Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 25 (2006), 467–477

Colombeau Generalized Functions and Solvability of Differential Operators

Khaled Benmeriem and Chikh Bouzar

Abstract. The aim of this paper is to prove that the well known non solvable Mizohata type partial differential equations have Colombeau generalized solutions which are distributions if and only if they are solvable in the space of Schwartz distributions. Therefore the Colombeau generalized solvability includes both a new solution concept and new mathematical objects as solutions.

Keywords. Colombeau generalized functions, regularized derivatives, Mizohata type operators, solvability of differential operators

Mathematics Subject Classification (2000). Primary 46F30, secondary 35D05, 35A07

1. Introduction

Colombeau generalized functions were introduced , see [4], in connection with the so-called problem of multiplication of Schwartz distributions [15]. They were developed and applied in important nonlinear problems, see [2, 5] and [14]. General methods of construction of such generalized functions were given in [1] and [11]. The authors of [12] have tackled the linear counterpart of this theory.

The theory of Colombeau generalized functions provides new solutions of partial differential equations; these new solutions can be divided into two categories:

1) there are classical functions or distributions which are solutions (in one of the new senses provided by this theory) of partial differential equations without solution in the sense of distributions, e.g., see [2, 5, 6, 7] and [10].

2) there are also new objects (such as the square of the Dirac delta distribution,. . .) which can be solutions of equations.

K. Benmeriem: Centre Universitaire de Mascara, Mascara, Algeria;
benmeriemkhaled@yahoo.fr
C. Bouzar: Department of Mathematics, University of Oran Essenia, Algeria;
bouzar@yahoo.com

In [6] the fundamental concept of regularized derivatives was studied and results on global solvability, in the framework of this theory, of the Cauchy problem for large classes of regularized partial differential equations have been given. In particular, the well-known non solvable Mizohata differential equations with regularized derivatives become solvable in the Colombeau algebra. It is then interesting to show the relation between Colombeau generalized solutions and distributional solutions if they exist.

The paper deals, in the framework of the simplified Colombeau algebra, with a class of differential operators non solvable in distributions theory. We show that their Colombeau generalized solutions as regularized differential equations are in relations with distributional solutions if and only if they are solvable in the space of Schwartz distributions. Therefore in the general case in which there are no distributional solution, the new solutions from [6] are not associated with classical objects, even if they are solutions in a new sense: an enlargement of the reservoir of mathematical objects that could be solutions is really needed.

Acknowledgement. The authors thank the referee for the suggested ameliorations of the paper.

2. Simplified algebra of Colombeau

In this section we recall the simplified Colombeau algebra of generalized functions and some needed notions of this theory, for a deep study see [4, 5] and [14]. Let Ω be a non void open subset of $\mathbb{R}^d$ and $I =]0,1[$, define $\chi_M(\Omega)$ as the space of elements $(u_\varepsilon)_\varepsilon$ of $\chi(\Omega) = (C^\infty(\Omega))^I$ such that, for every compact set $K \subset \Omega$, for all $\alpha \in \mathbb{Z}_+^d$, there exists $m > 0$,

$$\sup_{x\in K} |\partial^\alpha u_\varepsilon| \leq O\left(\varepsilon^{-m}\right), \qquad \text{as } \varepsilon \to 0.$$

By $\mathcal{N}(\Omega)$ we denote the elements $(u_\varepsilon)_\varepsilon \in \chi_M(\Omega)$ satisfying for all $K \subset \Omega$, for all $\alpha \in \mathbb{Z}_+^d$, for all $q > 0$,

$$\sup_{x\in K} |\partial^\alpha u_\varepsilon| \leq O\left(\varepsilon^{q}\right), \qquad \text{as } \varepsilon \to 0.$$

An element of $\chi_M(\Omega)$ is called moderate and an element of $\mathcal{N}(\Omega)$ is called null. It is easy to prove that $\chi_M(\Omega)$ is an algebra and $\mathcal{N}(\Omega)$ is an ideal of $\chi_M(\Omega)$.

Definition 2.1. The simplified algebra of Colombeau defined on Ω, denoted $\mathcal{G}_s(\Omega)$, is the quotient algebra

$$\mathcal{G}_s(\Omega) = \frac{\chi_M(\Omega)}{\mathcal{N}(\Omega)}.$$

The algebra of Colombeau $\mathcal{G}_s(\Omega)$ is a commutative and associative differential algebra containing $D'(\Omega)$ as a subspace and $C^\infty(\Omega)$ as subalgebra, see for details [4, 5] and [14], where others important properties of this algebra are studied.

Recall the notion of association relation in the Colombeau algebra $\mathcal{G}_s(\Omega)$, a generalized function $u \in \mathcal{G}_s(\Omega)$ and a distribution $T \in D'(\Omega)$ are called associated, denoted $u \approx T$, if there exists $(u_\varepsilon)_\varepsilon$ a representative of u such that, for all $\phi \in C_0^\infty(\Omega)$,

$$\lim_{\varepsilon\to 0} \int u_\varepsilon(x)\,\phi(x) = \langle T, \phi\rangle .$$

We introduce, for our need, an association relation less stronger than the classical association.

Definition 2.2. A generalized function $u \in \mathcal{G}_s(\Omega)$ and a distribution $T \in D'(\Omega)$ are called locally associated at $x_0 \in \Omega$, denoted $u \approx_{x_0} T$, if there exists $(u_\varepsilon)_\varepsilon$ a representative of u and $\omega \subset \Omega$ an open neighborhood of x_0, such that, for all $\phi \in C_0^\infty(\omega)$,

$$\lim_{\varepsilon\to 0} \int u_\varepsilon(x)\,\phi(x) = \langle T, \phi\rangle .$$

The proof of the following result is easy.

Proposition 2.3. *Let $u \in \mathcal{G}_s(\Omega)$ and $T \in D'(\Omega)$, then $u \approx_{x_0} T$, for all $x_0 \in \Omega$, if and only if $u \approx T$.*

3. Regularized partial differential equations

For the concept of regularized derivatives of Colombeau generalized functions and its application to general Cauchy problems see [6]. Denote by $\mathcal{H}$ the set of non-decreasing functions $h : I \to I$, such that $\lim_{\varepsilon\to 0} h(\varepsilon) = 0$. Let $\rho \in C_0^\infty(\mathbb{R}^d)$ and $\int \rho(x)\,dx = 1$, we define the sequence $(\rho_\varepsilon)_\varepsilon$ by $\rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho\left(\frac{x}{\varepsilon}\right), \varepsilon \in I$.

Definition 3.1. Let $u \in \mathcal{G}_s\left(\mathbb{R}^d\right)$ and $h \in \mathcal{H}$, the partial regularized derivative of u with respect to x_j, denoted $\left(\widetilde{\partial}_{x_j}\right)_h u$, is defined by

$$\left(\widetilde{\partial}_{x_j}\right)_h u = \mathrm{cl}\left(\partial_{x_j} u_\varepsilon * \rho_{h(\varepsilon)}\right)_{\varepsilon\in I} \quad ,$$

where $(u_\varepsilon)_\varepsilon$ is a representative of u.

Remark 3.2. We have $\left(\widetilde{\partial}_{x_j}\right)_h^0 u = u$ and for $\alpha \in \mathbb{Z}_+^d$,

$$\widetilde{\partial}_h^\alpha u = \left(\widetilde{\partial}_{x_1}\right)_h^{\alpha_1} \circ \left(\widetilde{\partial}_{x_2}\right)_h^{\alpha_2} \circ \ldots \circ \left(\widetilde{\partial}_{x_d}\right)_h^{\alpha_d} u \ .$$

It is clear that $\widetilde{\partial}_h^{\alpha} u$ may be defined by the representative $(\partial^{\alpha} u_{\varepsilon} * \rho_{h(\varepsilon)}^{[\alpha]})_{\varepsilon}$, where $\rho_{h(\varepsilon)}^{[\alpha]} = \rho_{h(\varepsilon)} * \rho_{h(\varepsilon)} * \ldots * \rho_{h(\varepsilon)}$, the convolution is taken $|\alpha|$ times. The notion of regularized derivative is well defined and its class is independent of the choice of the representative $(u)_{\varepsilon}$.

In order to study the existence and uniqueness of Colombeau generalized solutions of Cauchy problems with partial regularized derivatives, one introduces the algebra of generalized functions suitable to this context.

We denote by $D_{L^{\infty}}\left(\overline{\Omega}\right)$ the algebra of restrictions to $\overline{\Omega}$ of smooth functions defined on $\mathbb{R}^d$ with all derivatives bounded. With the same method of construction of the simplified algebra of Colombeau, we define the simplified algebra of global generalized functions, denoted $\mathcal{G}_{s,g}\left(\overline{\Omega}\right)$, by the quotient algebra

$$\mathcal{G}_{s,g}\left(\overline{\Omega}\right) = \frac{\mathcal{E}_{M,s,g}\left[\overline{\Omega}\right]}{\mathcal{N}_{s,g}\left[\overline{\Omega}\right]} \quad ,$$

where

$$\begin{aligned} \mathcal{E}_{M,s,g}\left[\overline{\Omega}\right] &= \left\{ (u_{\varepsilon})_{\varepsilon} \in \mathcal{E}_{s,g}\left[\overline{\Omega}\right] : \forall \alpha \in \mathbb{Z}_+^d, \exists p > 0, \|\partial^{\alpha} u_{\varepsilon}\|_{L^{\infty}(\overline{\Omega})} \leq O\left(\varepsilon^{-p}\right) \right\} \\ \mathcal{N}_{s,g}\left[\overline{\Omega}\right] &= \left\{ (u_{\varepsilon})_{\varepsilon} \in \mathcal{E}_{s,g}\left[\overline{\Omega}\right] : \forall \alpha \in \mathbb{Z}_+^d, \forall q > 0, \|\partial^{\alpha} u_{\varepsilon}\|_{L^{\infty}(\overline{\Omega})} \leq O\left(\varepsilon^{q}\right) \right\} \end{aligned}$$

and $\mathcal{E}_{s,g}[\overline{\Omega}] = (D_{L^{\infty}}(\overline{\Omega}))^I$. It is easy to see that $\mathcal{E}_{M,s,g}[\overline{\Omega}]$ is a differential subalgebra of $\mathcal{E}_{s,g}[\overline{\Omega}]$ and $\mathcal{N}_{s,g}[\overline{\Omega}]$ is an ideal of $\mathcal{E}_{M,s,g}[\overline{\Omega}]$.

Proposition 3.3. *Let $u \in D_{L^{\infty}}(\mathbb{R}^d), \alpha \in \mathbb{Z}_+^d$ and $h \in \mathcal{H}$, if $h(\varepsilon) = O(\varepsilon), \varepsilon \to 0$, then*

$$\widetilde{\partial}_h^{\alpha} u = \partial^{\alpha} u \quad \text{in } \mathcal{G}_{s,g}\left(\mathbb{R}^d\right).$$

Remark 3.4. In general if $u \in \mathcal{G}_s(\mathbb{R}^d)$, $\widetilde{\partial}_h^{\alpha} u \neq \partial^{\alpha} u$ in $\mathcal{G}_{s,g}(\mathbb{R}^d)$. For example, denote H the Heaviside function on $\mathbb{R}$, then

$$\widetilde{\partial}_h H \neq H' \quad \text{in } \mathcal{G}_{s,g}\left(\mathbb{R}\right).$$

Let $T > 0$, $h \in \mathcal{H}$ and $W = [-T, T] \times \mathbb{R}^d$, the regularized derivative of an element u of $\mathcal{G}_{s,g}(W)$ with respect to x_j is defined as

$$\left(\widetilde{\partial}_{x_j}\right)_h u = \mathrm{cl}\left(\partial_{x_j} u_{\varepsilon}(t, .) * \rho_{h(\varepsilon)}\right)_{\varepsilon \in I},$$

where $(u_{\varepsilon})_{\varepsilon \in I}$ is a representative of u and $\rho \in \mathcal{S}(\mathbb{R}^d)$ satisfies

i) $\int \rho(x)\, dx = 1$
ii) $\int x^{\alpha} \rho(x)\, dx = 0, \forall \alpha \in \mathbb{N}^d$.

Now we consider in $\mathcal{G}_{s,g}(W)$ the following linear Cauchy problem

$$\begin{cases} \partial_t u + \sum_{|\alpha| \leq m} a_\alpha \widetilde{\partial}_h^\alpha u = f \\ \qquad u(0,x) = u_0(x), \end{cases} \tag{1}$$

where $a_\alpha \in D_{L^\infty}(W), f \in \mathcal{G}_{s,g}(W)$ and $u_0 \in \mathcal{G}_{s,g}(\mathbb{R}^d)$.

One of the main results of the paper [6] is the following.

Theorem 3.5. *The linear Cauchy problem* (1) *admits a global unique solution* $u \in \mathcal{G}_{s,g}(W)$ *if there exists* $p \in \mathbb{Z}_+$ *such that*

$$e^{C.h(\varepsilon)^{-m}} = O\left(\varepsilon^{-p}\right),$$

where $c_\alpha = \left\| \partial^\alpha \rho^{[\alpha]} \right\|_{L^1(\mathbb{R}^d)}$ *and* $C = \sum_{|\alpha| \leq m} c_\alpha \|a_\alpha\|_{L^\infty(W)}$.

4. Non solvable differential operators

The differential operators

$$M = \frac{\partial}{\partial t} + ib(t) \frac{\partial}{\partial x}, \tag{2}$$

where $b \in C^\infty(\mathbb{R})$ satisfies the condition

$$tb(t) > 0, \quad \forall t \in \mathbb{R}^*, \tag{3}$$

are called *differential operators of Mizohata type.* We know, see [13], that such operators M are not locally solvable at the origin in the framework of Schwartz distributions. A construction of a function $f \in C_0^\infty(\mathbb{R}^2)$ such that there is no locally distributional solution at the origin of the equation $Mu = f$ is given in [3].

Remark 4.1. It is well known that the operator (2) with the condition (3) is reduced to the Mitzohata operator $\frac{\partial}{\partial t} + it\frac{\partial}{\partial x}$ if and only if $b(0) = 0$ and $b'(0) \neq 0$, see Trèves [16]. In our case the function $b(t)$ may have a zero at the origin of infinite order.

In this section we give a necessary and sufficient condition for local solvability of the equation

$$Mu(t,x) = f(t,x), \tag{4}$$

where $f \in C_0^\infty(\mathbb{R}^2)$. Let $B(t) = \int_0^t b(s)\,ds$ and define the function Kf by

$$Kf(x) = \int_0^{+\infty} \int_{-\infty}^{+\infty} e^{i(x+iB(s))\xi} \widehat{f}(s,\xi)\, ds d\xi,$$

where $\widehat{f}(t,\xi)$ is the Fourier transform of $f(t,x)$ with respect to the variable x.

Theorem 4.2. *The equation* (4) *admits a local distributional solution at the origin of* $\mathbb{R}^2$ *if and only if the function* Kf *is real analytic at the origin of* $\mathbb{R}$.

Proof. To solve the equation (4) we formally apply the Fourier transformation with respect to x, then

$$\frac{\partial \widehat{u}}{\partial t}(t,\xi) - b(t)\,\xi\widehat{u}(t,\xi) = \widehat{f}(t,\xi),$$

hence

$$\widehat{u}(t,\xi) = \int_{t_0}^{t} e^{(B(t)-B(s))\xi}\widehat{f}(s,\xi)\,ds. \tag{5}$$

To recover u we must apply the inverse Fourier transformation to $\widehat{u}$, so the choice of t_0 is important. In (5), we choose t_0 such that $(B(t)-B(s))\,\xi \leq 0$, $\forall \xi \in \mathbb{R}$. By the condition (3) the function B is increasing for $t > 0$ and decreasing for $t < 0$.
For $\xi < 0$, we choose $t_0 = 0$, and we define u by

$$\widehat{u}(t,\xi) = \int_{0}^{t} e^{(B(t)-B(s))\xi}\widehat{f}(s,\xi)\,ds.$$

For $\xi > 0$, we take

$$\widehat{u}(t,\xi) = \begin{cases} -\int_t^{+\infty} e^{(B(t)-B(s))\xi}\widehat{f}(s,\xi)\,ds, & t > 0 \\ \int_{-\infty}^{t} e^{(B(t)-B(s))\xi}\widehat{f}(s,\xi)\,ds, & t < 0\,. \end{cases}$$

In this case the function $\widehat{u}$ admits a jump at $t = 0$ given by

$$\widehat{u}(+0,\xi) - \widehat{u}(-0,\xi) = -\int_{-\infty}^{+\infty} e^{-B(s)\xi}\widehat{f}(s,\xi)\,ds.$$

Consequently, we obtain in the distributional sense

$$\frac{\partial \widehat{u}}{\partial t}(t,\xi) - b(t)\,\xi\widehat{u}(t,\xi) = \widehat{f}(t,\xi) + [\widehat{u}(0_+,\xi) - \widehat{u}(0_-,\xi)]\,\delta(t) \quad , \tag{6}$$

where δ is the Dirac measure at 0. The inverse Fourier transform of (6) with respect to ξ gives

$$Mu(t,x) = f(t,x) - \delta(t)\,Kf(x)\,.$$

Let $H(t)$ be the Heaviside function, then

$$M\left(u(t,x) + H(t)\,Kf(x)\right) = f(t,x) + ib(t)\,H(t)\,(Kf(x))'\,.$$

The term $ib(t)\,H(t)\,(Kf(x))'$ in the last equation is eliminated thanks to the following function:

$$v(t,x) = \begin{cases} i\int_0^t b(s)\,(Kf)'\big(x - i(B(t)-B(s))\big)ds, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

which is well defined as $Kf(x)$ is assumed to be real analytic at the origin. Therefore the function v admits an holomorphic extension to a neighborhood ω of the origin of $\mathbb{C}$. Further the function v satisfies the equation $Mv(t,x) = ib(t)H(t)(Kf)'(x)$. Define

$$w(t,x) = u(t,x) + H(t)\,Kf(x) - v(t,x),$$

then $Mw(t,x) = f(t,x)$, i.e., $w(t,x)$ is a solution of the equation (4).

The constructed solution w is of class C^∞ in a neighborhood of the origin. Indeed, we remark that, if $t \neq 0$, the operator M is elliptic so w is C^∞ when $t \neq 0$. To show that w is C^∞ we study the case $t = 0$. We have

$$w(t,x) = H(t)\,A(t,x) + H(-t)\,B(t,x),$$

where

$$\begin{aligned} A(t,x) = &\int_{-\infty}^{0}\int_{0}^{t} e^{(ix+B(t)-B(s))\xi}\widehat{f}(s,\xi)\,ds\,d\xi \\ &-\int_{0}^{+\infty}\int_{t}^{+\infty} e^{(ix+B(t)-B(s))\xi}\widehat{f}(s,\xi)\,ds\,d\xi \\ &+\int_{0}^{+\infty}\int_{-\infty}^{+\infty} e^{i(x+iB(s))\xi}\widehat{f}(s,\xi)\,ds\,d\xi \\ &- i\int_{0}^{t} b(s)\,(Kf)'\big(x - i\,(B(t)-B(s))\big)ds \end{aligned}$$

and

$$B(t,x) = \int_{-\infty}^{0}\int_{0}^{t} e^{(ix+B(t)-B(s))\xi}\widehat{f}(s,\xi)\,dsd\xi + \int_{0}^{+\infty}\int_{-\infty}^{t} e^{(ix+B(t)-B(s))\xi}\widehat{f}(s,\xi)\,dsd\xi.$$

It is clear that $w(t,x)$ is C^∞ with respect to the variable x. Moreover we have

$$\begin{aligned} \partial_t^j\partial_x^k w(t,x) &= H(t)\,\partial_t^j\partial_x^k A(t,x) + H(-t)\,\partial_x^j\partial_t^k B(t,x) \\ &\quad + \sum_{i=0}^{j-1}\delta^{(i)}(t)\left(\partial_t^{j-1-i}\partial_x^k w(0_+,x) - \partial_t^{j-1-i}\partial_x^k w(0_-,x)\right) \\ &= H(t)\,\partial_t^j\partial_x^k A(t,x) + H(-t)\,\partial_x^j\partial_t^k B(t,x) \\ &\quad + \sum_{i=0}^{j-1}\delta^{(i)}(t)\left(\partial_t^{j-1-i}\partial_x^k A(0_+,x) - \partial_t^{j-1-i}\partial_x^k B(0_-,x)\right) \quad . \end{aligned}$$

We also have

$$\begin{aligned} B(t,x) - A(t,x) = &\int_{0}^{+\infty}\int_{-\infty}^{+\infty} e^{(ix+B(t)-B(s))\xi}\widehat{f}(s,\xi)\,ds\,d\xi - Kf(x) + \\ &+ i\int_{0}^{t} b(s)\,(Kf)'\big(x - i\,(B(t)-B(s))\big)ds, \end{aligned}$$

then for all $l,k\in\mathbb{Z}_+,\partial_t^l\partial_x^k\left(B\left(t,x\right)-A\left(t,x\right)\right)$ is a finite sum of the following terms:

$$b^{(l_1)}\left(t\right)b^{l_2}\left(t\right)\left[i^k\int_0^{+\infty}\int_{-\infty}^{+\infty}\xi^{l_3+k}e^{(ix+B(t)-B(s))\xi}\widehat{f}\left(s,\xi\right)dsd\xi\right.$$
$$\left.-\left(-i\right)^{l_3}\left(Kf^{(l_3+k)}\left(x\right)-i\int_0^t b\left(s\right)\left(Kf\right)^{(l_3+1)}\left(x-i\left(B\left(t\right)-B\left(s\right)\right)\right)ds\right)\right],$$

where l_1, l_2 and l_3 depend only on l. It is clear that these terms equal all zero when $t=0$, then

$$\partial_t^j\partial_x^k w\left(t,x\right)=H\left(t\right)\partial_t^j\partial_x^k A\left(t,x\right)+H\left(-t\right)\partial_x^j\partial_t^k B\left(t,x\right)$$

which give $w\in C^\infty$.

The proof of the necessity of the analyticity of Kf. Let us suppose that there is $u\in C^1$ such that $Mu=f$ in a neighborhood Ω of the origin and let $\chi\in C_0^\infty\left(\mathbb{R}^2\right),\chi\equiv 1$ in a neighborhood of the origin and $\operatorname{supp}\chi\subset\Omega$. So (χf) satisfies locally $Mu=\chi f$ and therefore the function $Kf(x)$ can be written in the form

$$Kf(x)=\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^{+\infty}e^{i(x-y+iB(s))\xi-\varepsilon\xi^2}\left(\chi f\right)(s,y)\frac{d\xi}{2\pi}\,dy\,ds\,.$$

Consider the integral

$$K_\varepsilon f(x)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^{+\infty}e^{i(x-y+iB(s))\xi-\varepsilon\xi^2}\left(\chi f\right)(s,y)\frac{d\xi}{2\pi}\,dy\,ds\ ,$$

then

$$\begin{aligned}K_\varepsilon f(x)=&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^{+\infty}e^{i(x-y+iB(s))\xi-\varepsilon\xi^2}M(\chi u)\,(s,y)\,\frac{d\xi}{2\pi}\,dy\,ds\\&-\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^{+\infty}e^{i(x-y+iB(s))\xi-\varepsilon\xi^2}u(M\chi)\,(s,y)\,\frac{d\xi}{2\pi}\,dy\,ds\,.\end{aligned}\tag{7}$$

An integration by parts of the first term of the second member gives

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^{+\infty}e^{i(x-y+iB(s))\xi-\varepsilon\xi^2}M(\chi u)\,(s,y)\,\frac{d\xi}{2\pi}\,dy\,ds=0,$$

hence

$$K_\varepsilon f(x)=-\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^{+\infty}e^{i(x-y+iB(s))\xi-\varepsilon\xi^2}\frac{d\xi}{2\pi}u(s,y)M\chi\,(s,y)\ dy\,ds\ .$$

Consider now the deformation of the path of integration with respect to ξ in $K_\varepsilon f(x)$ by taking the contour Γ defined by

$$\zeta = \rho\left(1+\frac{i}{2}\frac{x-y}{|x-y|}\right), \quad \rho>0.$$

Hence, for any $\varepsilon>0$ fixed, we have

$$K_\varepsilon f(x) = -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_\Gamma e^{i(x-y+iB(s))\zeta-\varepsilon\zeta^2}\frac{d\zeta}{2\pi}u(s,y)M\chi(s,y)\,dy\,ds. \tag{8}$$

The function $K_\varepsilon f(x)$ is analytic in x for each fixed ε. It remains to show that $\lim_{\varepsilon\to 0} K_\varepsilon f(x)$ is analytic at the origin. For this need, we have to estimate uniformly the expression $K_\varepsilon f(x)$. Since $M\chi = 0$ in a neighborhood of the origin, as $\chi\equiv 1$ in this neighborhood, so in (8) the integral with respect to s and y is taken outside a rectangle, i.e., either $|s|>c_1$ or $|y|>c_2$. Consequently, $B(s)>c_3$ or $|x-y|>c_4$, where $(c_j)_{j=1}^4$ are positive constants not depending on s, y , x. Then

$$\operatorname{Im}\left((x-y+iB(s))\,\zeta - i\varepsilon\zeta^2\right) \geq c\rho + \frac{3}{4}\varepsilon\rho^2,$$

from this estimate we conclude that $\lim_{\varepsilon\to 0} K_\varepsilon f(x)$ is analytic with respect to x in a neighborhood of the origin.

Now suppose that $u\in D'\backslash C^1$ and u is a solution of $Mu=f$, then u is a C^∞ function of t with values in $D'(\mathbb{R})$, see [9, Theorem 4.4.8]. By the local structure of a distribution, we can assume that, there exists a function $v\in C^1(\mathbb{R}^2)$ such that $u=\partial_x^N v$. As $M\left(\partial_x^N v\right) = \partial_x^N (Mv)$, we may substitute $\partial_x^N v$ for u in (7) and proceed in a same way to obtain the general result. □

5. Differential operators of Mizohata type in $\mathcal{G}_{s,g}$

Consider in $\mathcal{G}_{s,g}(W)$, $W=[-T,T]\times\mathbb{R}$, the following equation

$$\partial_t U + ib(t)\,\widetilde{\partial}_{xh} U = f \quad \text{ in } \mathcal{G}_{s,g}(W), \tag{9}$$

where $f\in C_0^\infty(\mathbb{R}^2)$, $b\in D_{L^\infty}(\mathbb{R})$ and $tb(t)>0, t\in\mathbb{R}^*$. Let $h\in\mathcal{H}$ such that $h(\varepsilon)=O(\varepsilon), \varepsilon\to 0$, then we have the following results.

Theorem 5.1. *Let $U=\mathrm{cl}\,(u_\varepsilon)_\varepsilon\in\mathcal{G}_{s,g}(W)$ be a solution of (9) which is locally associated to a distribution $v\in D'(\omega)$ at the origin, then the function Kf is analytic in a neighborhood of the origin of $\mathbb{R}$.*

Proof. Let us suppose that a solution $U = \mathrm{cl}\,(u_\varepsilon)_\varepsilon$ of (9) is locally associated to a distribution v at the origin, then there is a neighborhood $\widetilde{\omega}$ of the origin such that, for all $\phi \in C_0^\infty(\widetilde{\omega})$,

$$\lim_{\varepsilon \to 0} \int u_\varepsilon (t,x)\, \phi (t,x) = \langle v, \phi \rangle .$$

We have

$$(\partial_t u_\varepsilon)_\varepsilon + ib\,(t) \left(\partial_x u_\varepsilon * \rho_{h(\varepsilon)}\right)_\varepsilon - f \in \mathcal{N}_{s,g}\,[W]\,,$$

where the convolution takes place in the x-variable at fixed t, so

$$\lim_{\varepsilon \to 0} \left((\partial_t u_\varepsilon)_\varepsilon + ib\,(t) \left(\partial_x u_\varepsilon * \rho_{h(\varepsilon)}\right)_\varepsilon - f\right) = 0 \quad \text{ in } D'\,(\widetilde{\omega})\,.$$

Since for all $h \in \mathcal{H}$, the sequence $(\rho_{h(\varepsilon)})_\varepsilon$ converges to the Dirac measure, as $\varepsilon \to 0$, then for all $\phi \in C_0^\infty\,(\widetilde{\omega})$ we have

$$\lim_{\varepsilon \to 0} \int \left(\partial_x u_\varepsilon * \rho_{h(\varepsilon)}\right)_\varepsilon (t,x)\, \phi\,(t,x) = \langle \partial_x v, \phi \rangle\,,$$

hence $\partial_t v + ib\,(t)\,\partial_x v = f$ in $D'\,(\widetilde{\omega})$, i.e., v is a solution of the equation

$$Mu = f \text{ in } D'\,(\widetilde{\omega})\,,$$

consequently, by Theorem 4.2, the function Kf is analytic in neighborhood of the origin of $\mathbb{R}$. □

Theorem 5.2. *If Kf is analytic in a neighborhood of the origin of $\mathbb{R}$, then* (9) *admits a solution $U = \mathrm{cl}\,(u_\varepsilon)_\varepsilon \in \mathcal{G}_{s,g}\,(W)$ which is locally associated to a distribution $v \in D'\,(\omega)$ at the origin.*

Proof. Let us suppose that the function Kf is analytic in a neighborhood of the origin, then there exists $v \in D'(\omega)$ such that $Mv = f$ in $D'(\omega)$. Moreover, see the proof of the Theorem 4.2, v is of class C^∞ in ω. Let $\Omega \Subset \omega$, then $v \in D_{L^\infty}(\overline{\Omega})$ and by Proposition 3.3 we have

$$\partial_t v + ib\,(t) \left(\widetilde{\partial_x}\right)_h v = f \quad \text{ in } \mathcal{G}_{s,g}\left(\overline{\Omega}\right).$$

Let U be a solution of the equation (9) in $\mathcal{G}_{s,g}\,(W)$ with the initial data given by $U\,(0,x) = [v\,(0,x)]$, then in $\mathcal{G}_{s,g}\left(\overline{\Omega}\right)$ we have

$$\begin{cases} \partial_t\,(U - v) + ib\,(t) \left(\widetilde{\partial_x}\right)_h (U - v) = 0 \\ \qquad\qquad\qquad\qquad (U - v)\,(0,x) = 0. \end{cases}$$

The uniqueness of the generalized solution in the Theorem 3.5 gives $U - v = 0$ in $\mathcal{G}_{s,g}(\overline{\Omega})$, hence $U \approx_0 v$. □

For the Mizohata equations under consideration, the new generalized solutions from the method in [6] can be associated with distributions only in the case these equations are solvable in the sense of distributions theory. This follows at once from Theorems 4.2, 5.1 and 5.2.

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Received December 10, 2004; revised October 9, 2005