Non-Compact and Sharp Embeddings of Logarithmic Bessel Potential Spaces into Hölder-Type Spaces

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Abstract. In our recent paper [Compact and continuous embeddings of logarithmic Bessel potential spaces. Studia Math. 168 (2005), 229 – 250] we have proved an embedding of a logarithmic Bessel potential space with order of smoothness $\sigma$ less than one into a space of $\lambda(\cdot)$-Hölder-continuous functions. We show that such an embedding is not compact and that it is sharp.

Keywords. Generalized Lorentz-Zygmund spaces, logarithmic Bessel potential spaces, Hölder-continuous functions, embeddings

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1. Introduction

In the recent paper [8] we have derived embeddings of Bessel potential spaces with smoothness $\sigma \in (0,1)$, modelled upon generalized Lorentz-Zygmund spaces, into spaces of $\lambda(\cdot)$-Hölder-continuous functions. Here we discuss non-compactness and sharpness of those embeddings.

To be more specific, we need some notation. Given two (quasi-)Banach spaces $X$ and $Y$, we write $X \hookrightarrow Y$ or $X \hookrightarrow \hookrightarrow Y$ if $X \subset Y$ and the natural embedding is continuous or compact, respectively.

Let $p,q \in (0,\infty], m \in \mathbb{N}, \alpha_1,\ldots,\alpha_m \in \mathbb{R}$ and let $\Omega$ be a measurable subset of $\mathbb{R}^n$ (with respect to $n$-dimensional Lebesgue measure). The generalized
Lorentz-Zygmund (GLZ) space \( L_{p,q;\alpha_1,\ldots,\alpha_m}(\Omega) \) consists of all measurable (real or complex) functions \( f \) on \( \Omega \) such that the quantity
\[
\|f\|_{p,q;\alpha_1,\ldots,\alpha_m} := \left\| t^{\frac{1}{p}-\frac{1}{q}} \left( \prod_{j=1}^{m} \ell_j^*(t) \right) f^*(t) \right\|_{q,(0,\infty)}
\]
is finite. Here \( \ell_1, \ldots, \ell_m \) are (logarithmic) functions defined on \((0,\infty)\) by
\[
\ell_1(t) = \ell(t) = 1 + |\log t|, \quad \ell_j(t) = 1 + \log \ell_{j-1}(t) \quad (j > 1),
\]
\( f^* \) denotes the non-increasing rearrangement of \( f \) given by
\[
f^*(t) = \inf \{ \lambda > 0 : \{ x \in \Omega; |f(x)| > \lambda \} \}_{\infty} \leq t, \quad t \geq 0,
\]
\( |G|_n \) stands for the \( n \)-volume of a measurable subset \( G \) of \( \mathbb{R}^n \) and \( \| \cdot \|_{q,(a,b)} \) is the usual \( L^q \)-quasi-norm on an interval \((a,b) \subseteq \mathbb{R} \). (For more details about the spaces \( L_{p,q;\alpha_1,\ldots,\alpha_m}(\Omega) \) see [2]–[7], [9], and [11].)

The Bessel kernel \( g_\sigma, \sigma > 0, \) is defined to be that function on \( \mathbb{R}^n \) whose Fourier transform \( \widehat{g}_\sigma \) is
\[
\widehat{g}_\sigma(\xi) = (2\pi)^{-\frac{n}{2}} (1 + |\xi|^2)^{-\frac{n}{2}}, \quad \xi \in \mathbb{R}^n,
\]
where by the Fourier transform \( \widehat{f} \) of a function \( f \) we mean
\[
\widehat{f}(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ixy} f(y) \, dy, \quad x \in \mathbb{R}^n.
\]
Let \( \sigma > 0, p \in (1,\infty), q \in [1,\infty], \alpha_1, \ldots, \alpha_m \in \mathbb{R} \). The logarithmic Bessel potential space \( H^\sigma L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n) \) is defined by
\[
H^\sigma L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n) := \{ u = g_\sigma * f ; f \in L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n) \},
\]
and is equipped with the (quasi-)norm
\[
\|u\|_{\sigma;p,q;\alpha_1,\ldots,\alpha_m} := \|f\|_{p,q;\alpha_1,\ldots,\alpha_m}. \tag{1}
\]
(By \( f * g \) we mean the convolution of functions \( f \) and \( g \).)

Let \( \mathcal{L} \) be the class of all continuous functions \( \lambda : (0,\infty) \to (0,\infty) \) which are increasing on some interval \((0,\delta)\), with \( \delta = \delta(\lambda) > 0 \), and satisfy \( \lim_{\lambda \to 0,\lambda(t) = 0} \). Let \( \lambda \in \mathcal{L} \) and let \( \Omega \) be a domain in \( \mathbb{R}^n \). The space \( C^{0,\lambda(\cdot)}(\Omega) \) of \( \lambda(\cdot) \)-Hölder-continuous functions consists of all those functions \( u \in C(\overline{\Omega}) \) for which the norm
\[
\|u\|_{C^{0,\lambda(\cdot)}(\Omega)} := \sup_{x \in \Omega} |u(x)| + \sup_{\substack{x,y \in \Omega \ x \neq y}} \frac{|u(x) - u(y)|}{\lambda(|x-y|)}
\]
is finite. Here $C(\Omega)$ stands for the family of all functions which are bounded and uniformly continuous on $\Omega$. (For more information about such spaces see [1] or [10].)

We write $A \lesssim B$ (or $A \gtrsim B$) if $A \leq cB$ (or $cA \geq B$) for some positive constant $c$ independent of appropriate quantities involved in the expressions $A$ and $B$, and $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$. If $p \in [1, \infty]$, the conjugate number $p'$ is defined by $\frac{1}{p} + \frac{1}{p'} = 1$ with the understanding that $1' = \infty$ and $\infty' = 1$.

In [8] we have extended Theorem 4.9 of [5] (to the range $\sigma \in (0, 1)$) and proved the following embedding.

**Theorem 1.** Let $0 < \sigma < 1$, $\frac{n}{\sigma} < p < \infty$, $1 < q < \infty$, $m \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ and let

$$\lambda(t) = t^{n-\frac{n}{p}} \prod_{j=1}^{m} t^{-\alpha_j}, \quad t > 0.$$  

Then

\[ H^\sigma L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n) \hookrightarrow C^{0,\lambda}(\mathbb{R}^n). \] (2)

The aim of this paper is to show that the embedding of $H^\sigma L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n)$ into $C^{0,\lambda}(\mathbb{R}^n)$, where $\Omega$ is a nonempty domain in $\mathbb{R}^n$, cannot be compact and that the embedding (2) is sharp with respect to the function $\lambda$.

## 2. Main result and proofs

Our main result reads as follows.

**Theorem 2.** Let the assumptions of Theorem 1 be satisfied. Let $n \geq 2$ and $\Omega \subseteq \mathbb{R}^n$ be a nonempty domain. Then the embedding

\[ H^\sigma L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n) \hookrightarrow C^{0,\lambda}(\mathbb{R}^n) \] (3)

is not compact. Moreover, if a function $\mu \in \mathcal{L}$ satisfies $\frac{\mu}{\lambda} \in \mathcal{L}$, then the embedding

\[ H^\sigma L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n) \hookrightarrow C^{0,\mu}(\mathbb{R}^n) \] (4)

does not hold.

To prove Theorem 2, we need some preliminary work. We modify the idea from [7] to construct suitable test functions. Assume that $\mathcal{G}$ is a function with the following properties:

- $\mathcal{G}$ is positive and continuous on $(0, 1]$; \hspace{1cm} (4)
- $t \mathcal{G}(t)$ is nonincreasing on $(0, r_0]$, where $r_0 \in (0, 1]$ is a fixed number; \hspace{1cm} (5)
- $\mathcal{G}(\frac{t}{2}) \preceq \mathcal{G}(t)$, $t \in (0, 1]$ \hspace{1cm} (6)
(notice that the assumption (5) is stronger than (4.2) of [7]). We use mollifiers to assign to the function \( G \) a family of functions \( \{G_r\} \). Let \( \varphi \in C_0^\infty(\mathbb{R}) \) be a non-negative function such that \( \int_{\mathbb{R}} \varphi = 1 \) and \( \text{supp} \varphi = [-1, 1] \). We define the function \( \varphi_\varepsilon, \varepsilon > 0, \) by

\[
\varphi_\varepsilon(t) := \frac{1}{\varepsilon} \varphi\left(\frac{t}{\varepsilon}\right), \quad t \in \mathbb{R},
\]

and we put

\[
\psi := \chi_{[-2+\frac{1}{\varepsilon}, 1+\frac{1}{\varepsilon}] * \varphi_\varepsilon}.
\]

Now, we extend \( G \) by zero outside the interval \((0, 1]\) and we define functions \( G_r, r \in (0, 1) \), by

\[
G_r(t) := \left((\chi_{[r, \infty]} \psi G) * \varphi_\varepsilon\right)(t), \quad t \in \mathbb{R}. \tag{7}
\]

For any \( r \in (0, \frac{1}{4}) \), let \( a_r \) be a positive number, let

\[
h_r(x) := a_r G_r(|x|), \quad x \in \mathbb{R}^n, \tag{8}
\]

and

\[
u_r(x) := x_1 (g_\sigma * h_r)(x), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{9}
\]

Our first aim is to show that the functions \( \nu_r \) belong to the source space in (3). To this end, we shall need the following result.

**Lemma 1** (cf. Lemma 4.1 of [7]). Let \( r \in (0, \frac{1}{4}) \) and let \( G_r \) be the functions defined by (7), where \( G \) satisfies (4)–(6). Then

\[
G_r \in C_0^\infty(\mathbb{R}), \quad \text{supp}G_r \subset [\frac{r}{2}, 1] \quad \text{and} \quad G_r \geq 0. \tag{10}
\]

Moreover, there are positive constants \( C_1 \) and \( C_2 \) (independent of \( r \) and \( t \)) such that

\[
G_r(t) \leq C_1 G(t) \chi_{[\frac{r}{2}, 1]}(t), \quad t \in (0, 1] \tag{11}
\]

\[
G_r(t) \geq C_2 G(t), \quad t \in [2r, \frac{1}{2}].
\]

We shall make use of the next assertions.

**Lemma 2.** Let \( h \) belong to the Schwartz space \( \mathcal{S} \), \( \sigma \geq 0, j \in \{1, \ldots, n\} \) and let \( \mathcal{R}_j \) be the Riesz transform. Then there exists a finite measure \( \nu \) on \( \mathbb{R}^n \) such that, for any \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \)

\[
x_j (g_\sigma * h)(x) = -\sigma (2\pi)^{-\frac{n}{2}} \left[ g_\sigma * (\mathcal{R}_j (\nu * g_1 * h)) \right](x) + \left[ g_\sigma * (y_j h(y)) \right](x).
\]

**Proof.** The equality can be derived analogously to (4.48) in [7]. \( \square \)
Lemma 3 (cf. Cor. 4.12 of [7]). Let $1 < p < \infty$, $1 \leq q \leq \infty$, $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ and let $\nu$ be the measure from Lemma 2. Then, for all $f \in L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n)$,
\[
\|g_n * f\|_{p,q;\alpha_1,\ldots,\alpha_m} \lesssim \|f\|_{p,q;\alpha_1,\ldots,\alpha_m}, \quad \alpha \geq 0,
\]
\[
\|R_j f\|_{p,q;\alpha_1,\ldots,\alpha_m} \lesssim \|f\|_{p,q;\alpha_1,\ldots,\alpha_m}, \quad j = 1, \ldots, n,
\]
\[
\|\nu * f\|_{p,q;\alpha_1,\ldots,\alpha_m} \lesssim \|f\|_{p,q;\alpha_1,\ldots,\alpha_m}
\]

Proof. The assumption $p > \frac{n}{n-1}$ and the equality $\frac{1}{p} = \frac{1}{\tilde{p}} - \frac{1}{n}$ imply that $\tilde{p} \in (1, n)$. Thus, the result follows on applying Theorem 3.1 of [7].

Lemma 4. Let $n \geq 2$, $p > \frac{n}{n-1}$, $q \in [1, \infty]$, $\frac{1}{p} = \frac{1}{\tilde{p}} - \frac{1}{n}$, $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$. Then, for all $f \in L_{\tilde{p},q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n)$,
\[
\|g_1 * f\|_{p,q;\alpha_1,\ldots,\alpha_m} \lesssim \|f\|_{p,q;\alpha_1,\ldots,\alpha_m}.
\]

Proof. The assumption $p > \frac{n}{n-1}$ and the equality $\frac{1}{p} = \frac{1}{\tilde{p}} - \frac{1}{n}$ imply that $\tilde{p} \in (1, n)$. Thus, the result follows on applying Theorem 3.1 of [7].

Lemma 5. Let $p, q \in (1, \infty)$, $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$. Let $g$ be a positive function which is continuous in $(0, 1]$ and nonincreasing in some interval $(0, r_0) \subset (0, 1]$. Then, for all $r \in (0, r_0)$,
\[
\|g(|y|)\chi_{[r,1]}(|y|)\|_{p,q;\alpha_1,\ldots,\alpha_m} \lesssim \mathcal{V}_1(r) + \mathcal{V}_2(r),
\]

where
\[
\mathcal{V}_1(r) := \left\| t^{\frac{n-1}{p}} \left( \prod_{j=1}^{m} \ell_j^{\alpha_j}(t) \right) g(t) \right\|_{q;r(1)}
\]
\[
\mathcal{V}_2(r) := r^{\frac{n}{\tilde{p}}} \left( \prod_{j=1}^{m} \ell_j^{\alpha_j}(r) \right) g(r).
\]

Proof. The estimate can be proved analogously to the estimate (4.3) in Lemma 4.1 of [4].

The next lemma provides the upper estimate of $\|u_r\|_{\sigma,p,q;\alpha_1,\ldots,\alpha_m}$, where $u_r$ are the functions given by (9).

Lemma 6. Let $n \geq 2$, $p > \frac{n}{n-1}$, $q \in (1, \infty)$, $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$. Then the functions $u_r$, $r \in (0, r_0)$, defined by (9) (with $\mathcal{G}$ given by (4)--(6)), satisfy
\[
\|u_r\|_{\sigma,p,q;\alpha_1,\ldots,\alpha_m} \lesssim a_r(\mathcal{W}_1(r/2) + \mathcal{W}_2(r/2)),
\]

where
\[
\mathcal{W}_1(r) := \left\| t^{\frac{n+1}{p}} \left( \prod_{j=1}^{m} \ell_j^{\alpha_j}(t) \right) \mathcal{G}(t) \right\|_{q;r(1)}
\]
\[
\mathcal{W}_2(r) := r^{\frac{n+1}{\tilde{p}}} \left( \prod_{j=1}^{m} \ell_j^{\alpha_j}(r) \right) \mathcal{G}(r).
\]
Proof. Since \( u_r \in \mathcal{S} \) (cf. (10) and the fact that \( g_\sigma * f \in \mathcal{S} \) for \( f \in \mathcal{S} \) and \( \sigma > 0 \)), we can use Lemma 2 and the definition in (1) to get

\[
\|u_r\|_{p,q;\alpha_1,\ldots,\alpha_m} 
\lesssim \|g_\sigma * \mathcal{R}_1(\nu * g_1 * h_r)\|_{p,q;\alpha_1,\ldots,\alpha_m} + \|g_\sigma * (y_1 h_r(y))\|_{p,q;\alpha_1,\ldots,\alpha_m} 
\]

(12)

Applying Lemma 3, Lemma 4, (8) and (11) to the first term, we obtain

\[
\|\mathcal{R}_1(\nu * g_1 * h_r)\|_{p,q;\alpha_1,\ldots,\alpha_m} \lesssim \|g_1 * h_r\|_{p,q;\alpha_1,\ldots,\alpha_m} \leq \|h_r\|_{p,q;\alpha_1,\ldots,\alpha_m} \leq a_r \|\mathcal{G}(\|y\|)\chi_{[\frac{1}{2},1]}(\|y\|)\|_{p,q;\alpha_1,\ldots,\alpha_m}.
\]

Moreover, using Lemma 5 with \( g = \mathcal{G} \) (observe that this function satisfies the assumptions of Lemma 5) and the identity \( \frac{n}{p} = \frac{n}{p} + 1 \), we arrive at

\[
\|\mathcal{G}(\|y\|)\chi_{[\frac{1}{2},1]}(\|y\|)\|_{p,q;\alpha_1,\ldots,\alpha_m} \lesssim \mathcal{W}_1(r/2) + \mathcal{W}_2(r/2).
\]

Consequently,

\[
\|\mathcal{R}_1(\nu * g_1 * h_r)\|_{p,q;\alpha_1,\ldots,\alpha_m} \lesssim a_r \left[ \mathcal{W}_1(r/2) + \mathcal{W}_2(r/2) \right].
\]

(13)

Furthermore, we use (8), (11) and Lemma 5 with \( g(t) = t \mathcal{G}(t) \) to get

\[
\|y_1 h_r(y)\|_{p,q;\alpha_1,\ldots,\alpha_m} \leq \|y_1 h_r(y)\|_{p,q;\alpha_1,\ldots,\alpha_m} \leq a_r \|\mathcal{G}(\|y\|)\chi_{[\frac{1}{2},1]}(\|y\|)\|_{p,q;\alpha_1,\ldots,\alpha_m} 
\]

(14)

Finally, by (12), (13) and (14) we obtain the result. \( \square \)

To prove the non-compactness of the embedding (3), we shall need the following assertion.

Lemma 7. Let \( \sigma \in (0, n) \), \( R \in (0, \frac{1}{2}) \) and let

\[
a_r \leq C \quad \text{for all } r \in (0, \frac{1}{4}) \text{ with some } C \in (0, \infty). \]

(15)

Moreover, let the function \( \mathcal{G} \) from (4)–(6) and the numbers \( a_r \) satisfy

\[
a_r \int_{2r}^{\frac{R}{2}} t^{\sigma - 1} \mathcal{G}(t) \, dt \to \infty \quad \text{as } \quad r \to 0_+.
\]

(16)

Then there exist \( \varepsilon = \varepsilon(\sigma) \in (0, \frac{1}{2}) \), \( r_1 = r_1(R) \in (0, \frac{R}{4}) \) and a positive constant \( c \) (independent of \( R \) and \( r_1 \)) such that for the functions \( u_r \) defined by (9), (8) and (7),

\[
\left| [u_r(x) - u_R(x)] - [u_r(0) - u_R(0)] \right| \geq c r a_r \int_{2r}^{\frac{R}{2}} t^{\sigma - 1} \mathcal{G}(t) \, dt
\]

(17)

for every \( r \in (0, r_1) \) and \( x = (\varepsilon r, 0, \ldots, 0) \in \mathbb{R}^n \).
Proof. The result immediately follows from Lemma 4.5 of [7].

Now, we are ready to prove the main result.

Proof of Theorem 2. We can suppose without loss of generality that

\[ B := \{ x \in \mathbb{R}^n; |x| \leq 1 \} \subset \Omega. \]  

(18)

Let \( r \in (0, \frac{1}{4}) \). Take \( \gamma < 0 \) and put

\[ G(t) = t^{\gamma_1 - \gamma} - n \prod_{j=1}^{m} \ell_j^{-\alpha_j}(t), \quad t \in (0,1], \quad \text{and} \quad a_r = r^{-\gamma}. \]

The function \( G \) satisfies (4)–(6). Thus, by Lemma 6,

\[ \| u_r \|_{\sigma,p,q;\alpha_1,\ldots,\alpha_m} \leq r^{-\gamma} \left[ \left( \int_{\frac{1}{2}}^1 \tau^{\gamma_1 - \gamma} - n \prod_{j=1}^{m} \ell_j^{-\alpha_j}(t) dt \right)^{\frac{1}{q}} + r^{\gamma} \right] \leq 1 \quad \text{for all} \quad r \in (0, r_0), \]  

(19)

where \( u_r \) are the functions given by (9). (Observe, that the assumptions \( \sigma \in (0, 1) \) and \( n \geq 2 \) yield \( p > n \sigma > n > \frac{n}{n-1} \).)

Taking \( R \in (0, \frac{1}{4}) \), we can see that the conditions (15) and (16) are satisfied and so, by Lemma 7, there exists \( \varepsilon \in (0, \frac{1}{2}) \) and \( r_1 \in (0, R) \) and a positive constant \( c \) (independent of \( R \) and \( r_1 \)) such that

\[ \left| u_r(x) - u_R(x) \right| - \left| u_r(0) - u_R(0) \right| \geq c r^{1-\gamma} \int_{2r}^{\frac{3}{2}} t^{1+\gamma - \frac{n}{p}} \prod_{j=1}^{m} \ell_j^{-\alpha_j}(t) dt \approx r^{\alpha - \frac{n}{p}} \prod_{j=1}^{m} \ell_j^{-\alpha_j}(r) = \lambda(r) \]

for every \( r \in (0, r_1) \) and \( x = (\varepsilon r, 0, \ldots, 0) \). Consequently, for any fixed \( R \in (0, \frac{1}{4}) \) and every sufficiently small positive \( r \),

\[ \| u_r - u_R \|_{C_0,\lambda(\cdot)(\Omega)} \geq \frac{\| u_r(x) - u_R(x) \| - \| u_r(0) - u_R(0) \|}{\lambda(\varepsilon r)} \geq c \frac{\lambda(r)}{\lambda(\varepsilon r)} \geq c_0, \]  

(20)

where \( c \) and \( c_0 \) are positive constants independent of \( R \) and \( r \).

Finally, consider the sequence of functions \( \{ u_{1/k} \}_{k=k_0}^{\infty} \) with \( k_0 \) sufficiently large. By (19), this sequence is bounded in \( H^p L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n) \) however, in view of (20), it has no Cauchy subsequence in \( C_0,\lambda(\cdot)(\Omega) \). Therefore, the embedding (3) is not compact.

To prove sharpness, suppose that there is a function \( \mu \in \mathcal{L} \) such that \( \frac{\xi}{\lambda} \in \mathcal{L} \) and \( H^p L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n) \hookrightarrow C_0,\mu(\cdot)(\Omega) \) for some nonempty domain \( \Omega \) in \( \mathbb{R}^n \). Take a ball \( B \subset \Omega \). Then

\[ H^p L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n) \hookrightarrow C_0,\mu(\cdot)(\overline{B}). \]  

(21)
Moreover, by Lemma 4.15 (iv) of [5], the condition \( \chi \in \mathcal{L} \) implies that
\[
C^{0,\mu(\cdot)}(\overline{B}) \hookrightarrow C^{0,\lambda(\cdot)}(\overline{B}).
\]
Combining this embedding with (21), we arrive at
\[
H^s L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n) \hookrightarrow C^{0,\lambda(\cdot)}(\overline{B}),
\]
which contradicts the non-compactness of the embedding (3) with \( \Omega = B \). \( \square \)

**References**


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