Quasi-Periodic Solutions  
in Nonlinear Asymmetric Oscillations  

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Abstract. The existence of Aubry–Mather sets and infinitely many subharmonic solutions to the following $p$-Laplacian like nonlinear equation

$$(p - 1)^{-1}(\phi_p(x'))' + [\alpha \phi_p(x^+) - \beta \phi_p(x^-)] + g(x) = h(t)$$

is discussed, where $\phi_p(u) = |u|^{p-2}u$, $p > 1$, $\alpha, \beta$ are positive constants satisfying $\alpha^{\frac{1}{p}} + \beta^{\frac{1}{p}} = \frac{2}{n}$ with $n \in \mathbb{N}$, $h$ is piece-wise two times differentiable and $2\pi\rho$-periodic, $g \in C^1(R)$ is bounded, $x^\pm = \max\{\pm x, 0\}$, $\pi_p = \frac{2\pi}{\rho \sin(\pi/p)}$.

Keywords. Aubry–Mather sets, $p$-Laplacian, resonance, quasi-periodic solutions

Mathematics Subject Classification (2000). 34C25

1. Introduction

In this paper, we consider the existence of Aubry–Mather sets and quasi-periodic solutions to the following $p$-Laplacian like nonlinear differential equation

$$(p - 1)^{-1}(\phi_p(x'))' + [\alpha \phi_p(x^+) - \beta \phi_p(x^-)] + g(x) = h(t) \quad (\rho = \frac{d}{dt}), \quad (1)$$

where $\phi_p(u) = |u|^{p-2}u$, $p > 1$ is a constant, $x^\pm = \max\{\pm x, 0\}$, $\alpha, \beta$ are positive constants satisfying

$$\alpha^{\frac{1}{p}} + \beta^{\frac{1}{p}} = \frac{2}{n}, \quad (2)$$

$h$ is piece-wise two times differentiable and $2\pi\rho$-periodic, $g \in C^1(R)$ is bounded and $\pi_p = \frac{2\pi}{\rho \sin(\pi/p)}$. If $p = 2$, then $\pi_2 = \pi$ and (1) reduces to second order differential equation

$$x'' + \alpha x^+ - \beta x^- + g(x) = h(t). \quad (3)$$

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The existence of Aubry–Mather sets and quasi-periodic solutions of (3) was discussed recently in [3] if \( g \in C^2 \) and the limits

\[
\lim_{x \to +\infty} g(x) = g(+\infty), \quad \lim_{x \to -\infty} g(x) = g(-\infty)
\]

exist and \( g \) satisfies some further approximate properties at infinity. Capitello and Liu [3], by applying a version of Aubry–Mather theory due to Pei [11], proved the existence of quasi-periodic solutions in generalized sense and multiplicity of subharmonic solutions to equation (3) under the so-called "resonance" situation, i.e., (2) holds for \( p = 2 \) and some \( n \in \mathbb{N} \).

Let \( C \) be the solution of the initial value problem

\[
x'' + \alpha x^+ - \beta x^- = 0, \quad x(0) = 1, \quad x'(0) = 0.
\]

Assume \( \alpha \neq \beta \) and \( \frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2m}{n}, \ m, n \in \mathbb{N} \). Alonso and Ortega [2] proved that if the function

\[
\phi_f(\theta) = \int_0^{2\pi} \left( C \left( \frac{mn}{n} + t \right) h(t) \right) dt
\]

has only simple zeros, then any solution \( x \) of the linear equation

\[
x'' + \alpha x^+ - \beta x^- = h
\]

with large initial values, that is, if \( |x(t_0)| + |x'(t_0)| \gg 1 \) for some \( t_0 \in \mathbb{R} \), goes to infinity in the future or in the past. Moreover, they showed the existence of \( h \) such that unbounded solutions and 2\( \pi \)-periodic solutions of (4) can coexist.

Fabry and Fonda [4] and Fabry and Mawhin [5] obtained, by degree methods, sufficient conditions for the existence of 2\( \pi \)-periodic solutions and of unbounded solutions as well as subharmonic solutions for (5) below, respectively. More precisely, in [5] the following equation is considered:

\[
x'' + \alpha x^+ - \beta x^- = g(x) + f(x) + h(t),
\]

and it is proved that if the function

\[
\Phi_h(\theta) = \frac{n}{\pi} \left[ \frac{g(+\infty)}{\alpha} - \frac{g(-\infty)}{\beta} \right] + \frac{1}{2\pi \sqrt{\alpha}} \int_0^{2\pi} C(t + \theta)h(t) dt
\]

has zeros and all of them are simple, then all solutions of (5) with large initial values are unbounded if the following resonance condition is satisfied:

\[
\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2}{n}, \quad n \in \mathbb{N},
\]
and $f$ has a sublinear primitive, that is, $\lim_{|x| \to \infty} \frac{1}{|x|} \int_0^x f(s)ds = 0$. Later, the author of this paper [12] discussed the more general equation (1) and considered the following function:

$$\phi(\theta) = D_p - \frac{p}{\alpha^\frac{1}{p}} \int_0^{2\pi} h(mt)C_p(mt + \theta) dt,$$

where

$$D_p = \frac{2n}{m} B \left( \frac{2}{p}, 1 - \frac{1}{p} \right) \left[ \frac{g(+\infty)}{\alpha^\frac{1}{p}} - \frac{g(-\infty)}{\beta^\frac{1}{p}} \right],$$

$B(r, s) = \int_0^1 (1-t)^{r-1}t^{s-1}dt$ for $r > 0$, $s > 0$, and $C_p(t)$ is the $\frac{2\pi m}{n}$-periodic solution of the following initial value problem:

$$(p - 1)^{-1}(\phi_p(u'))' + [\alpha \phi_p(u^+) - \beta \phi_p(u^-)] = 0, \quad u(0) = 1, \quad u'(0) = 0,$$

if $\alpha$ and $\beta$ satisfy $\alpha^{-\frac{1}{p}} + \beta^{-\frac{1}{p}} = \frac{2m}{n}$, $m, n \in \mathbb{N}$.

It was shown in [12] that if the function $\phi(\theta)$ has no zero for all $\theta \in \mathbb{R}$, then all solutions of (1) are bounded. For more recent results on boundedness and existence $2\pi$-periodic solutions of (1) and (3), we refer [1], [6]–[9], [11]–[15] and the references therein.

In the rest of this paper, we denote by $S$ the unique solution of the initial value problem

$$(p - 1)^{-1}(\phi_p(x'))' + [\alpha \phi_p(x^+) - \beta \phi_p(x^-)] = 0, \quad x(0) = 0, \quad x'(0) = 1. \quad (6)$$

**Definition 1.**

(A) A solution of $(x_\omega(t), x'_\omega(t))$ of (1) is called of *Mather type with rotation number* $\omega$ if $\omega = \frac{k}{m}$ is rational, the solutions $(x_\omega(t + 2i\pi), x'_\omega(t + 2i\pi))$, $1 \leq i \leq m-1$, are mutually unlinked periodic solutions of periodic $2m\pi$ and, in this case,

$$\lim_{\omega \to n} \min_{t \in \mathbb{R}}(|x_\omega(t)| + |x'_\omega(t)|) = +\infty.$$

(B) If $\omega$ is irrational, the solution $(x_\omega(t), x'_\omega(t))$ is either a usual quasi-periodic solution or a generalized one, that is, the closed set

$$\{(x_\omega(2i\pi), x'_\omega(2i\pi)), \ i \in \mathbb{Z}\}$$

is a *Denjoy’s minimal set*.

The main results of this paper are formulated in the following theorems:
Theorem 1. Assume $h \in L^\infty[0, 2\pi_p]$ is $2\pi_p$-periodic and $g \in C^1(\mathbb{R})$ is bounded and satisfies the following conditions: the limits $\lim_{x \to +\infty} g(x) = g(\infty)$ and $\lim_{x \to -\infty} g(x) = g(-\infty)$ exist and $g$ satisfies

$$g(x) = g(\pm \infty) + c^\pm |x|^{-(p-1)\sigma} \text{sgn} x + O(|x|^{-(p-1)\sigma-1})$$

for $|x| \gg 1$, where $\sigma \in (0, \frac{1}{p-1})$ is a constant and $c^\pm$ are constants satisfying

$$D_0 := \frac{c^+}{\alpha \frac{2}{-(p-1)\sigma}} + \frac{c^-}{\beta \frac{2}{-(p-1)\sigma}} \neq 0.$$

Define a $2\pi_p$-periodic function $\lambda_1$ as

$$\lambda_1(t) = \int_0^{2\pi_p} S(\theta) h(t + \theta) d\theta - \frac{2}{p} \left( \frac{g(\infty)}{\alpha^\frac{2}{p}} - \frac{g(-\infty)}{\beta^2/p} \right) B \left( \frac{2}{p}, 1 \right),$$

where $q = \frac{p}{p-1}$ is the conjugate exponent of $p$. Let one of the following conditions be satisfied:

1. $\lambda_1(t) \neq 0$ for all $t \in \mathbb{R}$;
2. either (a) $\lambda_1(t) \geq 0$ and $D_0 < 0$ or (b) $\lambda_1(t) \leq 0$ and $D_0 > 0$.

Then there exists an $\varepsilon_0 > 0$ such that for any $\omega \in (n, n + \varepsilon_0)$, equation (1) has a solution $(x_\omega(t), x'_\omega(t))$ of Mather type with rotation number $\omega$.

Theorem 2. Assume $g(x) \equiv 0$, $h$ is piece-wise two times differentiable and $2\pi_p$-periodic. Assume

$$\lambda_1(t) = \int_0^{2\pi_p} S(\theta) h(t + \theta) d\theta \equiv 0.$$  

For $p \neq 2$, define a $2\pi_p$-periodic function $\lambda_2(t)$ as

$$\lambda_2(t) = (p-2) \left[ \int_0^{2\pi_p} S(\theta) h(t + \theta) d\theta \int_0^0 S(\tau) h'(t + \tau) d\tau d\theta - \int_0^{2\pi_p} S^2(\theta) h^2(t + \theta) d\theta \right].$$

For $p = 2$, define a $2\pi$-periodic function $\lambda_3(t)$ as

$$\lambda_3(t) = -\frac{1}{2} \left[ \int_0^{2\pi} S^3(\theta) h^3(t + \theta) d\theta + \int_0^{2\pi} S(\theta) h''(t + \theta) \left( \int_0^0 S(\tau) h(t + \tau) d\tau \right)^2 d\theta \right]
- \int_0^{2\pi} S^2(\theta) h(t + \theta) h'(t + \theta) \int_0^0 S(\tau) h(t + \tau) d\tau d\theta.$$

Then the conclusions of Theorem 1 are true, if one of the following conditions holds:

1. $p \neq 2$, $\lambda_2(t) \neq 0$ for all $t \in \mathbb{R}$;
2. $p = 2$, $\lambda_3(t) \neq 0$ for all $t \in \mathbb{R}$. 
2. Generalized polar coordinates transformation

If we introduce a new variable $y = \phi_p(x')$, then (1) is equivalent to the planar system

$$
\begin{align*}
x' &= \phi_p(y), \\
y' &= (p - 1)[-\alpha \phi_p(x^+) + \beta \phi_p(x^-) + h(t) - g(x)],
\end{align*}
$$

(7)

where $q = \frac{p}{p-1}$ is the conjugate exponent of $p$. Let $u = \sin_p t$ be the solution of the initial value problem

$$
(\phi_p(u'))' + (p - 1)\phi_p(u) = 0, \quad u(0) = 0, \quad u'(0) = 1
$$

which for $t \in [0, \frac{\pi}{2}]$ can be expressed implicitly by

$$
t = \int_0^{\sin_p t} \frac{ds}{(1 - s^p)^{\frac{1}{p}}}.
$$

Then it follows from [10] that $u = \sin_p t$ can be extended to $\mathbb{R}$ as a $2\pi_p$-periodic odd $C^2$-function which satisfies $\sin_p t : [0, \frac{\pi}{2}] \to [0, 1]$ and $\sin_p(\pi_p - t) = -\sin_p t$ for $t \in [\frac{\pi}{2}, \pi_p]$, $\sin_p(2\pi_p - t) = \sin_p t$ for $t \in [\pi_p, 2\pi_p]$.

Let the function $S$ be the unique solution of problem (6), then it is not difficult to verify that $S \in C^2(R)$ is $\frac{2\pi_p}{n}$-periodic and can be expressed as

$$
S(t) = \begin{cases} 
\alpha^{-\frac{1}{p}} \sin_p \alpha^{\frac{1}{p}} t, & t \in [0, \alpha^{-\frac{1}{p}} \pi_p) \\
-\beta^{-\frac{1}{p}} \sin_p \beta^{\frac{1}{p}} (t - \alpha^{-\frac{1}{p}} \pi_p), & t \in [\alpha^{-\frac{1}{p}} \pi_p, \frac{2\pi_p}{n}],
\end{cases}
$$

(8)

from which it is easy to verify that the following equality holds:

$$
|S'(t)|^p + \alpha(S^+(t))^p + \beta(S^-(t))^p \equiv 1, \quad t \in R.
$$

(9)

For $\rho > 0$, $\theta (\mod 2\pi_p)$, we define the generalized polar coordinates transformation $(\rho, \theta) \rightarrow (x, y)$ as

$$
x = \rho^{\frac{1}{p}} S\left(\frac{\theta}{n}\right), \quad y = \rho^{\frac{1}{p}} \phi_p\left(S'\left(\frac{\theta}{n}\right)\right).
$$

Under this transformation and by using (9), (7) is changed into the planar system

$$
\rho' = p \rho^{\frac{1}{p}} S'\left(\frac{\theta}{n}\right) (h(t) - g(x)), \quad \theta' = n - n \rho^{-\frac{1}{p}} S\left(\frac{\theta}{n}\right) (h(t) - g(x)).
$$

(10)

If we define $r = \rho^{\frac{1}{p}}$, then (10) can be further simplified as

$$
r' = (p - 1) S'\left(\frac{\theta}{n}\right) (h(t) - g(x)), \quad \theta' = n \left[1 - r^{-1} S\left(\frac{\theta}{n}\right) (h(t) - g(x))\right],
$$

(11)

where $x = r^{\frac{1}{p-1}} S\left(\frac{\theta}{n}\right)$. 

where in [3], we can write (11) in the following equivalent form:

\[
\begin{align*}
\frac{dt}{d\theta} &= \frac{1}{n(1-r^{-1}S\left(\frac{\theta}{n}\right)(h(t) - g(x)))} \\
\frac{dr}{d\theta} &= \frac{(p-1)S'\left(\frac{\theta}{n}\right)(h(t) - g(x))}{n(1-r^{-1}S\left(\frac{\theta}{n}\right)(h(t) - g(x)))}.
\end{align*}
\]  
(12)

Now let \((r(\theta; r_0, t_0), t(\theta; r_0, t_0))\) be the solution of (12) with initial value \((r_0, t_0)\) where \(t_0 \in I\) and \(\theta \in [0, 2\pi]\). Then for \(r_0 \gg 1\), we obtain \(r(\theta) \geq r_0/2 \gg 1\) and (12) can be written as

\[
\begin{align*}
\frac{dr}{d\theta} &= \frac{p-1}{n} \left[ S'\left(\frac{\theta}{n}\right)(h(t) - g(x)) + r^{-1}(\theta)S'\left(\frac{\theta}{n}\right)S\left(\frac{\theta}{n}\right)(h(t) - g(x))^2 + \cdots \right] \\
\frac{dt}{d\theta} &= \frac{1}{n} \left[ 1 + r^{-1}(\theta)S\left(\frac{\theta}{n}\right)(h(t) - g(x)) + r^{-2}(\theta)S^2\left(\frac{\theta}{n}\right)(h(t) - g(x))^2 + \cdots \right].
\end{align*}
\]  
(13)

where \(x = x(\theta) = r_0^{-1}S\left(\frac{\theta}{n}\right) + O(1)\).

3. Lemmas

For the proof of theorems, we need the following lemmas:

**Lemma 1.** Assume the conditions of Theorem 1 hold, then we have

\[
\begin{align*}
r_1 &= r_0 + \mu_0(t_0) + O(r_0^{-1}) \\
t_1 &= t_0 + 2\pi p + \lambda_1(t_0)r_0^{-1} + \lambda_{1+\sigma}r_0^{-(1+\sigma)} + O(r_0^{-2}),
\end{align*}
\]  
(14)

where \(r_1 = r(2\pi p; r_0, t_0)\), \(t_1 = t(2\pi p; r_0, t_0)\) and

\[
\begin{align*}
\mu_0(t) &= (p-1) \int_0^{2\pi p} S'(\theta)f(t + \theta)d\theta \\
\lambda_1(t) &= \int_0^{2\pi p} S(\theta)f(t + \theta)d\theta - \frac{2}{p} \left( \frac{g(0) - g(-\infty)}{\alpha^2} \right) B\left( \frac{2}{p}, \frac{1}{q} \right) \\
\lambda_{1+\sigma} &= -\frac{2}{p} \left( \frac{c^+}{\alpha^p r} + \frac{c^-}{\beta^p r} \right) B\left( \frac{r+1}{p}, \frac{1}{q} \right) \\
&= -D_0 \frac{2}{p} B\left( \frac{r+1}{p}, \frac{1}{q} \right)
\end{align*}
\]

where \(\tau = 1 - (p-1)\sigma \in (0, 1)\). Moreover, we have \(\mu_0(t) = -(p-1)\lambda_1(t)\).
Proof. It follows from (13) and for \( t_0 \in \mathbb{R} \) and \( \theta \in [0, 2n\pi_p] \), we have
\[
    r(\theta) = r_0 + O(1), \quad t(\theta) = t_0 + \frac{\alpha}{p} + O(r_0^{-1}).
\]  
(15)

For \( r_0 \gg 1 \), substituting (15) into (13), then integrating over \([0, 2n\pi_p]\) and letting \( r_1 = r(2n\pi_p) \), \( t_1 = t(2n\pi_p) \), we obtain
\[
    r_1 = r_0 + \mu_0(t_0) + O(r_0^{-1})
    \quad t_1 = t_0 + 2\pi_p + \lambda_1(t_0)r_0^{-1} + \lambda_{1+\sigma}r_0^{-(1+\sigma)} + O(r_0^{-2}),
\]  
(16)

where
\[
    \mu_0(t) = \frac{p-1}{n} \int_0^{2n\pi_p} S'(\frac{\theta}{n}) h(t + \frac{\theta}{n}) d\theta 
    - \frac{p-1}{n} g(+\infty) \int I S'(\frac{\theta}{n}) d\theta 
    - \frac{p-1}{n} g(-\infty) \int J S'(\frac{\theta}{n}) d\theta
    = (p-1) \int_0^{2\pi_p} S'(\theta) h(t + \theta) d\theta,
\]

and
\[
    \lambda_1(t) = \frac{1}{n} \left[ \int_0^{2n\pi_p} S(\frac{\theta}{n}) h(t + \frac{\theta}{n}) d\theta - g(+\infty) \int I S(\frac{\theta}{n}) d\theta - \frac{1}{n} g(-\infty) \int J S(\frac{\theta}{n}) d\theta \right]
    = \int_0^{2\pi_p} S(\theta) h(t + \theta) d\theta - g(+\infty) \int_0^{\pi_p} S(\theta) d\theta + g(-\infty) \int_{\pi_p}^{2\pi_p} |S(\theta)| d\theta,
\]

where \( I = \{ \theta \in [0, 2n\pi_p] : S(\frac{\theta}{n}) > 0 \} \) and \( J = \{ \theta \in [0, 2n\pi_p] : S(\frac{\theta}{n}) < 0 \} \).

By using the similar method used in [12], we can show that
\[
    \int_0^{\pi_p} S(\theta) d\theta = \frac{2}{\alpha^2} B \left( \frac{2}{p}, \frac{1}{q} \right)
    \int_{\pi_p}^{2\pi_p} S(\theta) d\theta = \frac{1}{\beta^2} B \left( \frac{2}{p}, \frac{1}{q} \right).
\]

From above equalities, we obtain the expressions of \( \mu_1(t) \) and \( \lambda_1(t) \).

Next, we calculate the value \( \lambda_{1+\sigma} \). From (16) and the expression of \( S \) in (8), we obtain
\[
    \lambda_{1+\sigma} = -c^- \int_0^{\pi_p} (S(\theta))^\tau d\theta - c^+ \int_{\pi_p}^{2\pi_p} |S(\tau)|^\tau d\theta
    \quad = - \left( \frac{c^-}{\alpha} \frac{1}{\alpha} + \frac{c^+}{\beta} \right) \int_0^{\pi_p} (\sin_p \theta)^\tau d\theta
\]
and
\[ \int_0^{2\pi} (\sin \theta)^\tau d\theta = 2 \int_0^{\pi} (\sin \theta)^\tau d\theta = \frac{2}{p} B \left( \frac{\tau + 1}{p}, \frac{1}{q} \right), \]
which yields the expression of \( \lambda_{1+\sigma} \). Now the integration by parts yields \( \mu_0(t) = -(p-1)\lambda_1(t) \). \( \square \)

**Lemma 2.** Assume the conditions of Theorem 2 hold, then we have
\[ r_1 = r_0 + \mu_1(t_0)r_0^{-1} + O(r_0^{-2}) \]
\[ t_1 = t_0 + 2\pi + \lambda_2(t_0)r_0^{-2} + O(r_0^{-3}), \]
where
\[ \mu_1(t) = -(p-1) \int_0^{2\pi} S(\theta)h''(t+\theta) \int_0^\theta S(\tau)h(t+\tau) d\tau d\theta \]
\[ -2(p-1) \int_0^{2\pi} S^2(\theta)h(t+\theta)h(t+\theta) d\theta \]
\[ \lambda_2(t) = (p-2) \int_0^{2\pi} S(\theta)h'(t+\theta) \int_0^\theta S(\tau)h(t+\tau) d\tau d\theta \]
\[ - (p-2) \int_0^{2\pi} S^2(\theta)h^2(t+\theta) d\theta. \]
Moreover, we have \((p-2)\mu_1(t) = (p-1)\lambda_2(t)\).

**Proof.** Substituting (15) into (13) and integrating over \([0, \theta] \subset [0, 2\pi] \) we, obtain
\[ r(\theta) = r_0 + \mu_0(t_0, \theta) + O(r_0^{-1}) \]
\[ t(\theta) = t_0 + \frac{a}{n} + \lambda_1(t_0, \theta)r_0^{-1} + O(r_0^{-2}) \]
\[ r^{-1}(\theta) = r_0^{-1} - \mu_0(t_0, \theta)r_0^{-2} + O(r_0^{-3}), \]
where
\[ \mu_0(t, \theta) = \frac{p-1}{n} \int_0^\theta S'(\frac{\zeta}{n})h(t + \frac{\zeta}{n}) d\tau \]
\[ \lambda_1(t, \theta) = \frac{1}{n} \int_0^\theta S(\frac{\zeta}{n})h(t + \frac{\zeta}{n}) d\tau. \]
Substituting (18)–(19) into (13) and integrating over \([0, 2n\pi] \), we get
\[ r_1 = r_0 + \mu_0(t_0) + \mu_1(t_0)r_0^{-1} + O(r_0^{-2}) \]
\[ t_1 = t_0 + 2\pi + \lambda_1(t_0)r_0^{-1} + \lambda_2(t_0)r_0^{-2} + O(r_0^{-3}), \]
where \( \lambda_1(t) = \lambda_1(t, 2n\pi_p) \), \( \mu_0(t) = \mu_0(t, 2n\pi_p) \),

\[
\mu_1(t) = \frac{p - 1}{n} \int_0^{2n\pi_p} S'(\frac{\theta}{n}) h'(t + \frac{\theta}{n}) \lambda_1(t, \theta) d\theta \\
+ \frac{p - 1}{n} \int_0^{2n\pi_p} S(\frac{\theta}{n}) S'(\frac{\theta}{n}) h^2(t + \frac{\theta}{n}) d\theta \\
= -(p - 1) \int_0^{2\pi} S(\theta) h''(t + \theta) \int_0^\theta S(\tau) h(t + \tau) d\tau d\theta \\
- 2(p - 1) \int_0^{2\pi} S^2(\theta) h(t + \theta) h'(t + \theta) d\theta - (p - 1) \lambda_1(t) \lambda_1'(t)
\]

and

\[
\lambda_2(t) = \frac{1}{n} \int_0^{2n\pi_p} S^2(\frac{\theta}{n}) h^2(t + \frac{\theta}{n}) d\theta - \frac{1}{n} \int_0^{2n\pi_p} S(\frac{\theta}{n}) h(t + \frac{\theta}{n}) \mu_0(t, \theta) d\theta \\
+ \frac{1}{n} \int_0^{2n\pi_p} S(\frac{\theta}{n}) h'(t + \frac{\theta}{n}) \lambda_1(t, \theta) d\theta \\
= (p - 2) \int_0^{2\pi} S(\theta) h(t + \theta) \int_0^\theta S(\tau) h'(t + \tau) d\tau d\theta \\
- (p - 2) \int_0^{2\pi} S^2(\theta) h^2(t + \theta) d\theta + \lambda_1(t) \lambda_1'(t).
\]

From these equalities we obtain after some elementary calculation

\[
(p - 2) \mu_1(t) = (p - 1) \left[ \lambda_2'(t) - \frac{p}{2} (\lambda_1'(t))^2 - (p - 1) \lambda_1(t) \lambda_1''(t) \right],
\]

which implies that, for \( \lambda_1(t) \equiv 0 \), we have \( (p - 2) \mu_1(t) = (p - 1) \lambda_2'(t) \).

**Lemma 3.** Assume that the conditions of Theorem 2 hold, and \( p = 2 \). Then

\[
\begin{align*}
    r_1 &= r_0 + \mu_1(t_0) r_0^{-1} + O(r_0^{-2}) \\
    t_1 &= t_0 + 2\pi + \lambda_3(t_0) r_0^{-3} + O(r_0^{-4}),
\end{align*}
\]

(20)

where

\[
\mu_1(t) = - \int_0^{2\pi} S(\theta) h''(t + \theta) \int_0^\theta S(\tau) h(t + \tau) d\tau d\theta - 2 \int_0^{2\pi} S^2(\theta) h(t + \theta) h'(t + \theta) d\theta
\]

and \( \lambda_3(t) \) is given as in Theorem 2.

**Proof.** Substituting (15) into (13) and integrating over \( [0, \theta] \subset [0, 2n\pi] \), we obtain (18) with \( \mu_0, \lambda_1 \) given by (19) with \( p = 2 \). Substituting (18) into (13)
and integrating over $[0, \theta] \subset [0, 2\pi]$, we obtain

$$
\begin{align*}
    r(\theta) &= r_0 + \mu_0(t_0, \theta) + \mu_1(t_0, \theta)r_0^{-1} + O(r_0^{-2}) \\
    t(\theta) &= t_0 + \theta/n + \lambda_1(t_0, \theta)r_0^{-1} + \lambda_2(t_0, \theta)r_0^{-2} + O(r_0^{-3}) \\
    r^{-1}(\theta) &= r_0^{-1} - \mu_0(t_0, \theta)r_0^{-2} + (\mu_0^2(t_0, \theta) - \mu_1(t_0, \theta))r_0^{-3} + O(r_0^{-4}),
\end{align*}
$$

with

$$
\begin{align*}
    \mu_1(t, \theta) &= \frac{1}{n} \int_0^\theta S'(\frac{z}{n}) h'(t + \frac{z}{n}) \lambda_1(t, \tau) d\tau \\
    &\quad + \frac{1}{n} \int_0^\theta S(\frac{z}{n}) S'(\frac{z}{n}) h^2(t + \frac{z}{n}) d\tau \\
    &= \frac{1}{n} \int_0^\theta S'(\frac{z}{n}) h'(t + \frac{z}{n}) \int_0^\tau S(u/n) h(t + u/n) du d\tau \\
    &\quad + \frac{1}{n} \int_0^\theta S(\frac{z}{n}) S'(\frac{z}{n}) h^2(t + \frac{z}{n}) d\tau
\end{align*}
$$

and

$$
\begin{align*}
    \lambda_2(t, \theta) &= -\frac{1}{n} \int_0^\theta S(\frac{z}{n}) h(t + \frac{z}{n}) \mu_0(t, \tau) d\tau + \frac{1}{n} \int_0^\theta S^2(\frac{z}{n}) h^2(t + \frac{z}{n}) d\tau \\
    &\quad + \frac{1}{n} \int_0^\theta S(t) h(t + \tau) \lambda_1(t, \tau) d\tau \\
    &= \frac{1}{n} \int_0^\theta Sh \int_0^\tau Sh' du d\tau + \frac{1}{n^2} \int_0^\theta Sh' \int_0^\tau Sh du d\tau \\
    &= \frac{1}{n^2} \int_0^\theta S(\frac{z}{n}) h(t + \frac{z}{n}) \int_0^\tau S(\frac{z}{n}) h'(t + \frac{z}{n}) d\tau.
\end{align*}
$$

Substituting (21)–(23) into (13) again and integrating over $[0, 2\pi]$, we obtain

$$
\begin{align*}
    r_1 &= r_0 + \mu_0(t_0) + \mu_1(t_0)r_0^{-1} + O(r_0^{-2}) \\
    t_1 &= t_0 + 2\pi + \lambda_1(t_0)r_0^{-1} + \lambda_2(t_0)r_0^{-2} + \lambda_3(t_0)r_0^{-3} + O(r_0^{-4}),
\end{align*}
$$

where $\lambda_k(t) = \lambda_k(t, 2\pi), k = 1, 2, \mu_i(t) = \mu_i(t, 2\pi), i = 0, 1,$ and

$$
\begin{align*}
    \lambda_3(t) &= \frac{1}{n} \int_0^{2\pi} S^3 h^3 d\theta - \frac{2}{n} \int_0^{2\pi} S^2 h^2 \mu_0 d\theta + \frac{2}{n} \int_0^{2\pi} S^2 h' \lambda_1 d\theta \\
    &\quad + \frac{1}{n} \int_0^{2\pi} Sh(\mu_0^2 - \mu_1) d\theta - \frac{1}{n} \int_0^{2\pi} Sh' \lambda_1 \mu_0 d\theta \\
    &\quad + \frac{1}{n} \int_0^{2\pi} Sh'' \lambda_1^2 d\theta + \frac{1}{n} \int_0^{2\pi} Sh' \lambda_2 d\theta.
\end{align*}
$$
Now, substituting the expressions of $\mu_0$, $\mu_1$, $\lambda_1$ and $\lambda_2$ into (24) and using $\lambda_1(t) \equiv 0$, we obtain from Lemma 2 that $\lambda_2(t) \equiv 0$ and $\mu_0(t) = -\lambda_1(t) \equiv 0$. After some elementary calculation, we obtain the expression of $\lambda_3(t)$ given in Theorem 2.

\section{Proof of the theorems}

Now, we are ready to prove the main results of this paper.

\textit{Proof of Theorem 1.} Assume the conditions of Theorem 1 hold. If (I) is satisfied, then the Poincaré map $P : (t_0, r_0) \rightarrow (t_1, r_1)$ of the solutions of (13) has the following form:

\begin{equation}
\begin{align*}
t_1 &= t_0 + 2\pi p + \lambda_1(t_0)r_0^{-1} + O(r_0^{-2}) \\
r_1 &= r_0 + \mu_0(t_0) + O(r_0^{-1}),
\end{align*}
\end{equation}

with $\mu_0(t) = -(p - 1)\lambda_1'(t)$.

Now we introduce another action variable $u$ and a positive parameter $\varepsilon$ by $r = \frac{1}{u^2}$ with $u \in [1, 2]$. Then $r \gg 1 \Leftrightarrow \varepsilon \ll 1$. Under this transformation, (25) is changed to the following form:

\begin{equation}
\begin{align*}
t_1 &= t_0 + 2\pi p + \lambda_1(t_0)u_0\varepsilon + O(\varepsilon^2) \\
u_1 &= u_0 - \mu_0(t_0)u_0^2\varepsilon + O(\varepsilon^2).
\end{align*}
\end{equation}

Let $t_1 = t_0 + \varepsilon R(t_0, u_0, \varepsilon)$, $u_1 = u_0 + \varepsilon W(t_0, u_0, \varepsilon)$, then $R(t, u, \varepsilon) = \lambda_1(t)u + O(\varepsilon)$, $W(t, u, \varepsilon) = -\mu_0(t)u_0^2 + O(\varepsilon)$, and for $t \in [0, 2n\pi p]$, $u \in [1, 2]$, we have

\begin{equation} |R(t, u, \varepsilon)| + \left| \frac{\partial R(t, u, \varepsilon)}{\partial t} \right| + \left| \frac{\partial R(t, u, \varepsilon)}{\partial u} \right| \leq C_1 \end{equation}

and

\begin{equation} |W(t, u, \varepsilon)| + \left| \frac{\partial W(t, u, \varepsilon)}{\partial t} \right| + \left| \frac{\partial W(t, u, \varepsilon)}{\partial u} \right| \leq C_2 \end{equation}

for some constants $C_1$, $C_2$. Moreover, if $\min_{t \in \mathbb{R}} \lambda_1(t) = d_0 > 0$, we have for $\varepsilon \ll 1$, $u \in [1, 2]$,

\[ \frac{\partial R(t, u, \varepsilon)}{\partial u} \geq \frac{d_0}{2} > 0 \]

and if $\max_{t \in S^1} \lambda_1(t) = -d_1 < 0$, we have

\[ \frac{\partial R(t, u, \varepsilon)}{\partial u} \leq -\frac{d_1}{2} < 0. \]
In both cases, the Poincaré map of (25) is a monotone map. Going back to (13), we know that the Poincaré map \( Q : (\theta_0, r_0) \to (\theta_1, r_1) \) is also monotone if \( r_0 \gg 0 \). Using similar arguments as in [11, Section 4], we may construct a map \( \bar{Q} \) which is a global monotone twist homeomorphism of the cylinder \( S^1 \times \mathbb{R} \) and coincides with \( Q \) on \( S^1 \times [A_0, +\infty) \) with a fixed constant \( A_0 \gg 1 \), where \( S^1 = \mathbb{R}/2\pi \mathbb{Z} \). Therefore, the existence of Mather sets \( M_\omega \) of \( \bar{Q} \) is guaranteed by Aubry–Mather theory (see [11]). Moreover, for some small \( \varepsilon_0 > 0 \), such invariant sets with rotation \( \omega \in (n, n + \varepsilon_0) \) lie in the domain \( S^1 \times [A_0, +\infty) \). Hence they are just the Aubry–Mather sets of the Poincaré map of \( Q \). The above discussion shows the existence of Mather sets, this implies that (1) has a solution \((x_\omega(t), x'_\omega(t))\) of Mather type. Moreover, if \( \omega = \frac{m}{k} \) is a rational, the solutions \((x_\omega(t + 2i\pi p), x'_\omega(t + 2i\pi p))\), \( 1 \leq i \leq k - 1 \), are mutually unlinked periodic solutions of period \( 2k \pi p \) and \( \lim_{k \to +\infty} \min_{t \in \mathbb{R}} \|(x_\omega(t), x'_\omega(t))\| = +\infty \). If \( \omega \) is irrational, the solution \((x_\omega(t), x'_\omega(t))\) is either a usual quasi-periodic solution or a generalized one.

In case (II), by Lemma 1, the Poincaré map of (13) has the form of (14), under the same transformation \( r = \frac{1}{u \varepsilon} \), (14) is of the following form:

\[
\begin{align*}
t_1 &= t_0 + 2\pi p + \varepsilon R_1(t_0, u_0, \varepsilon) \\
u_1 &= u_0 + \varepsilon W_1(t_0, u_0, \varepsilon),
\end{align*}
\]

where \( R_1(t, u, \varepsilon) = \lambda_1(t) u + \lambda_{1+\sigma} u^{1+\sigma} \varepsilon + O(\varepsilon^1) \), and \( W_1(t, u, \varepsilon) = -\mu_0(t) u^2 + O(\varepsilon) \). It is easy to see that \( R_1 \) and \( W_1 \) satisfy the similar inequalities as (26) and (27). Moreover, for \( \lambda_1(t) \geq 0 \) and \( D_0 < 0 \), we have for \( \varepsilon \ll 1, t \in \mathbb{R}, u \in [1, 2], \lambda_{1+\sigma} > 0 \) and

\[
\frac{\partial R_1(t, u, \varepsilon)}{\partial u} = \lambda_1(t) + (1 + \sigma) \lambda_{1+\sigma} \varepsilon + O(\varepsilon^1) \geq \lambda_1(t) + \frac{1}{2}(1 + \sigma) \lambda_{1+\sigma} \varepsilon > 0.
\]

Similarly, for \( \lambda_1(t) \leq 0 \) and \( D_0 > 0 \), we have \( \lambda_{1+\sigma} < 0 \) and

\[
\frac{\partial R_1(t, u, \varepsilon)}{\partial u} = \lambda_1(t) + (1 + \sigma) \lambda_{1+\sigma} \varepsilon + O(\varepsilon^1) \leq \lambda_1(t) + \frac{1}{2}(1 + \sigma) \lambda_{1+\sigma} \varepsilon < 0.
\]

The rest proof is similar to that of case (I), so we omit it for simplicity. \( \Box \)

**Proof of Theorem 2 (a sketch).** By Lemma 2 and Lemma 3, the Poincaré map of (13) has the form of (17) or the form of (20). Under the transformation \( r = \frac{1}{u \varepsilon} \), (17) and (20) have the forms

\[
\begin{align*}
t_1 &= t_0 + 2\pi p + \lambda_2(t_0) u_0^2 \varepsilon^2 + O(\varepsilon^3) \\
u_1 &= u_0 - \mu_1(t_0) u_0^3 \varepsilon^2 + O(\varepsilon^3),
\end{align*}
\]
respectively, then it is not difficult to verify that for $0 < \varepsilon \ll 1$, $\frac{\partial R_k(t,u,\varepsilon)}{\partial u} \neq 0$ if $\lambda_k(t) \neq 0$, $t \in \mathbb{R}$ for $k = 2, 3$. The rest proofs are similar to that of Theorem 1, so we omit them for simplicity.

**Example 1.** Consider equation (1) with $\alpha = \beta = n = 1$, $h(t) \equiv 1$, $g(x) = \arctan x + |x|^{-\tau} \text{sgn} x$, where $\tau \in (0, 1)$. Then Theorem 1 implies that, for all $t \in \mathbb{R}$, $\lambda_1(t) = -\frac{2\pi}{p} B\left(\frac{1}{p}, \frac{1}{q}\right) < 0$. Now (I) of Theorem 1 implies that the conclusion of Theorem 1 holds.

**Example 2.** Consider the following equation

$$
(p - 1)^{-1}(\phi_p(x'))' + \phi_p(x) + |x|^{-\tau} \text{sgn} x - 2|x|^{-\tau} \text{sgn} x = 1,
$$

where $p > 1$, $\tau \in (0, 1)$. Then $\alpha = \beta = n = 1$, $c^+ = 1$, $c^- = -2$, $h(t) \equiv 1$, and it is easy to see that $S(t) = \sin_p t$, $\lambda_1(t) \equiv 0$ and $D_0 = c^+ + c^- < 0$. Now (II) of Theorem 1 implies that there exists $\varepsilon_0 > 0$ such that for any $\omega \in (n, n + \varepsilon_0)$, (28) has a solution $(x_\omega(t), x'_\omega(t))$ of Mather type with rotation number $\omega$.

**Example 3.** Consider a special case of (1):

$$
(p - 1)^{-1}(\phi_p(x'))' + \phi_p(x) = 1.
$$

In this example, $p \neq 2$, $\alpha = \beta = n = 1$, $g(x) \equiv 0$, $h(t) = 1$. Then it can be verified that $\lambda_1(t) \equiv 0$, $\lambda_2(t) = (2 - p) \int_0^{2\pi} \sin^2 \theta d\theta \neq 0$. Now Theorem 2 implies that there exists $\varepsilon_0 > 0$ such that for any $\omega \in (n, n + \varepsilon_0)$, (29) has a solution $(x_\omega(t), x'_\omega(t))$ of Mather type with rotation number $\omega$.

**Example 4.** Consider the following linear equation

$$
x'' + \alpha x^+ - \beta x^- = h(t),
$$

where $\alpha \neq \beta$ satisfying (2) with $p = 2$, $n = 1$, and $h$ is piecewise continuous and $2\pi$-periodic such that $h(t) = 1$, $t \in \left[0, \frac{\pi}{\sqrt{\alpha}}\right]$; $h(t) = \frac{\beta}{\alpha}$, $t \in \left(\frac{\pi}{\sqrt{\alpha}}, 2\pi\right]$. Then it follows from Theorem 2 that $\lambda_1(t) = \lambda_2(t) \equiv 0$ and $\lambda_3(t) \equiv \lambda_3(0) = -\frac{2}{3\alpha}(\alpha - \beta) \neq 0$. Hence Theorem 2 implies that the conclusion of Theorem 2 holds.

**Remark 1.** Let $p = 2$, Theorem 1 reduces to [3, Theorem 1], moreover, our assumption $D_0 \neq 0$ is weaker than the assumption $c^\pm \neq 0$ and $c^+ c^- > 0$. In case $g(x) \equiv 0$ and $\lambda_1(t) \equiv 0$, the result of [3] cannot be applied to equation (30), but Theorem 2 gives partial results. Therefore, our results are natural generalization and refinements of the result of [3].
References


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