Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 26 (2007), 407–416
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Interpolation and Transmutation

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Abstract. We show that the existence of a transmutation between two self-adjoint operators L_1 and L_2 is equivalent to the existence of an interpolation operator in the spectral variable. This equivalence helps construct a transmutation operator between abstract self-adjoint operators.

Keywords. Sampling, interpolation, transmutation

Mathematics Subject Classification (2000). Primary 41A05, secondary 94A20

1. Introduction

We are concerned with the existence of a transmutation also known as a transformation operator between two given self-adjoint operators, L_1 and L_2 that act in the Hilbert spaces H_1 and H_2, respectively. Recall that a linear operator W is said to be a transmutation operator if $H_2 \overset{W}{\mapsto} H_1$ and

$$L_1 W = W L_2 \tag{1}$$

holds on a dense subspace of the Hilbert space H_2. If the operator W is invertible, then $L_1 = W L_2 W^{-1}$ and this helps reconstruct the operator L_1 from the knowledge of both L_2 and W. The concept of transmutation became an essential tool for the inverse spectral problem by the Gelfand Levitan theory, see [9, 12]. Further concepts and applications of transmutations can be found in the books by Carroll, see [5, 6]. Observe that (1) can also be seen as the homogeneous part of an operator equation in X

$$L_1 X - X L_2 = Y, \tag{2}$$

where Y, L_1 and L_2 are given operators. When L_1 and L_2 are bounded operators, one can prove the existence and uniqueness of a solution X, see [2, 13],

$$X = \frac{1}{2\pi i} \int_\Gamma (L_1 - \lambda I)^{-1} \, Y \, (L_2 - \lambda I)^{-1} \, d\lambda$$

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and (2) has a unique solution if and only if (1) has the trivial solution. Observe that equation (1), in the simple case when L_1 and L_2 are finite matrices with disjoint spectra, has the trivial solution $W = 0$, see also the Sylvester-Rosemblum theorem [2]. A simple way to see this classical result is to assume that if v is an eigenvector for L_2, i.e., $L_2 v = \lambda v$ where $\lambda \in \sigma_2$ and σ_i denotes the spectrum of L_i, for $i = 1, 2$. Then (1) implies $L_1 W v = W L_2 v = \lambda W v$, and so either $Wv = 0$ or $\lambda \in \sigma_1$. Since $\sigma_1 \cap \sigma_2 = \emptyset$, we must have $Wv = 0$ and the fact that v is an arbitrary eigenvector implies that $W = 0$.

It is also known that if L_1 and L_2 are unbounded operators uniqueness may not hold, see also examples using the shift operator in [2]. Observe that in the case where operators have continuous spectra, the above simple argument fails because eigenfunctions are now distributions see [10]. Let us define the linear operator τ_{12} by

$$\tau_{12}(X) := L_1 X - X L_2$$

and thus (2) becomes $\tau_{12}(X) = Y$. Then the existence and uniqueness of a solution X to (2) is equivalent to the invertibility of the operator τ_{12}. It turns out that the spectrum of τ_{12} always contains the direct sum $\overline{\sigma_1 - \sigma_2}$, see [1], and thus if $\sigma_1 \cap \sigma_2 \neq \emptyset$, then it is not invertible. In other words, any nontrivial bounded operator solution W for (1) belongs to the null space of the operator τ_{12}.

In this note we show that equation (1) has non trivial unbounded solutions even if $\sigma_1 \cap \sigma_2 = \emptyset$, which means that (2) has no uniqueness in the class of unbounded operators. More precisely we show that a nontrivial solution W for (1) exists if and only if a special interpolation operator between the spaces of the transforms does. When both operators are self-adjoint, the approach also allows for interpolation on the real line, and more precisely reconstructing values of a transform on σ_1 from its known values on σ_2. Most interesting cases will arise when the spectra are discrete and disjoint as the interpolation reduces to the well known idea of sampling, see [14, 16].

To motivate the approach, let us explain how to construct an explicit solution of (1) while $\sigma_1 \cap \sigma_2 = \emptyset$. Consider the unbounded self-adjoint differential operators

$$\begin{cases} L_1(f)(x) := -f''(x) + q_1(x) f(x), \ x \geq 0 \\ f'(0) - h_1 f(0) = 0 \end{cases}$$

$$\begin{cases} L_2(f)(x) := -f''(x) + q_2(x) f(x), \ x \geq 0 \\ f'(0) - h_2 f(0) = 0 \end{cases} \tag{3}$$

which act in the Hilbert space $H_2 = H_1 = L^2(0, \infty)$. For $i = 1, 2$, let us denote their eigenfunctionals by

$$L_i(y_i)(x, \lambda) = \lambda y_i(x, \lambda) \tag{4}$$

which we normalize by $y_i(0,\lambda) = 1$. By the Gelfand–Levitan theory, we can always construct q_1 and q_2 such that σ_1 and σ_2 are discrete and disjoint $\sigma_1 \cap \sigma_2 = \emptyset$, see [8]. On the other hand, we have the existence of transformation operators such that

$$y_i(x,\lambda) = \cos\left(x\sqrt{\lambda}\right) + \int_0^x K_i(x,t)\cos\left(t\sqrt{\lambda}\right) dt$$
$$\cos\left(x\sqrt{\lambda}\right) = y_i(x,\lambda) + \int_0^x H_i(x,t) y_i(t,\lambda)\, dt,$$

where K_i and H_i are continuous kernels. The next step is to compose the above mappings, as to eliminate $\cos(x\sqrt{\lambda})$ and write

$$\begin{aligned} y_2(x,\lambda) &= y_1(x,\lambda) + \int_0^x \left(H_1(x,t) + K_2(x,t)\right) y_1(t,\lambda)\, dt \\ &\quad + \int_0^x K_2(x,t) \int_0^t H_1(t,s) y_1(s,\lambda)\, ds\, dt \\ &= y_1(x,\lambda) + \int_0^x K_{12}(x,t) y_1(t,\lambda)\, dt, \end{aligned} \tag{5}$$

where K_{12} is continuous in (x,t), and so we can write

$$y_2(x,\lambda) = V\left(y_1\right)(x,\lambda). \tag{6}$$

The operator V then is an unbounded operator solution to (1) since $L_2 V = V L_1$ holds over the set $\{y_1(x,\lambda)\}_{\lambda\in\sigma_1}$ which is a complete set of functionals. To see the unboundedness of V observe that if $\lambda_n \in \sigma_1$, then $y_1(x,\lambda_n) \in L^2(0,\infty)$ while $y_2(x,\lambda_n) = V\left(y_1\right)(x,\lambda_n) \notin L^2(0,\infty)$ since the spectra are disjoint. This adds a simple counter example to the Sylvester-Rosemblum theorem in the case the operators are unbounded.

2. Notation

We shall assume that L_1 and L_2 are both unbounded self-adjoint operators acting in the separable Hilbert spaces H_1 and H_2, respectively. For the sake of simplicity, we assume that their respective spectra σ_1 and σ_2 are simple. Then by the spectral theorem, [15, p. 31], for $i = 1, 2$, each operator L_i generates an isomorphism or a transform F_i such that

$$H_i \overset{F_i}{\mapsto} L^2_{d\rho_i}$$

with

$$L^2_{d\rho_i} := \left\{ F \text{ measurable: } \int_{-\infty}^{\infty} \left|F(\lambda)\right|^2 d\rho_i\left(\lambda\right) < \infty \right\}$$

$$F_i\left(L_i f\right)(\lambda) = \lambda F_i\left(f\right)(\lambda) \quad \text{and} \quad \|f\|_i^2 = \int_{-\infty}^{\infty} \left|F_i\left(f\right)(\lambda)\right|^2 d\rho_i(\lambda),$$

where $\|\cdot\|_i$ is the norm in H_i, $i = 1, 2$. The function ρ_i is called the spectral function and defines a Lebesgue-Stieltjes measure $d\rho_i$. Thus it is non-decreasing, has a jump discontinuity at an eigenvalue only, is increasing on the continuous spectrum and its support $\operatorname{supp} d\rho_i = \sigma_i$. The existence of a spectral function guarantees that the spectrum is simple otherwise it is a matrix. In [10], one can find a more general setting for the spectral theory of operators in rigged Hilbert spaces, based on fact that when λ is in the continuous spectrum, the corresponding eigenfunctional is a generalized function.

Let us denote by $\operatorname{Dom}(W)$ the domain of the operator W. We begin with few definitions.

Definition 2.1. W is a *transformation operator* ((T.O.) for short) if

i) $W : H_2 \mapsto H_1$ and $\overline{\operatorname{Dom}(W)} = H_2$;

ii) the set $\Omega := \{f \in \operatorname{Dom}(W) \text{ and } L_2 f \in \operatorname{Dom}(W)\}$ is dense in H_2;

iii) $L_1 W(f) = W L_2(f)$ holds for any $f \in \Omega$.

The above definition agrees with the definition of a transformation operator as given in [11], except for its boundedness. We now define the interpolation operator which connects both transforms.

Definition 2.2. J is an *interpolation operator* ((I.O.) for short) if

1) is a densely closed linear operator $L^2_{d\rho_2} \overset{J}{\mapsto} L^2_{d\rho_1}$;

2) the set $S := \{F \in \operatorname{Dom}(J) \text{ and } \lambda F(\cdot) \in \operatorname{Dom}(J)\}$ is dense in $L^2_{d\rho_2}$;

3) for any $F \in S$ we have $\lambda J(F)(\lambda) = J(\lambda F)(\lambda)$.

At first sight the operator J is simply a mapping between two weighted L^2 spaces. The idea of interpolation is contained in the following:

Proposition 2.3. *If J is an I.O. then $\phi(\lambda)J(F)(\lambda) = J(\phi F)(\lambda)$ holds for any analytic function ϕ and $F \in L^2_{d\rho_2}$ with a compact support.*

Proof. Let $F \in L^2_{d\rho_2}$ have a compact support then for any $n \geq 0$ we have $\lambda^n F(\lambda) \in L^2_{d\rho_2}$, $\lambda^n F(\lambda) \in S$ and, by condition 3),

$$\lambda^n J(F)(\lambda) = J(\lambda^n F)(\lambda).$$

The next step we use the fact that any analytic function about the origin can be written as a power series $\phi(\lambda) = \sum_{n\geq 0} a_n \lambda^n$ and since J is closed operator we have

$$\sum_{n\geq 0} a_n \lambda^n J(F)(\lambda) = J\left(\sum_{n\geq 0} a_n \lambda^n F\right)(\lambda)$$
$$\phi(\lambda)J(F)(\lambda) = J(\phi F)(\lambda).$$

Also by translation we have $(\lambda - a)\, J\,(F)\,(\lambda) = J\,((\lambda - a)\,F)\,(\lambda)$ which extends the argument to any analytic function. While the function ϕF is known only over σ_2, ϕ is constructed over a new domain σ_1, whenever $J(F)\,(\lambda) \neq 0$, by the formula

$$\phi(\lambda) = J\,(\phi F)\,(\lambda)/J(F)\,(\lambda)\,.$$

Thus to define ϕ at different values say λ_0, we need to use a function F with $J(F)\,(\lambda_0) \neq 0$. □

On the other hand if J is a sampling operator in the classical sense then condition 3) $\lambda J\,(F)\,(\lambda) = J\,(\lambda F)\,(\lambda)$ is obvious as shown by the following simple example of an interpolation operator.

Let $\sigma_2 = \mathbb{Z}$ where $\mathbb{Z}$ is the set of integers and $\sigma_1 = \{\lambda_n\}$ where $\lambda_n \notin \mathbb{Z}$ and thus $\sigma_1 \cap \sigma_2 = \emptyset$. Let us recall the definition

$$PW_\pi = \left\{ F \text{ entire: } |F(\lambda)| \leq Me^{\pi|\Im(\lambda)|} \text{ and } \int_{-\infty}^{\infty} |F(x)|^2\, dx < \infty \right\}.$$

The Shannon–Whittacker–Kotelnikov sampling theorem [16] allows us to write down a mapping explicitly for $F \in PW_\pi$:

$$F(\mu) := \sum_{n\in\mathbb{Z}} F(n) \frac{\sin(\pi(\mu - n))}{\pi(\mu - n)} \quad \text{for} \sum_{n\in\mathbb{Z}} |F(n)|^2 < \infty. \tag{7}$$

Thus take the space $L^2_{d\rho_2}$ where the measure $\rho_2(\lambda) = [\lambda]$ represents the greatest integer function in λ. If $\{F(n)\}_{n\in\mathbb{Z}}$ is given, then $\{F(\lambda_n)\}_{n\in\mathbb{Z}}$ can be obtained from

$$J(F)(\lambda_n) := \sum_{k\in\mathbb{Z}} F(k) \frac{\sin(\pi(\lambda_n - k))}{\pi(\lambda_n - k)}. \tag{8}$$

A mapping $L^2_{d\rho_2} \overset{J}{\mapsto} L^2_{d\rho_1}$ can now be defined by the operation in (8) and by (7) we in fact have $J(F)(\lambda_n) = F(\lambda_n)$. It remains to see that condition 3) then holds since, for $\lambda F(\cdot) \in L^2_{d\rho_2}$, $J\,(\lambda F(\cdot))\,(\lambda_n) = \lambda_n F(\lambda_n) = \lambda_n J(F)(\lambda_n)$.

3. Interpolation

We now prove the main result.

Proposition 3.1. *Assume that L_i is an unbounded self adjoint operators acting in H_i with spectral functions ρ_i for $i = 1, 2$. Let J be a linear operator $L^2_{d\rho_2} \overset{J}{\mapsto} L^2_{d\rho_1}$ and define*

$$W = F_1^{-1} J F_2. \tag{9}$$

Then W is a T.O. if and only if J is an I.O.

Proof. It is enough to show that the conditions in Definitions 2.2 and 2.1 are equivalent in their respective order. Since F_1 and F_2 are unitary operators it follows from (9) that W is densely defined if and only if J is densely defined. For the second point, we need to show that S is dense if and only if Ω is dense. From (9) we have

$$\psi \in \mathrm{Dom}(J) \Longleftrightarrow F_2^{-1}(\psi) \in \mathrm{Dom}(W)$$
$$\lambda\psi \in \mathrm{Dom}(J) \Longleftrightarrow L_2F_2^{-1}(\psi) \in \mathrm{Dom}(W)$$

and hence

$$\psi \in S \Longleftrightarrow F_2^{-1}(\psi) \in \Omega.$$

In other words S is dense in $L^2_{d\rho_2}$ if and only if Ω is dense in H_2. For the third condition, let $f \in \Omega$, then

$$L_1W(f) = L_1F_1^{-1}JF_2(f) = F_1^{-1}\lambda JF_2(f)$$
$$WL_2(f) = F_1^{-1}JF_2L_2(f) = F_1^{-1}J\lambda F_2(f)$$

which simply says that

$$L_1W(f) = WL_2(f) \quad \forall f \in \Omega \quad \Longleftrightarrow \quad J(\lambda F) = \lambda J(F) \quad \forall F \in S.$$

Therefore W is a T.O. if and only if J is an I.O. □

Once the connection between I.O. and T.O. has been established, we now show how to construct an I.O. in a particular case. For the sake of simplicity, we shall call upon the well known Gelfand and Levitan theory, see [9] and [8].

From the given kernel defined by (5) define its adjoint $W = V'$, i.e., $W : L^2(0,\infty) \mapsto L^2(0,\infty)$:

$$Wf(x) = f(x) + \int_x^\infty K(t,x)f(t)\,dt. \tag{10}$$

Since the kernel $K(x,t)$, by the Gelfand-Levitan theory, is a continuous function in both variables W is densely defined as its domain includes for example $C_0(0,\infty)$.

Let us recall that the Gelfand–Levitan theory requires the spectral function ρ to satisfy the following conditions, where

$$\sigma(\lambda) := \begin{cases} \rho(\lambda) - \frac{2}{\pi}\sqrt{\lambda} & \text{if } \lambda \geq 0 \\ \rho(\lambda) & \text{if } \lambda < 0. \end{cases}$$

Theorem 3.2 (Gelfand–Levitan–Gasymov). *For $\rho(\lambda)$, a nondecreasing and right-continuous function to be the spectral function of* (3) *it is necessary and sufficient that it satisfies the following conditions:*

(A) *for* $f \in L^2_{dx}(0,\infty)$ *with compact support,*

$$\int_{-\infty}^{\infty} |E(f)(\lambda)|^2 \, d\rho(\lambda) = 0 \quad \Longrightarrow \quad f(x) \equiv 0,$$

where $E(f)(\lambda) := \int_0^\infty f(x) \cos\left(x\sqrt{\lambda}\right) dx$*;*

(B) $\int_{-\infty}^{N} \cos(x\sqrt{\lambda}) d\sigma(\lambda)$ *converges boundedly to* $\Psi(x)$ *as* $N \to \infty$ *and* Ψ *has two locally integrable derivatives.*

We have

Proposition 3.3. *Assume that* $\rho_i(\lambda)$ *satisfy conditions* (A) *and* (B)*, then the operator* $J : L^2_{\rho_2} \mapsto L^2_{\rho_1}$ *defined by*

$$J(F)(\lambda) := \int_0^\infty W(f)(x) y_1(x,\lambda) \, dx, \tag{11}$$

where $f(x) := \int_0^\infty F(\lambda) y_2(x,\lambda) \, d\rho_2(\lambda)$ *and* $f \in C_0^2[0,\infty)$ *is an interpolation operator in the sense of Definition* 2.2.

Proof. Conditions (A) and (B) ensure the existence of potentials $q_i(x)$ for the solution of the inverse spectral problem and the recovered differential operators (4) generate unitary transforms

$$F_i : L^2_{dx}(0,\infty) \mapsto L^2_{\rho_i}, \quad i = 1,2$$

$$F_i(f)(\lambda) := \int_0^\infty f(x) y_i(x,\lambda) \, dx \quad \text{and} \quad f(x) := \int F_i(f)(\lambda) y_i(x,\lambda) \, d\rho_i(\lambda).$$

The operator J can be defined via $L^2(0,\infty)$ as in (9):

$$J := F_1 W F_2^{-1}, \tag{12}$$

where W is defined by (10). We now verify that the three conditions for J to be an I.O. are satisfied. By (12), $F \in \mathrm{Dom}(J)$ if and only if $F_2^{-1}(F) \in \mathrm{Dom}(W)$. Since F_2 is a unitary operator and $\mathrm{Dom}(W)$ is dense in $L^2(0,\infty)$ it follows that J is also densely defined in $L^2_{\rho_2}$.

For the second condition it is enough to show that $\Omega = \{f \in \mathrm{Dom}(W)$ and $P_2 f \in \mathrm{Dom}(W)\}$ is dense in $L^2(0,\infty)$. If $f \in C_0^2(0,\infty)$, then $f \in \mathrm{Dom}(W)$ and $L_2(f) = -f'' + q_2 f \in C_0(0,\infty)$ and so $L_2 f \in \mathrm{Dom}(W)$. Thus $C_0^2(0,\infty) \subset \Omega$, and from the density of $C_0^2(0,\infty)$ it follows that Ω is also dense in $L^2(0,\infty)$. It remains to see that S is unitarily equivalent to Ω:

$$f \in \Omega \Longleftrightarrow F_2(f) \in S$$

and therefore it is also dense in $L^2_{\rho_2}$.

The last condition to verify is if $F \in S$ then $\lambda J(F)(\lambda) = J(\lambda F(\cdot))(\lambda)$. Let $F(\lambda) := F_2(f)(\lambda)$ where $f \in C_0^2(0,\infty)$. We then have $F \in S$ and $\lambda F(\lambda) = F_2(L_2 f)(\lambda)$, and it follows by (11) and the adjoint of V defined in (10) that

$$\begin{aligned}
\lambda J(F)(\lambda) &= \lambda \int_0^\infty W(f)(x) y_1(x,\lambda)\, dx \\
&= \lambda \int_0^\infty V'(f)(x)\, y_1(x,\lambda)\, dx \\
&= \lambda \int_0^\infty f(x)\, V(y_1)(x,\lambda)\, dx \\
&= \lambda \int_0^\infty f(x)\, y_2(x,\lambda)\, dx \\
&= \int_0^\infty f(x)\, L_2(y_2)(x,\lambda)\, dx \\
&= \int_0^\infty L_2(f)(x)\, y_2(x,\lambda)\, dx \\
&= \int_0^\infty L_2(f)(x)\, V(y_1)(x,\lambda)\, dx \\
&= \int_0^\infty WL_2(f)(x)\, y_1(x,\lambda)\, dx \\
&= J(\lambda F(\cdot))(\lambda). \qquad \square
\end{aligned}$$

Corollary 3.4. *Let the conditions of Proposition* 3 *hold, then* W *is a nontrivial solution of the operator equation* $WL_2 = L_1 W$.

Proof. Since J is an I.O. operator, it follows from Proposition 2 that W is a T.O. □

We now end this section by observing that if two given abstract self-adjoint operators P_1 and P_2 are similar to L_1 and L_2, in (3), in the sense they have the same spectral functions, then they "share" the same existing I.O. between L_1 and L_2. Indeed from the similarities relations

$$L_1 = UP_1U^{-1},\ L_2 = RP_2R^{-1} \ \text{ and } \ WL_2 = L_1W$$

it follows that $WRP_2R^{-1} = UP_1U^{-1}W$, i.e., $U^{-1}WRP_2 = P_1U^{-1}WR$. Thus $U^{-1}WR$ is the new T.O. for P_1 and P_2.

Corollary 3.5. *Assume that* P_i *is an unbounded self adjoint operator acting in* H_i *with transform* $\widetilde{F}_i$ *and its spectral function* ρ_i, *for* $i = 1, 2$, *satisfies conditions* (A) *and* (B) *in the Gelfand–Levitan–Gasymov theorem, then a T.O.* $\widetilde{W}$ *between* P_1 *and* P_2 *is simply given by*

$$\widetilde{W}\psi := \widetilde{F}_1^{-1} \int_0^\infty Wf(x)y_1(x,\lambda)\, dx \quad \textit{and} \quad f(x) = \int \widetilde{F}_2(\psi)(\lambda) y_2(x,\lambda)\, d\rho_2(\lambda),$$

where $y_i(x,\lambda)$ *are the eigenfunctionals of* L_i *defined in* (4).

Proof. Since the Gelfand–Levitan theory already provides a standard I.O., see (11), it follows by Proposition 2, that

$$\widetilde{F}_1 \widetilde{W} \widetilde{F}_2^{-1} = J = F_1 W F_2^{-1} \quad \text{and} \quad \widetilde{W} = \widetilde{F}_1^{-1} F_1 W F_2^{-1} \widetilde{F}_2,$$

and thus $\widetilde{W}(\psi) = \widetilde{F}_1^{-1} F_1 W(f)$ where $f = F_2^{-1} \widetilde{F}_2(\psi)$, and W is given by (10). □

Thus we have seen that the use of spectral functions allowed us to extend the Rosemblum–Sylvester theorem to unbounded operators, and furthermore it provides a new constructive approach to the solution of operator equation of type (1).

Acknowledgement. The author sincerely thanks the referee for the valuable comments.

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Received September 16, 2005; revised June 22, 2006