it is a matrix. In [10], one can find a more general setting for the spectral theory of operators in rigged Hilbert spaces, based on fact that when \( \lambda \) is in the continuous spectrum, the corresponding eigenfunctional is a generalized function.

Let us denote by \( \text{Dom}(W) \) the domain of the operator \( W \). We begin with few definitions.

**Definition 2.1.** \( W \) is a transformation operator ((T.O.) for short) if

i) \( W : H_2 \rightarrow H_1 \) and \( \overline{\text{Dom}(W)} = H_2 \);

ii) the set \( \Omega := \{ f \in \text{Dom}(W) \text{ and } L_2 f \in \text{Dom}(W) \} \) is dense in \( H_2 \);

iii) \( L_1 W(f) = WL_2(f) \) holds for any \( f \in \Omega \).

The above definition agrees with the definition of a transformation operator as given in [11], except for its boundedness. We now define the interpolation operator which connects both transforms.

**Definition 2.2.** \( J \) is an interpolation operator ((I.O.) for short) if

1) is a densely closed linear operator \( L^2_{d\rho_2} \xrightarrow{J} L^2_{d\rho_1} \);

2) the set \( S := \{ F \in \text{Dom}(J) \text{ and } \lambda F(\cdot) \in \text{Dom}(J) \} \) is dense in \( L^2_{d\rho_2} \);

3) for any \( F \in S \) we have \( \lambda J(F)(\lambda) = J(\lambda F)(\lambda) \).

At first sight the operator \( J \) is simply a mapping between two weighted \( L^2 \) spaces. The idea of interpolation is contained in the following:

**Proposition 2.3.** If \( J \) is an I.O. then \( \phi(\lambda) J(F)(\lambda) = J(\phi F)(\lambda) \) holds for any analytic function \( \phi \) and \( F \in L^2_{d\rho_2} \) with a compact support.

**Proof.** Let \( F \in L^2_{d\rho_2} \) have a compact support then for any \( n \geq 0 \) we have \( \lambda^n F(\lambda) \in L^2_{d\rho_2} \), \( \lambda^n F(\lambda) \in S \) and, by condition 3),

\[
\lambda^n J(F)(\lambda) = J(\lambda^n F)(\lambda).
\]

The next step we use the fact that any analytic function about the origin can be written as a power series \( \phi(\lambda) = \sum_{n \geq 0} a_n \lambda^n \) and since \( J \) is closed operator we have

\[
\sum_{n \geq 0} a_n \lambda^n J(F)(\lambda) = J\left( \sum_{n \geq 0} a_n \lambda^n F \right)(\lambda)
\]

\[
\phi(\lambda) J(F)(\lambda) = J(\phi F)(\lambda).
\]
Also by translation we have \((\lambda - a) J(F)(\lambda) = J((\lambda - a) F)(\lambda)\) which extends
the argument to any analytic function. While the function \(\phi F\) is known only
over \(\sigma_2\), \(\phi\) is constructed over a new domain \(\sigma_1\), whenever \(J(F)(\lambda) \neq 0\), by the
formula
\[
\phi(\lambda) = J(\phi F)(\lambda)/J(F)(\lambda).
\]
Thus to define \(\phi\) at different values say \(\lambda_0\), we need to use a function \(F\) with
\(J(F)(\lambda_0) \neq 0\).

On the other hand if \(J\) is a sampling operator in the classical sense then
condition 3) \(\lambda J(F)(\lambda) = J(\lambda F)(\lambda)\) is obvious as shown by the following simple
example of an interpolation operator.

Let \(\sigma_2 = \mathbb{Z}\) where \(\mathbb{Z}\) is the set of integers and \(\sigma_1 = \{\lambda_n\}\) where \(\lambda_n \notin \mathbb{Z}\) and
thus \(\sigma_1 \cap \sigma_2 = \emptyset\). Let us recall the definition
\[
PW_\pi = \left\{ F \text{ entire: } |F(\lambda)| \leq M e^{\pi|\Im(\lambda)|} \text{ and } \int_{-\infty}^{\infty} |F(x)|^2 \, dx < \infty \right\}.
\]
The Shannon–Whittacker–Kotelnikov sampling theorem [16] allows us to write
down a mapping explicitly for \(F \in PW_\pi\):
\[
F(\mu) := \sum_{n \in \mathbb{Z}} F(n) \frac{\sin(\pi(\mu - n))}{\pi(\mu - n)} \quad \text{for } \sum_{n \in \mathbb{Z}} |F(n)|^2 < \infty.
\]
Thus take the space \(L^2_{d\rho_2}\) where the measure \(\rho_2(\lambda) = [\lambda]\) represents the greatest
integer function in \(\lambda\). If \(\{F(n)\}_{n \in \mathbb{Z}}\) is given, then \(\{F(\lambda_n)\}_{n \in \mathbb{Z}}\) can be obtained from
\[
J(F)(\lambda_n) := \sum_{k \in \mathbb{Z}} F(k) \frac{\sin(\pi(\lambda_n - k))}{\pi(\lambda_n - k)}.
\]
A mapping \(L^2_{d\rho_2} \overset{J}{\rightarrow} L^2_{d\rho_1}\) can now be defined by the operation in (8) and by (7)
we in fact have \(J(F)(\lambda_n) = F(\lambda_n)\). It remains to see that condition 3) then
holds since, for \(\lambda F(\cdot) \in L^2_{d\rho_2}\), \(J(\lambda F(\cdot))(\lambda_n) = \lambda_n F(\lambda_n) = \lambda_n J(F)(\lambda_n)\).

3. Interpolation

We now prove the main result.

**Proposition 3.1.** Assume that \(L_i\) is an unbounded self adjoint operators acting
in \(H_i\) with spectral functions \(\rho_i\) for \(i = 1, 2\). Let \(J\) be a linear operator \(L^2_{d\rho_2} \overset{J}{\rightarrow} L^2_{d\rho_1}\) and define
\[
W = F_1^{-1} J F_2.
\]
Then \(W\) is a T.O. if and only if \(J\) is an I.O.
Proof. It is enough to show that the conditions in Definitions 2.2 and 2.1 are equivalent in their respective order. Since \( F_1 \) and \( F_2 \) are unitary operators it follows from (9) that \( W \) is densely defined if and only if \( J \) is densely defined. For the second point, we need to show that \( S \) is dense if and only if \( \Omega \) is dense. From (9) we have

\[
\psi \in \text{Dom}(J) \iff F_2^{-1}(\psi) \in \text{Dom}(W)
\]

and hence

\[
\psi \in S \iff F_2^{-1}(\psi) \in \Omega.
\]

In other words \( S \) is dense in \( L^2_{d\rho_2} \) if and only if \( \Omega \) is dense in \( H_2 \). For the third condition, let \( f \in \Omega \), then

\[
L_1W(f) = L_1F_1^{-1}JF_2(f) = F_1^{-1}\lambda JF_2(f)
\]

\[
WL_2(f) = F_2^{-1}JF_2L_2(f) = F_2^{-1}J\lambda JF_2(f)
\]

which simply says that

\[
L_1W(f) = WL_2(f) \quad \forall f \in \Omega \iff J(\lambda F) = \lambda J(F) \quad \forall F \in S.
\]

Therefore \( W \) is a T.O. if and only if \( J \) is an I.O.

Once the connection between I.O. and T.O. has been established, we now show how to construct an I.O. in a particular case. For the sake of simplicity, we shall call upon the well known Gelfand and Levitan theory, see [9] and [8].

From the given kernel defined by (5) define its adjoint \( W = V' \), i.e., \( W : L^2(0, \infty) \rightarrow L^2(0, \infty) \):

\[
Wf(x) = f(x) + \int_x^\infty K(t, x)f(t) \, dt.
\]

(10)

Since the kernel \( K(x, t) \), by the Gelfand-Levitan theory, is a continuous function in both variables \( W \) is densely defined as its domain includes for example \( C_0(0, \infty) \).

Let us recall that the Gelfand–Levitan theory requires the spectral function \( \rho \) to satisfy the following conditions, where

\[
\sigma(\lambda) := \begin{cases} 
\rho(\lambda) - \frac{2}{\pi} \sqrt{\lambda} & \text{if } \lambda \geq 0 \\
\rho(\lambda) & \text{if } \lambda < 0.
\end{cases}
\]

Theorem 3.2 (Gelfand–Levitan–Gasymov). For \( \rho(\lambda) \), a nondecreasing and right-continuous function to be the spectral function of (3) it is necessary and sufficient that it satisfies the following conditions:
(A) for \( f \in L^2_{dx}(0, \infty) \) with compact support,
\[
\int_{-\infty}^{\infty} |E(f)(\lambda)|^2 \, d\rho(\lambda) = 0 \implies f(x) \equiv 0,
\]
where \( E(f)(\lambda) := \int_{0}^{\infty} f(x) \cos(x\sqrt{\lambda}) \, dx \);

(B) \( \int_{-\infty}^{\infty} \cos(x\sqrt{\lambda}) \, d\sigma(\lambda) \) converges boundedly to \( \Psi(x) \) as \( N \to \infty \) and \( \Psi \) has two locally integrable derivatives.

We have

**Proposition 3.3.** Assume that \( \rho_i(\lambda) \) satisfy conditions (A) and (B), then the operator \( J : L^2_{\rho_2} \to L^2_{\rho_1} \) defined by
\[
J(F)(\lambda) := \int_{0}^{\infty} W(f)(x)y_1(x, \lambda) \, dx,
\]
where \( f(x) := \int_{0}^{\infty} F(\lambda)y_2(x, \lambda) \, d\rho_2(\lambda) \) and \( f \in C^2_0[0, \infty) \) is an interpolation operator in the sense of Definition 2.2.

**Proof.** Conditions (A) and (B) ensure the existence of potentials \( q_i(x) \) for the solution of the inverse spectral problem and the recovered differential operators (4) generate unitary transforms
\[
F_i : L^2_{dx}(0, \infty) \to L^2_{\rho_i}, \quad i = 1, 2
\]
\[
F_i(f)(\lambda) := \int_{0}^{\infty} f(x)y_i(x, \lambda) \, dx \quad \text{and} \quad f(x) := \int F_i(f)(\lambda)y_i(x, \lambda) \, d\rho_i(\lambda).
\]
The operator \( J \) can be defined via \( L^2(0, \infty) \) as in (9):
\[
J := F_1 W F_2^{-1},
\]
where \( W \) is defined by (10). We now verify that the three conditions for \( J \) to be an I.O. are satisfied. By (12), \( F \in \text{Dom}(J) \) if and only if \( F^{-1}(F) \in \text{Dom}(W) \). Since \( F_2 \) is a unitary operator and \( \text{Dom}(W) \) is dense in \( L^2(0, \infty) \) it follows that \( J \) is also densely defined in \( L^2_{\rho_2} \).

For the second condition it is enough to show that \( \Omega = \{ f \in \text{Dom}(W) \} \) is dense in \( L^2(0, \infty) \). If \( f \in C^2_0(0, \infty) \), then \( f \in \text{Dom}(W) \) and \( L_2(f) = -f'' + q_2 f \in C_0(0, \infty) \) and so \( L_2 f \in \text{Dom}(W) \). Thus \( C^2_0(0, \infty) \subset \Omega \), and from the density of \( C^2_0(0, \infty) \) it follows that \( \Omega \) is also dense in \( L^2(0, \infty) \). It remains to see that \( S \) is unitarily equivalent to \( \Omega \):
\[
f \in \Omega \iff F_2(f) \in S
\]
and therefore it is also dense in \( L^2_{\rho_2} \).
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The last condition to verify is if \( F \in S \) then \( \lambda J (F) (\lambda) = J (\lambda F (\cdot)) (\lambda) \). Let \( F(\lambda) := F_2 (f) (\lambda) \) where \( f \in C_0^\infty (0, \infty) \). We then have \( F \in S \) and \( \lambda F (\lambda) = F_2 (L_2 f) (\lambda) \), and it follows by (11) and the adjoint of \( V \) defined in (10) that

\[
\lambda J (F) (\lambda) = \lambda \int_0^\infty W(f) (x) y_1(x, \lambda) \, dx
= \lambda \int_0^\infty V'(f) (x) y_1(x, \lambda) \, dx
= \lambda \int_0^\infty f(x) V(y_1) (x, \lambda) \, dx
= \lambda \int_0^\infty f(x) y_2(x, \lambda) \, dx
= \int_0^\infty f(x) L_2 (y_2) (x, \lambda) \, dx
= \int_0^\infty L_2 (f) (x) y_2(x, \lambda) \, dx
= \int_0^\infty L_2 (f) (x) V(y_1) (x, \lambda) \, dx
= \int_0^\infty W L_2 (f) (x) y_1(x, \lambda) \, dx
= J (\lambda F (\cdot)) (\lambda).
\]

\[\square\]

**Corollary 3.4.** Let the conditions of Proposition 3 hold, then \( W \) is a nontrivial solution of the operator equation \( W L_2 = L_1 W \).

**Proof.** Since \( J \) is an I.O. operator, it follows from Proposition 2 that \( W \) is a T.O. \[\square\]

We now end this section by observing that if two given abstract self-adjoint operators \( P_1 \) and \( P_2 \) are similar to \( L_1 \) and \( L_2 \), in the sense they have the same spectral functions, then they “share” the same existing I.O. between \( L_1 \) and \( L_2 \). Indeed from the similarities relations

\[
L_1 = UP_1 U^{-1}, \quad L_2 = RP_2 R^{-1} \quad \text{and} \quad WL_2 = L_1 W
\]

it follows that \( WRP_2 R^{-1} = UP_1 U^{-1} W \), i.e., \( U^{-1} WRP_2 = P_1 U^{-1} W R \). Thus \( U^{-1} W R \) is the new T.O. for \( P_1 \) and \( P_2 \).

**Corollary 3.5.** Assume that \( P_i \) is an unbounded self adjoint operator acting in \( H_i \) with transform \( \tilde{F}_i \) and its spectral function \( \rho_i \), for \( i = 1, 2 \), satisfies conditions (A) and (B) in the Gelfand–Levitan–Gasymov theorem, then a T.O. \( \tilde{W} \) between \( P_1 \) and \( P_2 \) is simply given by

\[
\tilde{W} \psi := \tilde{F}_1^{-1} \int_0^\infty W f(x) y_1(x, \lambda) \, dx \quad \text{and} \quad f(x) = \int \tilde{F}_2 (\psi) (\lambda) y_2(x, \lambda) \, d\rho_2 (\lambda),
\]

where \( y_i(x, \lambda) \) are the eigenfunctionals of \( L_i \) defined in (4).
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Proof. Since the Gelfand–Levitan theory already provides a standard I.O., see (11), it follows by Proposition 2, that

\[ \tilde{F}_1 \tilde{W} \tilde{F}_2^{-1} = J = F_1 W F_2^{-1} \quad \text{and} \quad \tilde{W} = \tilde{F}_1^{-1} F_1 W F_2^{-1} \tilde{F}_2, \]

and thus \( \tilde{W} (\psi) = \tilde{F}_1^{-1} F_1 W (f) \) where \( f = F_2^{-1} \tilde{F}_2 (\psi) \), and \( W \) is given by (10).

Thus we have seen that the use of spectral functions allowed us to extend the Rosemblum–Sylvester theorem to unbounded operators, and furthermore it provides a new constructive approach to the solution of operator equation of type (1).

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References


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