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Higher Order Teodorescu Operators and Cauchy–Pompeiu Type Formulas

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Abstract. Using the fundamental solution of the Helmholtz equation we construct the explicit form of the fundamental solution for powers of the factors of the Helmholtz operator in quaternionic analysis. These results lend assistance aid to investigate some properties of higher order Teodorescu operators. A fundamental solution can also be constructed for the product of Helmholtz operators. This is used to prove Cauchy–Pompeiu type representation formulas in quaternionic analysis for the general polynomial operator $\prod_{\nu=1}^{j}(D+\alpha_\nu)^{k_\nu}$, where $\alpha_\nu \neq \alpha_\mu$ if $\nu \neq \mu$.

Keywords: *Cauchy–Pompeiu representation, iterated Helmholtz equation, quaternionic analysis, Teodorescu operator*

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1. Introduction

The importance of complex analysis for mathematical physics is that the Cauchy–Riemann differential operator and its complex conjugate provide a factorization of the two-dimensional Laplace operator. The Cauchy–Riemann system can be written in complex form as

$$\frac{\partial W}{\partial \bar{z}} = 0, \tag{1.1}$$

where $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. Similarly $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ is used. The Laplace operator $\Delta = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$ arises from various applications in many partial differential equations in mathematical physics e.g., from the Poinsson, the wave and the heat equation. For equation (1.1), two over $\mathbb{R}$ linearly independent fundamental solutions

$$W_1 = 2\frac{\partial \ln|z|}{\partial z} = \frac{1}{z}, \qquad W_2 = 2i\frac{\partial \ln|z|}{\partial z} = \frac{i}{z}$$

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are connected with the kernel of Cauchy-type. In case of the inhomogeneous Cauchy–Riemann equation

$$\frac{\partial W}{\partial \bar{z}} = f,$$

they lead to the Cauchy–Pompeiu integral representation formulas. The area integral appearing in the complex Cauchy–Pompeiu representation defines a weakly singular integral operator T. Its properties were studied by I. N. Vekua [26]. We refer to Begehr [8, 9, 27] or Bojarski [12] for solving complex first order partial differential equations based on properties as well of T as of the strongly singular integral operator of Ahlfors-Beurling type Π (see, e.g., [1, 26]). Higher order Cauchy–Pompeiu representations were developed in [4, 5, 6, 7]. Then, by repeated applications of the T-operator, second order complex equations have been investigated by Begehr [3], Dzhuraev [13, 14] and a complex fourth order equation is studied by Wen and Kang [28]. In complex analysis, Begehr and Hile have used the idea of generalizing the T-operator in order to handle with the higher order differential equations. This generalization is realized in term of the $T_{m,n}$-operators and their properties (see [10, 11]) . The $T_{m,n}$-operators, in fact, are useful in the study of some boudary value problems for generalized polyanalytic functions of order n in the Sobolev space $W_p^1(\Omega)$ (see [22, 23]), or for complex elliptic partial differential equations of higher order (see [2]).

However, a lot of physical problems are not only particular circumstances in two dimensions. For example, an overwhelming majority of physically meaningful problems can not be reduced to two-dimensional models. But, the Dirac operator D generalizing the Cauchy–Riemann operator in higher dimensional spaces provides a factorization of the Laplace operator. Likewise the Helmholtz operator, $(\Delta + \alpha^2)$, $\alpha \in \mathbb{C}$, can be factorized in quaternionic analysis by a certain first order partial differential operator $D_\alpha := D + \alpha$ and $D_{-\alpha} := D - \alpha$ where $D := \sum_{k=1}^{3} e_k \frac{\partial}{\partial x_k}$. Powers of these operators and of the Helmholtz operator lead to model equations of higher order. Inspired by the above-mentioned results, we want to develop further these ideas for the Helmholtz operator and its factors in quaternionic analysis.

From the quaternionic form of the Stokes theorem Cauchy–Pompeiu representation formulas related to both factors of the Helmholtz operator are provided. By iteration they lead to a second order Cauchy–Pompeiu formula related to the Helmholtz equation. Further iterations lead to higher order Cauchy–Pompeiu representations related to powers of the factors of the Helmholtz operator and of the Helmholtz operator itself. These results were published in [19]. Following the above-mentioned techniques in complex analysis, these iterations result as well in fundamental solutions to the higher order operators as in defining higher order Pompeiu integral operators. These are denoted by integral operators $T_{\alpha,n}, T_{r,\alpha,n}$. In quaternionic analysis they are called higher order Teodorescu operators. The aim of the present paper is to investigate some properties of higher order Teodorescu operators. It demands to provide the fundamental solutions for the operator D_α^n with $n \in \mathbb{N}$ explicitly. We refer to [10, 18] for the Teodorescu transform in the case $\alpha = 0$. They are operators of Calderon–Zygmund type and do not cause problems. However, in the general cases $\alpha \neq 0$, the situation becomes more complicated. We can not immediately apply the theory of Calderon and Zygmund. How to overcome these difficulties in investigating the properties of $T_{\alpha,1}$ are shown in [16]. Here a list of properties of $T_{\alpha,n}$ are outlined. As the operators T, $T_{m,n}$ have been widely used to study various boundary value problems for higher order equations in complex analysis, the $T_{\alpha,n}$ should give useful tools in investigating similar problems which can be reduced to such problems and systems to those of the Helmholtz equation in quaternionic analysis.

We also generalize our results from [19] to representation formulas for solutions to the general inhomogeneous polynomial equation $\prod_{\nu=1}^{j}(D+\alpha_\nu)^{k_\nu} f = g$ in Ω with $\alpha_1, \alpha_2, \ldots, \alpha_j$ are mutually different complex constants. The main idea for obtaining these results is an essential process for constructing fundamental solutions from Sommen and Xu [25, 29]. These integral representations of Cauchy–Pompeiu type for the general inhomogeneous polynomial $\prod_{\nu=1}^{j}(D+\alpha_\nu)^{k_\nu}$ operator pave the way for investigating the boundary value problem of classical Vekua type.

2. Preliminaries

We begin with the definition of the algebra of quaternion. Let $\{e_0, e_1, e_2, e_3\}$ be an orthonormal basis of $\mathbb{R}^4$ such that $x \in \mathbb{R}^4$ is represented as $x = \sum_{k=0}^{3} x_k e_k$, $x_k \in \mathbb{R}$, $0 \leq k \leq 3$. The part $x_0 e_0 =: \mathrm{Sc}(x)$ is called the scalar part of x and $\vec{x} = \sum_{k=1}^{3} x_k e_k =: \mathrm{Vec}(x)$ the vector part of x. A product is defined in $\mathbb{R}^4$ which satisfies the conditions

(i) $e_1^2 = e_2^2 = e_3^2 = -1$,

(ii) $e_1 e_2 = -e_2 e_1 = e_3;\ e_2 e_3 = -e_3 e_2 = e_1;\ e_3 e_1 = -e_1 e_3 = e_2$.

The element e_0 is regarded as the usual unit, that is, $e_0 = 1$. For $x, y \in \mathbb{R}^4$ we define

$$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 \qquad \text{and} \qquad [\vec{x} \times \vec{y}] = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

Then, the algebraic rules (i), (ii) yield the quaternionic product

$$xy = x_0 y_0 - \langle \vec{x}, \vec{y} \rangle + x_0 \vec{y} + \vec{x} y_0 + [\vec{x} \times \vec{y}].$$

We are now prepared to give the definition of the algebra of real quaternions.

Definition 2.1. The tuple $(\mathbb{R}^4, \cdot)$ is called the *algebra of real quaternions*. We signify $(\mathbb{R}^4, \cdot)$ by $\mathbb{H}(\mathbb{R})$.

The quaternion $\bar{x} = x_0 - \vec{x}$ is called the conjugate to x. The number $|x|$ defined by $|x|^2 := x\bar{x}$ is named the absolute value of x. Note that $\vec{x}^2 = -|\vec{x}|^2$.

Definition 2.2. A *complex quaternion (biquaternions)* x is an object of the form $x = \sum_{k=0}^{3} x_k e_k,\ x_k \in \mathbb{C},\ 0 \le k \le 3$, with the commutation rule for the usual complex imaginary unit i with the quaternionic imaginary unit $e_k,\ k = 1, 2, 3$, $ie_k = e_k i$. The algebra of complex quaternions will be denoted by $\mathbb{H}(\mathbb{C})$.

Note that any $x \in \mathbb{H}(\mathbb{C})$ can be represented as $x = \mathrm{Re}x + i\mathrm{Im}x$, where $\mathrm{Re}x = \sum_{k=0}^{3} \mathrm{Re}x_k e_k$ and $\mathrm{Im}x = \sum_{k=0}^{3} \mathrm{Im}x_k e_k$ belong to $\mathbb{H}(\mathbb{R})$. Then the conjugate to x also belongs to $\mathbb{H}(\mathbb{C})$. It can be written as $\bar{x} = \overline{\mathrm{Re}x} + i\overline{\mathrm{Im}x}$. The norm $|x|_{\mathbb{H}(\mathbb{C})}$, where $x \in \mathbb{H}(\mathbb{C})$, is defined by

$$|x|_{\mathbb{H}(\mathbb{C})} := \sqrt{|x_0|^2 + |x_1|^2 + |x_2|^2 + |x_3|^2}, \tag{2.1}$$

where $x_k \in \mathbb{C},\ |x_k|^2 = x_k \bar{x}_k$, $\bar{x}_k$ stands for the usual complex conjugation. It is easily seen that (2.1) represents a natural Euclidean metric in $\mathbb{R}^8$ and can be expressed as $|x|^2_{\mathbb{H}(\mathbb{C})} = |\mathrm{Re}x|^2 + |\mathrm{Im}x|^2$.

Lemma 2.3. [21, Chap. 1, Lemma 2] *Let x and y be complex quaternions. Then* $|xy|_{\mathbb{H}(\mathbb{C})} \le \sqrt{2}\,|x|_{\mathbb{H}(\mathbb{C})}|y|_{\mathbb{H}(\mathbb{C})}$.

We next recall some basic facts of spaces of complex quaternion-valued functions. By the *isomorphic embedding* we can identify $(x_1, x_2, x_3) = \vec{x} \in \mathbb{R}^3$ with $x = \sum_{k=1}^{3} x_k e_k \in \mathbb{H}(\mathbb{R}) \subset \mathbb{H}(\mathbb{C})$.

We now consider functions f defined in a domain Ω of $\mathbb{R}^3$ with values in $\mathbb{H}(\mathbb{C})$. Those functions may be written as $f(x) = \sum_{k=0}^{3} f_k(x) e_k,\ f_k(x) \in \mathbb{C},\ x \in \Omega$. Properties such as continuity, differentiability, integralbility, and so on, which are described to f have to be possessed by all components $f_k(x)$ which are complex-valued functions defined on Ω. Let $\mathcal{B}(\Omega)$ be a function space of complex functions defined on Ω. For example, $\mathcal{B}$ may be $C^k,\ C^{(k,\varepsilon)},\ L_p,\ W_p^k$ and so on. We then define a function space

$$\mathcal{B}(\Omega, \mathbb{H}(\mathbb{C})) := \big\{ f : \Omega \to \mathbb{H}(\mathbb{C}) :\ \text{all components of } f \text{ belong to } B(\Omega) \big\}.$$

If $\mathcal{B}(\Omega)$ is normed with norm $\|\cdot\|_{\mathcal{B}}$, then we can define a norm on $\mathcal{B}(\Omega, \mathbb{H}(\mathbb{C}))$ by

$$\|f\|_{\mathcal{B}} = \left(\sum_{k=0}^{3} \|f_k\|^2_{\mathcal{B}} \right)^{\frac{1}{2}} \quad \text{for } f \in \mathcal{B}(\Omega, \mathbb{H}(\mathbb{C})).$$

If $\mathcal{B}(\Omega)$ is a Banach space, then the space $\mathcal{B}(\Omega, \mathbb{H}(\mathbb{C}))$ defined in this manner is also a complex Banach space. In this way the usual Banach spaces of these functions are denoted by $C^k(\Omega, \mathbb{H}(\mathbb{C})),\ C^{(k,\varepsilon)}(\Omega, \mathbb{H}(\mathbb{C})),\ L_p(\Omega, \mathbb{H}(\mathbb{C})), W_p^k(\Omega, \mathbb{H}(\mathbb{C}))$ and so on.

Next, we will introduce some basic notations of the Sobolev spaces [24, Section VI]) used in our discussions. Let $\tilde{L}_p(\Omega, \mathbb{H}(\mathbb{C}))$ be the set of all continuous functions $f : \Omega \to \mathbb{H}(\mathbb{C})$ for which $\|f\|_{L_p} := \big(\int_\Omega |f(x)|^p_{\mathbb{H}(\mathbb{C})} \, dx \big)^{\frac{1}{p}}$ is finite with $p \in (1, +\infty)$. We will need the following Hölder's inequality for functions with value in $\mathbb{H}(\mathbb{C})$: Let $f \in \tilde{L}_p(\Omega, \mathbb{H}(\mathbb{C}))$, $g \in \tilde{L}_q(\Omega, \mathbb{H}(\mathbb{C}))$, where $1 < p, q < +\infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$fg \in \tilde{L}_1(\Omega, \mathbb{H}(\mathbb{C})) \quad \text{and} \quad \|fg\|_{L_1} \le \|f\|_{L_p} \|g\|_{L_q}.$$

Remark 2.4. Let k be a non-negative integer and let $1 \le p \le +\infty$. From Sobolev's imbedding theorem ([24, Chapter 6, §6.4]) more general imbedding theorems for Sobolev spaces W_p^k can be established. It is shown that

$$\begin{aligned} &W_p^k(\Omega, \mathbb{H}(\mathbb{C})) \subset L_{\frac{3p}{3-kp}}(\Omega, \mathbb{H}(\mathbb{C})) \quad \text{for } kp < 3 \\ &W_p^k(\Omega, \mathbb{H}(\mathbb{C})) \subset C_b(\Omega, \mathbb{H}(\mathbb{C})) \qquad \text{for } kp > 3, \end{aligned} \tag{2.2}$$

where $C_b(\Omega, \mathbb{H}(\mathbb{C}))$ is the set of all bounded continuous functions on $\bar{\Omega}$.

Let us review the concept of the Moisil–Teodorescu differential operator and the quaternionic Stokes formula which will be used throughout this paper. Let $f \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$. The *Moisil-Teodorescu differential operator* is given by $Df := \sum_{k=1}^3 e_k \partial_k f$ where $\partial_k := \frac{\partial}{\partial x_k}$. If we write $f = f_0 + \vec{f}$, then one gets by a straightforward calculation

$$Df = -\text{div}\vec{f} + \text{grad} f_0 + \text{rot}\vec{f}. \tag{2.3}$$

Note that the Moisil-Teodorescu operator was introduced as acting from the left-hand side. The correponding operator acting from the right-hand side will be denoted by D_r. That is $D_r f = \sum_{k=1}^3 \partial_k f e_k$ and in vector form the application of D_r can be represented as

$$D_r f = -\text{div}\vec{f} + \text{grad} f_0 - \text{rot}\vec{f}. \tag{2.4}$$

Let the operator $D_\alpha = D + \alpha I$ be given, where α is an arbitrary complex constant and I is the identity operator. As the Laplacian also the Helmholtz operator can be factorized in quaternionic analysis as

$$\Delta + \alpha^2 = -D_\alpha D_{-\alpha} = -D_{-\alpha} D_\alpha. \tag{2.5}$$

The following quaternionic Stokes formula is taken from [21, Theorem 2], (see also [18, Proposition 3.22]). Let f and g belong to $C^1(\Omega, \mathbb{H}(\mathbb{C})) \cap C(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$, $\Omega \subset \mathbb{R}^3$ a regular domain, $\Gamma := \partial\Omega$. Then

$$\int_\Omega \big[(D_r f(y)) g(y) + f(y) D g(y)\big] \, dy = \int_\Gamma f(y) \vec{n}(y) g(y) \, d\Gamma_y, \tag{2.6}$$

where $\vec{n} := \sum_{k=1}^3 n_k e_k$ denotes the outward unitary normal vector on Γ.

Remark 2.5. From (2.6) with $\alpha \in \mathbb{C}$ we have

$$\begin{aligned} \int_\Omega \big[(D_{r,-\alpha} f(y))g(y) + f(y)(D_\alpha g(y))\big]\, dy &= \int_\Gamma f(y)\vec{n}(y)g(y)\ d\Gamma_y \\ \int_\Omega \big[(D_{r,\alpha} f(y))g(y) + f(y)(D_{-\alpha} g(y))\big]\, dy &= \int_\Gamma f(y)\vec{n}(y)g(y)\, d\Gamma_y. \end{aligned} \tag{2.7}$$

The above equalities are powerful tools in the strategy of this work.

3. Higher order Teodorescu operators

In complex analysis, a special case of the Cauchy-Pompeiu formula is the Cauchy representation of analytic functions which is deduced from the Gauss theorem. Analogously to this, the Cauchy-Pompeiu integral representation in quaternionic analysis which is also a consequence of the quaternionic Stokes formula.

3.1. The quaternionic Cauchy–Pompeiu type formulas. Using the equality (2.5) and the fundamental solution of the Helmholtz equation, a fundamental solution for the factors of the Helmholtz operator can be constructed. Indeed, if we assume that ϑ is a fundamental solution of the Helmholtz operator, i.e., a function satisfying $(\Delta + \alpha^2)\vartheta(x) = \delta(x)$, where $\delta(x)$ is the Dirac delta distribution, then $K_\alpha(x) = -(D-\alpha)\vartheta(x)$ is a fundamental solution of D_α and $K_{-\alpha}(x) = -(D+\alpha)\vartheta(x)$ is a fundamental solution of $D_{-\alpha}$, i.e., $D_\alpha K_\alpha(x) = \delta(x)$ and $D_{-\alpha}K_{-\alpha}(x) = \delta(x)$.

As discussed in [21, p. 27] a unique fundamental solution to the Helmholtz operator related to its physical meaning is $\vartheta(x) = -\frac{e^{i\alpha|x|}}{4\pi|x|}$. Since $\vartheta(x)$ is a scalar function and using formulas (2.3), (2.4) we have $D_\alpha\vartheta(x) = D_{r,\alpha}\vartheta(x)$. From formula (2.3) by a straightforward computation we get

$$\begin{aligned} K_\alpha(x) &= -\operatorname{grad}\vartheta(x) + \alpha\vartheta(x) = \Big(\alpha + \frac{x}{|x|^2} - i\alpha\frac{x}{|x|}\Big)\Big(-\frac{e^{i\alpha|x|}}{4\pi|x|}\Big) \\ K_{-\alpha}(x) &= -\operatorname{grad}\vartheta(x) - \alpha\vartheta(x) = \Big(-\alpha + \frac{x}{|x|^2} - i\alpha\frac{x}{|x|}\Big)\Big(-\frac{e^{i\alpha|x|}}{4\pi|x|}\Big) \end{aligned}$$

We introduce here the quaternionic Cauchy–Pompeiu formulas which are related to the factors of the Helmholtz operator:

$$f(x) = -\int_\Gamma K_\alpha(x-y)\vec{n}(y)f(y)\, d\Gamma_y + \int_\Omega K_\alpha(x-y)D_\alpha f(y)\, dy \tag{3.1}$$

$$f(x) = -\int_\Gamma K_{-\alpha}(x-y)\vec{n}(y)f(y)\, d\Gamma_y + \int_\Omega K_{-\alpha}(x-y)D_{-\alpha} f(y)\, dy \tag{3.2}$$

for $f \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \cap C(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$. The formulas (3.1), (3.2) express a differentiable function through its boundary values and its first-order derivatives.

3.2. Higher order Teodorescu operators. The fundamental solution for the operator D_α^n with $n \in \mathbb{N}$ will be constructed by a method as given in [29, Chap. 4], (see also [19, 25]). The advantage of our method, using induction, is that it yields explicit kernel functions. For the reader's convenience, we will present this lemma and our result about the Cauchy–Pompeiu integral for the higher order D_α^n operator (see [19]).

Lemma 3.1. [19, Lemma 2.5] *Let $K_\alpha(x)$ be a fundamental solution for the operator D_α, i.e, a quaternionic function satisfying $D_\alpha K_\alpha(x) = \delta(x)\,(\alpha \neq 0)$ in distributional sense, and $K_\alpha(x)$ be infinitely often differentiable with respect to α. Then the functions $K_\alpha^{(n)}(x)$, $n \in \mathbb{N}$, determined by the recurrence fomulas*

$$K_\alpha^{(1)}(x) = K_\alpha(x), \quad K_\alpha^{(k)}(x) = \frac{-1}{k-1}\frac{\partial}{\partial\alpha}K_\alpha^{(k-1)}(x) \tag{3.3}$$

for all $k \in \mathbb{N}^*$, satisfy in distributional sense the equations

$$(D+\alpha)^n K_\alpha^{(n)}(x) = \delta(x).$$

We will recall our main result in [19] for later use.

Theorem 3.2. [19, Theorem 3.2] *Let $f \in C^n(\Omega, \mathbb{H}(\mathbb{C})) \cap C^{n-1}(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$. Then*

$$\begin{aligned} f(x) = &-\sum_{k=1}^{n}\int_\Gamma K_\alpha^{(k)}(x-y)\vec{n}(y)D_{\alpha,y}^{k-1}f(y)\,d\Gamma_y \\ &+\int_\Omega K_\alpha^{(n)}(x-y)D_{\alpha,y}^n f(y)\,dy. \end{aligned} \tag{3.4}$$

Looking at the kernel function $K_\alpha^{(1)}(x)$ we decompose it in the following way:

$$K_\alpha^{(1)}(x) = \left(\alpha - i\alpha\frac{x}{|x|}\right)\left(-\frac{e^{i\alpha|x|}}{4\pi|x|}\right) + \frac{x}{|x|^2}\left(-\frac{e^{i\alpha|x|}}{4\pi|x|}\right).$$

By a straightforward calculation in applying equalities (3.3) in the above lemma and using induction the following corollary is proved.

Corollary 3.3. *Let α be a complex constant with $\alpha \neq 0$. Then the function*

$$K_\alpha^{(n)}(x) = \frac{(-1)^{n-1}}{(n-1)!}\Big[(n-1)-(n-2)\frac{ix}{|x|}+i\alpha|x|+\alpha x\Big](i|x|)^{n-2}\left(-\frac{e^{i\alpha|x|}}{4\pi|x|}\right) \tag{3.5}$$

is a fundamental solution of the operator D_α^n.

The last term in equality (3.4) suggests the following definition of higher order Teodorescu operators.

Definition 3.4. For a bounded domain Ω in $\mathbb{R}^3$ with piecewise sufficiently smooth boundary Γ, we formally define operators $T_{\alpha,n}, T_{r,\alpha,n}$, where $\alpha \in \mathbb{C}$ acting on $\mathbb{H}(\mathbb{C})$-valued functions f defined in Ω, according to

$$(T_{\alpha,n}f)(x) := \int_\Omega K_\alpha^{(n)}(x-y)f(y)\,dy$$
$$(T_{r,\alpha,n}f)(x) := \int_\Omega f(y)K_\alpha^{(n)}(x-y)\,dy,$$

where, the kernel functions $K_\alpha^{(n)}$ are defined by formula (3.5). The operators $T_{\alpha,n},\ T_{r,\alpha,n}$ are called *higher order Teodorescu operators.*

We now begin with investigating mapping properties of $T_{\alpha,n}$.

3.3. Existence and continuity of integrals. In this subsection, we will prove the existence and continuity of $T_{\alpha,n}$. The operator $T_{r,\alpha,n}$ has analogous properties, where this operator is acting from the right on the function. We also refer the readers to [15, 17] for more details in the discussion of some properties of the integral operator $T_{\alpha,1}$ with a real number α, where $T_{\alpha,1}$ acts on real quaternion-valued functions. The use of complex quaternions as well as α a complex number does not cause changes of the mapping properties of $T_{\alpha,1}$, as was shown in [16]. Moreover, the kernel $K_\alpha^{(n)}(x)$ of the operator $T_{\alpha,n}$ has a singularity of order 2 at most, and thus it will not affect essentially the properties induced by $K_\alpha^{(1)}(x)$. Nevertheless, the following properties will be provided more explicitly again for $T_{\alpha,n}, n \geq 1$.

Lemma 3.5. *Under the same assumptions as in Definition* 3.4*, for* $f \in L_1(\Omega, \mathbb{H}(\mathbb{C}))$*, the integral* $F(x) = \int_\Omega \frac{1}{|x-y|^{3-\gamma}} f(y)\,dy$ *is in* $L_1(\Omega, \mathbb{H}(\mathbb{C}))$ *for all* $0 < \gamma \in \mathbb{R}$.

Proof. Notice that here we consider a bounded domain Ω and $\gamma > 0$. Firstly, looking at the integral $\int_\Omega \big(\int_\Omega \frac{1}{|x-y|^{3-\gamma}}\,dx\big)|f(y)|_{\mathbb{H}(\mathbb{C})}\,dy$, L. Hedberg has shown that there exists a constant C such that $\int_\Omega \frac{1}{|x-y|^{3-\gamma}}\,dy \leq C(\mathrm{diam}B)^\gamma$ holds, where B is the smallest cube containing Ω (see [20]). Hence,

$$\int_\Omega \left(\int_\Omega \frac{1}{|x-y|^{3-\gamma}}\right)|f(y)|_{\mathbb{H}(\mathbb{C})}\,dy \leq M_{(\Omega,\gamma)} \int_\Omega |f(y)|_{\mathbb{H}(\mathbb{C})}\,dy = M_{(\Omega,\gamma)}\|f\|_{L_1},$$

where $M_{(\Omega,\gamma)}$ is a constant depending on Ω and on γ. Using Fubini's theorem, it follows

$$\begin{aligned}\int_\Omega \left(\int_\Omega \frac{1}{|x-y|^{3-\gamma}}\,dx\right)|f(y)|_{\mathbb{H}(\mathbb{C})}\,dy &= \int_\Omega \left(\int_\Omega \frac{1}{|x-y|^{3-\gamma}}|f(y)|_{\mathbb{H}(\mathbb{C})}\,dy\right)dx \\ &= \int_\Omega F(x)\,dx\,,\end{aligned}$$

where the involved integrals are finite and hence $F(x)$ is in $L_1(\Omega, \mathbb{H}(\mathbb{C}))$. ∎

We now come to our main result ensuring the well-definedness of the operator $T_{\alpha,n}$ from $L_1(\Omega,\mathbb{H}(\mathbb{C}))$ into $L_q(\Omega,\mathbb{H}(\mathbb{C}))$ for some relevant $q \geq 1$.

Theorem 3.6. *For $\alpha \in \mathbb{C}$ and $f \in L_1(\Omega,\mathbb{H}(\mathbb{C}))$, the integral $T_{\alpha,n}f(x)$ exists for almost all $x \in \Omega$.*

Proof. For $n = 1$, $f \in L_1(\Omega,\mathbb{H}(\mathbb{C}))$, and viewing the formula for $K_\alpha^{(1)}(x-y)$ we observe that

$$\begin{aligned}\left|K_\alpha^{(1)}(x-y)f(y)\right|_{\mathbb{H}(\mathbb{C})} &\leq \frac{\sqrt{2}}{4\pi}\left|K_\alpha^{(1)}(x-y)\right|_{\mathbb{H}(\mathbb{C})}\left|f(y)\right|_{\mathbb{H}(\mathbb{C})}\\ &\leq \frac{\sqrt{2}}{4\pi}\,e^{-\mathrm{Im}\alpha\ \mathrm{diam}\Omega}|f(y)|_{\mathbb{H}(\mathbb{C})}\left(2|\alpha|\frac{1}{|x-y|}+\frac{1}{|x-y|^2}\right).\end{aligned}$$

Using Lemma 3.5 leads to the existence of $(T_{\alpha,1}f)(x)$ for almost all $x \in \bar{\Omega}$.

In the case $n = 2$, let us consider the estimate

$$\begin{aligned}\left|K_\alpha^{(2)}(x-y)f(y)\right|_{\mathbb{H}(\mathbb{C})} &\leq \sqrt{2}\,|K_\alpha^{(2)}(x-y)|_{\mathbb{H}(\mathbb{C})}|f(y)|_{\mathbb{H}(\mathbb{C})}\\ &\leq \frac{\sqrt{2}}{4\pi}\,e^{-\mathrm{Im}\alpha\ \mathrm{diam}\Omega}|f(y)|_{\mathbb{H}(\mathbb{C})}\left(2|\alpha|+\frac{1}{|x-y|}\right).\end{aligned}$$

Using Lemma 3.5 again in the case $\gamma \geq 2$, the existence of $(T_{\alpha,2}f)(x)$ for almost all $x \in \bar{\Omega}$ is proved.

If $n \geq 3$ is fixed, we have

$$\begin{aligned}\left|K_\alpha^{(n)}(x-y)f(y)\right|_{\mathbb{H}(\mathbb{C})} &\leq \sqrt{2}\,|K_\alpha^{(n)}(x-y)|_{\mathbb{H}(\mathbb{C})}|f(y)|_{\mathbb{H}(\mathbb{C})}\\ &\leq \frac{\sqrt{2}}{4\pi}\,e^{-\mathrm{Im}\alpha\ \mathrm{diam}\Omega}|f(y)|_{\mathbb{H}(\mathbb{C})}\left(2|\alpha||x-y|^{n-2}+|x-y|^{n-3}\right).\end{aligned}$$

Looking at the right-hand side of this inequality, we can easily see that $T_{\alpha,n}f(x)$ for $n \geq 3$ has no singularity. Hence, the existence of the integrals $T_{\alpha,n}f$ follows. ∎

Theorem 3.7. *Let the assumptions of Definition* 3.4 *be satisfied. In addition, let f be a complex quaternion-valued function in $L_1(\Omega,\mathbb{H}(\mathbb{C}))$. Then the integral $T_{\alpha,n}f(x)$ converges absolutely for all x in Ω. Moreover, if*

(i) $1 \leq q < \frac{3}{2}$, *when* $n = 1$
(ii) $1 \leq q < 3$, *when* $n = 2$
(iii) $1 \leq q \leq +\infty$, *when* $n \geq 3$,

then $T_{\alpha,n}f \in L_q(\Omega,\mathbb{H}(\mathbb{C}))$ with $\|T_{\alpha,n}f\|_{L_q} \leq M_{(\Omega,\alpha,n)}\|f\|_{L_1}$.

Proof. Firstly, we will define $W_\gamma(x)$ on $\bar{\Omega}$ according to $W_\gamma(x) := \int_\Omega \frac{|\omega(y)|}{|x-y|^\gamma}\,dy$, and let ω be an arbitrary function in $L_p(\Omega, \mathbb{H}(\mathbb{C}))$ where $\frac{1}{p} + \frac{1}{q} = 1$. Again the estimate of $|K_\alpha^{(n)}(x-y)|$ in Theorem 3.6 shows that the integral $T_{\alpha,n}f(x)$ converges absolutely. Now, using Hölder's inequality we obtain

$$\begin{aligned} W_\gamma(x) := \int_\Omega \frac{|\omega(y)|}{|x-y|^\gamma}\,dy &\le \left(\int_\Omega \left(\frac{1}{|x-y|^\gamma}\right)^q dy\right)^{\frac{1}{q}} \|\omega\|_{L_p} \\ &\le \left(\int_\Omega \frac{1}{|x-y|^\gamma}\,dy\right) \|\omega\|_{L_p}\,. \end{aligned}$$

By Lemma 3.5, hence the middle integral of this inequality exists for $q\gamma < 3$. Therefore, in the case of $n = 1$ the condition $1 \le q < \frac{3}{2}$ is sufficient for both values $\gamma = 1$ or $\gamma = 2$ (see the estimates in Theorem 3.6). For $n = 2$, by the estimate of $K_\alpha^{(2)}(x-y)$, we have to consider $\gamma = 1$. Thus the condition is $1 \le q < 3$. It is easily seen that $1 \le q \le +\infty$ is possible for all $n \ge 3$.

The assumtions for q together with $W_\gamma(x) \le \left(\int_\Omega \frac{1}{|x-y|^\gamma}\,dy\right)\|\omega\|_{L_p}$ yield $W_\gamma(x) \le M_{(\Omega,\gamma)}\|\omega\|_{L_p}$, where $M_{(\Omega,\gamma)}$ is a constant depending on Ω and γ but not on x (see the proof of Lemma 3.5). Hence, $W_\gamma(x)$ converges uniformly, i.e, W_γ is continuous on $\bar{\Omega}$ and for all $\omega \in L_p(\Omega, \mathbb{H}(\mathbb{C}))$. Then $W_\gamma(x) \in L_q(\Omega, \mathbb{R})$ for $\frac{1}{p} + \frac{1}{q} = 1$. Again, by Fubini's theorem we have

$$\int_\Omega \left(\int_\Omega \frac{|\omega(y)|}{|x-y|^\gamma}\,dy\right)|v(x)|_{\mathbb{H}(\mathbb{C})}\,dx = \int_\Omega \left(\int_\Omega \frac{|v(x)|}{|x-y|^\gamma}\,dx\right)|\omega(y)|_{\mathbb{H}(\mathbb{C})}\,dy\,,$$

where $\omega \in L_p(\Omega, \mathbb{H}(\mathbb{C}), v \in L_1(\Omega, \mathbb{H}(\mathbb{C})$. This is due to the fact that the latter integral represents a linear functional on $L_p(\Omega, \mathbb{R})$ gives $\left(\int_\Omega \frac{v(x)}{|x-y|^\gamma}\,dx\right) \in L_q(\Omega, \mathbb{R})$. Therefore, by the estimates in the above theorem this leads to $T_{\alpha,n}f \in L_q(\Omega, \mathbb{H}(\mathbb{C}))$ for every $f \in L_1(\Omega, \mathbb{H}(\mathbb{C})$.

Finally, using the same ideas gives explicit estimates of $\|T_{\alpha,n}\|_{L_1 \to L_q}$. Indeed, firstly notice that as well the function

$$\tilde{T}_{\alpha,n}f(x) = \int_\Omega \left|K_\alpha^{(n)}(x-y)\right|_{\mathbb{H}(\mathbb{C})}|f(y)|_{\mathbb{H}(\mathbb{C})}\,dy$$

is defined already, where $f \in L_1(\Omega, \mathbb{H}(\mathbb{C}))$, as

$$\begin{aligned} &\int_\Omega \left(\int_\Omega \left|K_\alpha^{(n)}(x-y)\right|_{\mathbb{H}(\mathbb{C})}|f(y)|_{\mathbb{H}(\mathbb{C})}\,dy\right)|v(x)|_{\mathbb{H}(\mathbb{C})}\,dx \\ &\qquad = \int_\Omega \left(\int_\Omega \left|K_\alpha^{(n)}(x-y)\right|_{\mathbb{H}(\mathbb{C})}|v(x)|_{\mathbb{H}(\mathbb{C})}\,dx\right)|f(y)|_{\mathbb{H}(\mathbb{C})}\,dy, \end{aligned}$$

where $v \in L_p(\Omega, \mathbb{H}(\mathbb{C}))$.

In the next step we consider $\left(\int_\Omega |K_\alpha^{(n)}(x-y)|_{\mathbb{H}(\mathbb{C})}|v(x)|_{\mathbb{H}(\mathbb{C})}\,dx\right)$. In the cases listed under conditions (i) – (iii), when $q=+\infty$ we have $p=1$. We then may apply Lemma 3.5 to a bounded domain large enough to contain $\bar{\Omega}$, and deduce that $\int_\Omega \left|K_\alpha^{(n)}(x-y)\right|_{\mathbb{H}(\mathbb{C})}|v(x)|_{\mathbb{H}(\mathbb{C})}\,dx \leq M_{(\alpha,\Omega,n)}\|v\|_{L_p}$. In the cases $1\leq q<+\infty$ and $1<p\leq+\infty$, with the list of conditions (i) – (iii) by Lemma 3.5 together with the estimates of $K_\alpha^{(n)}(x-y)$ in Theorem 3.6 we get

$$\int_\Omega \left|K_\alpha^{(n)}(x-y)\right|_{\mathbb{H}(\mathbb{C})}|v(x)|_{\mathbb{H}(\mathbb{C})}\,dx \leq \sup_{x\in\bar{\Omega}}\left(\int_\Omega \left|K_\alpha^{(n)}(x-y)\right|^q dx\right)^{\frac{1}{q}}\|v\|_{L_p}.$$

This leads to

$$\int_\Omega\left(\int_\Omega \left|K_\alpha^{(n)}(x-y)\right|_{\mathbb{H}(\mathbb{C})}|f(y)|_{\mathbb{H}(\mathbb{C})}\,dy\right)|v(x)|_{\mathbb{H}(\mathbb{C})}\,dx \leq M_{(\Omega,\alpha,n)}\|f\|_{L_1}\|v\|_{L_p}\,,$$

which completes the proof of the theorem. ∎

3.4. Differentiability of integrals. If we look for applications of the $T_{\alpha,n}$-operators, then we need their mapping properties within Sobolev spaces. To this purpose, in this subsection we will investigate differentiability of higher Teodorescu transforms. The Teodorescu transform in the case $\alpha=0$ do not cause many problems (see [10, 15], [18, Chap. 4] and references therein) because they are operators of Calderon–Zygmund type. However, in the cases $\alpha\neq 0$, the situation becomes more complicated. We can not immediately apply the theory of Calderon and Zygmund. This means that we have to give the estimate of the kernels of the higher Teodorescu operators in order to be able to use these theories. How to overcome these diffculties in the case $n=1$ was shown in [16]. The kernel $K_\alpha^{(n)}(x)$ of the operator $T_{\alpha,n}$ has a singularity of order 2 at most, and thus it will not affect essentially the properties induced by $K_\alpha^{(1)}(x)$. Therefore, the following properties are still true for $T_{\alpha,n},\ n\geq 1$.

Theorem 3.8. *Let $f\in C_c^1(\Omega,\mathbb{H}(\mathbb{C}))$, then*

(i) *for $k=1,2,3$,*

$$\partial_k(T_{\alpha,1}f)(x) = \int_\Omega \left[\partial_{k,x}K_\alpha^{(1)}(x-y)\right]f(y)\,dy + \bar{e}_k\frac{f(x)}{3}$$
$$\partial_k(T_{r,\alpha,1}f)(x) = \int_\Omega f(y)\left[\partial_{k,x}K_\alpha^{(1)}(x-y)\right]dy + \frac{f(x)}{3}\bar{e}_k,$$

(ii) *for $k=1,2,3$ and for all $n\geq 2$*

$$\partial_k(T_{\alpha,n}f)(x) = \int_\Omega \left[\partial_{k,x}K_\alpha^{(n)}(x-y)\right]f(y)\,dy$$
$$\partial_k(T_{r,\alpha,n}f)(x) = \int_\Omega f(y)\left[\partial_{k,x}K_\alpha^{(n)}(x-y)\right]dy.$$

Proof. (i) Its proof can be found in [16].

(ii) In the case $n = 2$, we have

$$T_{\alpha,2}f(x) = \int_\Omega \big(1 + i\alpha|x-y| + \alpha(x-y)\big)\,\frac{e^{i\alpha|x-y|}}{4\pi|x-y|}\,f(y)\,dy.$$

The kernel of this integral has a singularity of order 1, hence it can be allowed to exchange the differentiation and the integration. Therefore, (ii) holds for $n = 2$. For $n \geq 3$, the kernels of these integrals have no singularities, thus (ii) is easily seen. ∎

The following theorem can be proved by using the ideas in [16, Theorem 3.3] together with the above theorem and analogous estimations as in Theorem 3.6 for the kernel functions $\partial_k K_{\alpha,n}(x-y), n \geq 1$.

Theorem 3.9. *The operators* $\partial_k T_{\alpha,n} : L_p(\Omega, \mathbb{H}(\mathbb{C})) \to L_p(\Omega, \mathbb{H}(\mathbb{C}))$ *are well-defined and continuous for all* $n \geq 1$ *and* $1 \leq k \leq 3$.

In the following section, we now are able to come to our main result continuing the mapping properties of $T_{\alpha,n}$ between spaces of continuous functions.

3.5. Mapping properties of $T_{\alpha,n}$. In this section we will give the most important properties of $T_{\alpha,n}$ in Theorem 3.12 and Remark 3.13.

Theorem 3.10. *The operator* $T_{\alpha,n} : L_p(\Omega, \mathbb{H}(\mathbb{C})) \to W^1_p(\Omega, \mathbb{H}(\mathbb{C}))$ *is well-defined and continuous for all* $n \geq 1$.

Proof. From Theorem 3.9 we see that $\partial_k T_{\alpha,n} : L_p(\Omega, \mathbb{H}(\mathbb{C})) \to L_p(\Omega, \mathbb{H}(\mathbb{C}))$ is well-defined and continuous for $n \geq 1$. In order to show $T_{\alpha,n}f \in W^1_p(\Omega, \mathbb{H}(\mathbb{C}))$ for every $f \in L_p(\Omega, \mathbb{H}(\mathbb{C}))$, we at first verify that the operator $T_{\alpha,n}$ acts invariantly on $L_p(\Omega, \mathbb{H}(\mathbb{C}))$.

For the case $n = 1$, its proof can be found in [16]. In the case $n = 2$, if $1 \leq q \leq 3$ we have $T_{\alpha,2} \in \mathcal{L}(L_1(\Omega, \mathbb{H}(\mathbb{C})), W^1_q(\Omega, \mathbb{H}(\mathbb{C})))$. Consequently, we can say that $T_{\alpha,2} \in \mathcal{L}(L_q(\Omega, \mathbb{H}(\mathbb{C})), W^1_q(\Omega, \mathbb{H}(\mathbb{C})))$ for $1 \leq q \leq 3$. Using Sobolev's imbedding theorems (see inclusions (2.2)) leads to $T_{\alpha,2} \in \mathcal{L}(L_s(\Omega, \mathbb{H}(\mathbb{C})), L_s(\Omega, \mathbb{H}(\mathbb{C})))$ for $1 \leq s \leq +\infty$. The assertion of this theorem follows immediately from Theorem 3.7 for every $n \geq 3$. ∎

Theorem 3.11. *When* Ω *is a bounded domain in* $\mathbb{R}^3$, *then* $T_{\alpha,n} : C(\bar{\Omega}, \mathbb{H}(\mathbb{C})) \to C(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$ *is bounded and*

(i) $\|T_{\alpha,1}\|_{\mathcal{L}(C(\bar{\Omega}),C(\bar{\Omega}))} \leq \frac{\sqrt{2}}{4\pi} e^{-\operatorname{Im}\alpha\,\operatorname{diam}\Omega} \max\limits_{x\in\Omega} \left\{ \int_\Omega \big(2|\alpha|\frac{1}{|x-y|} + \frac{1}{|x-y|^2}\big)\,dy \right\}$

(ii) $\|T_{\alpha,2}\|_{\mathcal{L}(C(\bar{\Omega}),C(\bar{\Omega}))} \leq \frac{\sqrt{2}}{4\pi} e^{-\operatorname{Im}\alpha\,\operatorname{diam}\Omega} \max\limits_{x\in\Omega} \left\{ \int_\Omega \big(2|\alpha| + \frac{1}{|x-y|}\big)\,dy \right\}$

(iii) $\|T_{\alpha,n}\|_{\mathcal{L}(C(\bar{\Omega}),C(\bar{\Omega}))} \le \frac{\sqrt{2}}{4\pi} e^{-\operatorname{Im}\alpha \operatorname{diam}\Omega} \max_{x\in\Omega} \left\{ \int_\Omega \big(2|\alpha||x-y|^{n-2}+|x-y|^{n-3}\big)\, dy \right\}$ *for every* $n \ge 3$.

Proof. Let $f \in C(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$, and note that $C(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$ is dense in $L_1(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$. For $p > 3$, by the remarks preceding the theorem, together with Theorem 3.9 we have $f \in L_p(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$ and $T_\alpha f \in C_b(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$. With an arbitrarily fixed $x \in \bar{\Omega}$ we get the estimates

$$\begin{aligned}
\|T_{\alpha,1} f(x)\|_{\mathbb{H}(\mathbb{C})} &\le \sqrt{2} \int_\Omega |K_\alpha^{(1)}(x-y)| |f(y)|\, dy \\
&\le \frac{\sqrt{2}}{4\pi} \|f\|_{C(\bar{\Omega})} e^{-\operatorname{Im}\alpha \max_y\{|x-y|\}} \int_\Omega \left(2|\alpha| \frac{1}{|x-y|} + \frac{1}{|x-y|^2} \right) dy \\
\|T_{\alpha,2} f(x)\|_{\mathbb{H}(\mathbb{C})} &\le \sqrt{2} \int_\Omega |K_\alpha^{(2)}(x-y)| |f(y)|\, dy \\
&\le \frac{\sqrt{2}}{4\pi} \|f\|_{C(\bar{\Omega})} e^{-\operatorname{Im}\alpha \max_y\{|x-y|\}} \int_\Omega \left(2|\alpha| + \frac{1}{|x-y|} \right) dy,
\end{aligned}$$

and for $n \ge 3$,

$$\begin{aligned}
\|T_{\alpha,n} f(x)\|_{\mathbb{H}(\mathbb{C})} &\le \sqrt{2} \int_\Omega |K_\alpha^{(2)}(x-y)| |f(y)| dy \\
&\le \frac{\sqrt{2}}{4\pi} \|f\|_{C(\bar{\Omega})} e^{-\operatorname{Im}\alpha \max_y\{|x-y|\}} \int_\Omega \big(2|\alpha||x-y|^{n-2} + |x-y|^{n-3} \big) dy.
\end{aligned}$$

Taking the maximum with respect to $x \in \overline{\Omega}$, the norm inequalities (i) – (iii) hold. ∎

Theorem 3.12. *The following assertions hold:*

- **(i)** *The operator* $T_{\alpha,1}$ *is the algebraic right-inverse to the operator* D_α*, i.e., for any* $f \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \cap C(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$*, we have* $D_\alpha T_{\alpha,1} f(x) = f(x)$ *for every* $x \in \Omega$.
- **(ii)** $D_\alpha T_{\alpha,n} f(x) = T_{\alpha,n-1} f(x)$ *for every* $x \in \Omega, n \ge 2$.
- **(iii)** $D_\alpha^n T_{\alpha,n} f(x) = f(x)$ *for every* $x \in \Omega, n \ge 1$.

Proof. (i) In Theorem 3.8 it has been shown that $\partial T_{\alpha,1}$ is a strongly singular integral operator which has a singularity of order 3. Thus, (i) is proved by two techniques which can be found in [30, Theorem 2.6].

(ii) By Theorem 3.8, the operator $D_{\alpha,x}$ acting on $T_{\alpha,n} f(x)$ can be interchanged with integration for any $n \ge 2$ as in these cases the singularity at $y = x$ of the kernels $D_{\alpha,x} K_\alpha^{(n)}(x-y), n \ge 2$, is not worse than $O\big(\frac{1}{|x-y|^2}\big)$ allowing differentiation under the integral of $T_{\alpha,n} f$. By using Lemma 3.1 the identity (ii) holds.

(iii) By induction, together with (i), (ii) we obtain (iii). ∎

Remark 3.13. For each $n \geq 1$ the operator $T_{\alpha,n} : L_p(\Omega, \mathbb{H}(\mathbb{C})) \to W_p^n(\Omega, \mathbb{H}(\mathbb{C}))$ is well-defined and continuous. Consequently, we obtain $T_{\alpha,n} : W_p^k(\Omega, \mathbb{H}(\mathbb{C})) \to W_p^{k+n}(\Omega, \mathbb{H}(\mathbb{C}))$.

4. Integral representation of solutions to the general inhomogeneous polynomial equation

In this section, we will investigate Cauchy–Pompeiu type representation formulas in quaternionic analysis for the general polynomial operator $\prod_{\nu=1}^{j}(D+\alpha_\nu)^{k_\nu}$, where $\alpha_\nu \neq \alpha_\mu$ if $\nu \neq \mu$.

4.1. A fundamental solution for a general polynomial operator. Using the fundamental solution of the Helmholtz equation, a fundamental solution can be constructed for the product of Helmholtz operators. In a similar way as in [25, 29], starting with the fundamental solution $K_\alpha^{(1)}(x)$ for the D_α operator, a fundamental solution for the operator $\prod_{\nu=1}^{j}(D+\alpha_\nu)^{k_\nu}$ in Ω is constructed.

Lemma 4.1. *Let $\alpha_1, \alpha_2, \ldots, \alpha_j$ be mutually different complex constants. Then*

(i) *the function $K_{\alpha_1,\alpha_2}^{(1,1)}(x) := \frac{1}{\alpha_2-\alpha_1}\left(K_{\alpha_1}^{(1)}(x) - K_{\alpha_2}^{(1)}(x)\right)$ is a fundamental solution for the operator $D_{\alpha_1}D_{\alpha_2}$.*

(ii) *if $K_{\alpha_1,\alpha_2,\ldots,\alpha_{j-1},\alpha_j}^{(k_1,k_2,\ldots,k_{j-1},k_j-1)}(x)$ and $K_{\alpha_1,\alpha_2,\ldots,\alpha_{j-1},\alpha_j}^{(k_1,k_2,\ldots,k_{j-1}-1,k_j)}(x)$ are fundamental solutions for the operators $D_{\alpha_1}^{k_1}D_{\alpha_2}^{k_2}\ldots D_{\alpha_{j-1}}^{k_{j-1}}D_{\alpha_j}^{k_j-1}$ and $D_{\alpha_1}^{k_1}D_{\alpha_2}^{k_2}\ldots D_{\alpha_{j-1}}^{k_{j-1}-1}D_{\alpha_j}^{k_j}$, respectively, then the function*

$$K_{\alpha_1,\alpha_2,\ldots,\alpha_{j-1},\alpha_j}^{(k_1,k_2,\ldots,k_{j-1},k_j)}(x) = \frac{1}{\alpha_{j-1}-\alpha_j}\left(K_{\alpha_1,\alpha_2,\ldots,\alpha_{j-1},\alpha_j}^{(k_1,k_2,\ldots,k_{j-1}-1,k_j)}(x) - K_{\alpha_1,\alpha_2,\ldots,\alpha_{j-1},\alpha_j}^{(k_1,k_2,\ldots,k_{j-1},k_j-1)}(x)\right)$$

is a fundamental solution for the operator $D_{\alpha_1}^{k_1}D_{\alpha_2}^{k_2}\ldots D_{\alpha_{j-1}}^{k_{j-1}}D_{\alpha_j}^{k_j}$.

(iii) *it holds*

$$D_{\alpha_\nu}^{h_\nu}K_{\alpha_\ell,\alpha_{\ell+1},\ldots,\alpha_\nu,\ldots,\alpha_{m-1},\alpha_m}^{(k_\ell,k_{\ell+1},\ldots,k_\nu,\ldots,k_{m-1},k_m)}(x) = K_{\alpha_\ell,\alpha_{\ell+1},\ldots,\alpha_\nu,\ldots,\alpha_{m-1},\alpha_m}^{(k_\ell,k_{\ell+1},\ldots,(k_\nu-h_\nu),\ldots,k_{m-1},k_m)}(x)$$

in the distributional sense, where $\ell \leq \nu \leq m, 0 \leq h_\nu \leq k_\nu, k_\imath \in \mathbb{N}$, $\imath = \ell, \ldots, m$.

Proof. For the proofs of (i) and (ii) we refer the readers to [29]. In order to prove (iii) we also need the following notions. Since $D_r K_\alpha^{(k)}(x) = DK_\alpha^{(k)}(x)$ for all $k \in \mathbb{N}$, it is easy to see that

$$D_{r,x,\alpha_\nu}^{h_\nu}K_{\alpha_\ell,\alpha_{\ell+1},\ldots,\alpha_\nu,\ldots,\alpha_{m-1},\alpha_m}^{(k_\ell,k_{\ell+1},\ldots,k_\nu,\ldots,k_{m-1},k_m)}(x) = D_{x,\alpha_\nu}^{h_\nu}K_{\alpha_\ell,\alpha_{\ell+1},\ldots,\alpha_\nu,\ldots,\alpha_{m-1},\alpha_m}^{(k_\ell,k_{\ell+1},\ldots,k_\nu,\ldots,k_{m-1},k_m)}(x),$$

and for all $\phi \in C_c^\infty(\Omega, \mathbb{H}(\mathbb{C}))$ with the right $\mathbb{H}(\mathbb{C})$-distributions, we have

$$\begin{aligned}&\left\langle D^{h_\nu}_{r,x,\alpha_\nu} K^{(k_\ell,k_{\ell+1},\dots,k_\nu,\dots,k_{m-1},k_m)}_{\alpha_\ell,\alpha_{\ell+1},\dots,\alpha_\nu,\dots,\alpha_{m-1},\alpha_m}(x), \phi(x)\right\rangle \\ &\qquad = -\left\langle D^{h_\nu-1}_{r,x,\alpha_\nu} K^{(k_\ell,k_{\ell+1},\dots,k_\nu-1,\dots,k_{m-1},k_m)}_{\alpha_\ell,\alpha_{\ell+1},\dots,\alpha_\nu,\dots,\alpha_{m-1},\alpha_m}(x), D_{-\alpha_\nu}\phi(x)\right\rangle .\end{aligned}$$

Using (ii) step by step then (iii) is seen. ∎

Remark 4.2. From (i) and (ii) in the above lemma by a straightforward computation we obtain

$$K^{(1,1,\dots,1,1)}_{\alpha_1,\ \alpha_2,\dots,\alpha_{k-1},\alpha_k}(x) = \sum_{i=1}^{k} \prod_{\substack{\nu=1\\ \nu\neq i}}^{k} \frac{1}{(\alpha_\nu - \alpha_i)} K^{(1)}_{\alpha_i}(x).$$

These results are used to prove Cauchy–Pompeiu type representation formulas in quaternionic analysis for the general polynomial operator $\prod_{\nu=1}^{j}(D+\alpha_\nu)^{k_\nu}$, where $\alpha_\nu \neq \alpha_\mu$ if $\nu \neq \mu$.

4.2. Representation for the general polynomial operator. In this subsection, the representation formulas of solutions to the general inhomogeneous polynomial equation $\prod_{\nu=1}^{j}(D+\alpha_\nu)^{k_\nu} f = g$ in Ω is proved.

Theorem 4.3. *Let Ω be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega =: \Gamma$ and $f \in C^2(\Omega, \mathbb{H}(\mathbb{C})) \cap C^1(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$. Then*

$$\begin{aligned} f(x) = &-\int_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y) f(y)\, d\Gamma_y \\ &- \int_\Gamma K^{(1,1)}_{\alpha_1,\alpha_2}(x-y)\vec{n}(y) D_{\alpha_1,y} f(y)\, d\Gamma_y \\ &+ \int_\Omega K^{(1,1)}_{\alpha_1,\alpha_2}(x-y) D_{\alpha_1,y} D_{\alpha_2,y} f(y)\, dy, \end{aligned} \tag{4.1}$$

where $K^{(1,1)}_{\alpha_1,\alpha_2}(x)$ are given in Lemma 4.1 and α_1, α_2 are different complex constants.

Proof. Note that

$$\begin{aligned} D_{\alpha_2,x} D_{\alpha_1,x} &= D_{\alpha_1,x} D_{\alpha_2,x} \\ D_{r,-\alpha,y} K^{(1)}_{\alpha}(x-y) &= -D_{r,\alpha,x} K^{(1)}_{\alpha}(x-y). \end{aligned} \tag{4.2}$$

Applying the quaternionic Cauchy–Pompeiu formula (3.1) for $D_{\alpha_1,y} f(y)$ we obtain

$$\begin{aligned} &D_{\alpha_1,y} f(y) \\ &\quad = -\int_\Gamma K^{(1)}_{\alpha_2}(y-\tilde{y})\vec{n}(\tilde{y}) D_{\alpha_1,\tilde{y}} f(\tilde{y})\, d\Gamma_{\tilde{y}} + \int_\Omega K^{(1)}_{\alpha_2}(y-\tilde{y}) D_{\alpha_2,\tilde{y}} D_{\alpha_1,\tilde{y}} f(\tilde{y}) d\tilde{y}\,. \end{aligned}$$

This leads to

$$\begin{aligned} f(x) = &- \int_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)f(y)\, d\Gamma_y \\ &- \int_\Gamma \psi^{(1),\ (1)}_{(\alpha_1),(\alpha_2)}(x,\tilde{y})\vec{n}(\tilde{y})D_{\alpha_1,\tilde{y}}f(\tilde{y})\, d\Gamma_y \\ &+ \int_\Omega \psi^{(1),\ (1)}_{(\alpha_1),(\alpha_2)}(x,\tilde{y})D_{\alpha_1,\tilde{y}}D_{\alpha_2,\tilde{y}}f(\tilde{y})\, d\tilde{y}, \end{aligned} \tag{4.3}$$

where $\psi^{(1),\ (1)}_{(\alpha_1),(\alpha_2)}(x,\tilde{y}) = \int_\Omega K^{(1)}_{\alpha_1}(x-y)K^{(1)}_{\alpha_2}(y-\tilde{y})\, dy$. On the other hand, we have in the distributional sense that

$$D_{\alpha_1}K^{(1)}_{\alpha_1}(x) = \delta(x), \qquad D_{\alpha_2}K^{(1)}_{\alpha_2}(x) = \delta(x), \tag{4.4}$$

so that for any $\phi \in C^\infty_c(\Omega, \mathbb{H}(\mathbb{C}))$

$$\begin{aligned} &\big\langle (D+\alpha_1)K^{(1,1)}_{\alpha_1,\alpha_2}(x), \phi(x)\big\rangle \\ &\quad = \frac{1}{\alpha_2-\alpha_1}\big\langle (D+\alpha_1)[K^{(1)}_{\alpha_1}(x) - K^{(1)}_{\alpha_2}(x)], \phi(x)\big\rangle \\ &\quad = \frac{1}{\alpha_2-\alpha_1}\Big\{\big\langle (D+\alpha_1)K^{(1)}_{\alpha_1}(x), \phi(x)\big\rangle - \big\langle (D+\alpha_2-\alpha_2+\alpha_1)K^{(1)}_{\alpha_2}(x), \phi(x)\big\rangle\Big\} \\ &\quad = \big\langle K^{(1)}_{\alpha_2}(x), \phi(x)\big\rangle. \end{aligned}$$

Therefore, $D_{\alpha_1}K^{(1,1)}_{\alpha_1,\alpha_2}(x) = K^{(1)}_{\alpha_2}(x)$ in the sense of distributions. For $x, \tilde{y} \in \Omega$ with $x \neq \tilde{y}$ the quaternionic Cauchy–Pompeiu formula yields

$$K^{(1,1)}_{\alpha_1,\alpha_2}(x-\tilde{y}) = \psi^{(1),\ (1)}_{(\alpha_1),(\alpha_2)}(x,\tilde{y}) - \tilde{\psi}^{(1),\ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}),$$

where

$$\tilde{\psi}^{(1),\ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}) = \int_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(1,1)}_{\alpha_1,\alpha_2}(y-\tilde{y})\, d\Gamma_y.$$

Substituting this equality into equality (4.3), we obtain

$$\begin{aligned} f(x) = &- \int_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)f(y)\, d\Gamma_y - \int_\Gamma K^{(1,1)}_{\alpha_1,\alpha_2}(x-y)\vec{n}(y)D_{\alpha_1,y}f(y)\, d\Gamma_y \\ &+ \int_\Omega K^{(1,1)}_{\alpha_1,\alpha_2}(x-y)D_{\alpha_1,y}D_{\alpha_2,y}f(y)\, dy \\ &- \int_\Gamma \tilde{\psi}^{(1),\ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y})\vec{n}(\tilde{y})D_{\alpha_1,\tilde{y}}f(\tilde{y})\, d\Gamma_{\tilde{y}} \\ &+ \int_\Omega \tilde{\psi}^{(1),\ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y})D_{\alpha_1,\tilde{y}}D_{\alpha_2,\tilde{y}}f(\tilde{y})d\tilde{y}. \end{aligned}$$

Applying Stokes' formula (2.7) again gives

$$\int_\Gamma \tilde{\psi}^{(1),\ \ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y})\vec{n}(\tilde{y})D_{\alpha_1,\tilde{y}}f(\tilde{y})d\Gamma_{\tilde{y}} - \int_\Omega \tilde{\psi}^{(1),\ \ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y})D_{\alpha_2,\tilde{y}}D_{\alpha_1,\tilde{y}}f(\tilde{y})\,d\tilde{y}$$
$$= \int_\Omega D_{r,-\alpha_2,\tilde{y}}\tilde{\psi}^{(1),\ \ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y})D_{\alpha_1,\tilde{y}}f(\tilde{y})d\tilde{y}.$$

Using the definition of $K^{(1,1)}_{\alpha_1,\alpha_2}(x)$ and the equalities (4.2), (4.4), and noting that $\Gamma \ni y \neq x \in \Omega,\ \Gamma \ni y \neq \tilde{y} \in \Omega$, we obtain

$$D_{r,-\alpha_2,\tilde{y}}\tilde{\psi}^{(1),\ \ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}) = -\int_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(1)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y.$$

Since $\int_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(1)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y = 0$ (see the proof of [19, Theorem 3.1]) we have $D_{r,-\alpha_2,\tilde{y}}\tilde{\psi}^{(1),\ \ (1,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}) = 0$. This leads to (4.1). ∎

Remark 4.4. If $\alpha_2 = -\alpha_1$ we have the representation formula in terms of the Helmholtz operator as in [19, Theorem 4.1].

In order to obtain a generalization of the above theorem, we need the following lemma.

Lemma 4.5. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be mutually different complex constants. Then* $\sum_{i=1}^n \prod_{\nu=1,\nu\neq i}^n \frac{1}{(\alpha_\nu - \alpha_i)} = 0$ *for all* $2 \le n \in \mathbb{N}$.

Proof. For $n = 2$, we have $\sum_{i=1}^2 \prod_{\nu=1,\nu\neq i}^2 \frac{1}{(\alpha_\nu-\alpha_i)} = \frac{1}{\alpha_2-\alpha_1} + \frac{1}{\alpha_1-\alpha_2} = 0$. By direct caculation we also get $\sum_{i=1}^3 \prod_{\nu=1,\nu\neq i}^3 \frac{1}{(\alpha_\nu-\alpha_i)} = 0$.

In the case $n > 3$, we suppose that this lemma holds for some n. We now consider the function $f(x) = \sum_{i=1}^{n+1}(x-\alpha_i)\prod_{\nu=1,\nu\neq i}^{n+1}\frac{1}{(\alpha_\nu-\alpha_i)}$. Note that

$$f(\alpha_{n+1}) = \sum_{i=1}^{n+1}(\alpha_{n+1}-\alpha_i)\prod_{\substack{\nu=1\\ \nu\neq i}}^{n+1}\frac{1}{(\alpha_\nu-\alpha_i)} = \sum_{i=1}^{n}\prod_{\substack{\nu=1\\ \nu\neq i}}^{n}\frac{1}{(\alpha_\nu-\alpha_i)} = 0$$

by inductive hypothesis. Similarly, $f(\alpha_j) = 0$ for all $1 \le\ j \le n$. Therefore, $f(x)$ has $(n+1)$ zeroes $\alpha_j,\ 1 \le\ j \le n+1$. However, it is a polynomial of degree one. Thus, $f(x) \equiv 0$. Then, we have $f'(x) \equiv 0$. In other words, $f'(x) = \sum_{i=1}^{n+1}\prod_{\nu=1,\nu\neq i}^{n+1}\frac{1}{(\alpha_\nu-\alpha_i)} = 0$, i.e., the lemma holds. ∎

Theorem 4.6. *Let $f \in C^n(\Omega, \mathbb{H}(\mathbb{C})) \cap C^{n+1}(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$. Then*

$$\begin{aligned} f(x) = &-\int_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)f(y)\,d\Gamma_y \\ &-\sum_{j=2}^{n}\int_\Gamma K^{(1,1,\ldots,1,1)}_{\alpha_1,\alpha_2,\ldots,\alpha_{j-1},\alpha_j}(x-y)\vec{n}(y)\prod_{k=1}^{j-1}D_{\alpha_k,y}f(y)\,d\Gamma_y \\ &+\int_\Omega K^{(1,1,\ldots,1,1)}_{\alpha_1,\alpha_2,\ldots,\alpha_{n-1},\alpha_n}(x-y)\prod_{k=1}^{n}D_{\alpha_k,y}f(y)\,dy, \end{aligned} \tag{4.5}$$

where $K^{(1,1,\dots,1,1)}_{\alpha_1,\alpha_2,\dots,\alpha_{k-1},\alpha_k}(x)$ *is given in Remark* 4.2 *and* $\alpha_1,\alpha_2,\dots,\alpha_n$ *are mutually different complex constants.*

Proof. For $n=1$, formula (4.5) coincides with the Cauchy–Pompeiu representation. For the case $n=2$, we have already shown (4.5) in Theorem 4.3. In order to prove this formula for any $n>2$ assume it holds for $n-1$. By inductive hypothesis we have

$$\begin{aligned} f(x) = &-\int_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)f(y)\,d\Gamma_y \\ &-\sum_{j=2}^{n-1}\int_\Gamma K^{(1,1,\dots,1,1)}_{\alpha_1,\alpha_2,\dots,\alpha_{j-1},\alpha_j}(x-y)\vec{n}(y)\prod_{k=1}^{j-1}D_{\alpha_k,y}f(y)\,d\Gamma_y \\ &+\int_\Omega K^{(1,1,\dots,1,1)}_{\alpha_1,\alpha_2,\dots,\alpha_{n-1}}(x-y)\prod_{k=1}^{n-1}D_{\alpha_k,y}f(y)\,dy\,. \end{aligned}$$

Applying the Cauchy–Pompeiu formula (3.1) to $\prod_{k=1}^{n-1}D_{\alpha_k,y}f(y)$ gives

$$\begin{aligned} \prod_{k=1}^{n-1}D_{\alpha_k,y}f(y) = &-\int_\Gamma K^{(1)}_{\alpha_n}(y-\tilde{y})\vec{n}(\tilde{y})\prod_{k=1}^{n-1}D_{\alpha_k,\tilde{y}}f(\tilde{y})\,d\Gamma_{\tilde{y}} \\ &+\int_\Omega K^{(1)}_{\alpha_n}(y-\tilde{y})\prod_{k=1}^{n}D_{\alpha_k,\tilde{y}}f(\tilde{y})d\tilde{y}. \end{aligned}$$

It follows

$$\begin{aligned} f(x) = &-\int_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)f(y)\,d\Gamma_y \\ &-\sum_{j=2}^{n-1}\int_\Gamma K^{(1,1,\dots,1,1)}_{\alpha_1,\alpha_2,\dots,\alpha_{j-1},\alpha_j}(x-y)\vec{n}(y)\prod_{k=1}^{j-1}D_{\alpha_k,y}f(y)\,d\Gamma_y \\ &-\int_\Gamma \psi^{(1,1,\dots,1,1),\ \ (1)}_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1}),(\alpha_n)}(x,\tilde{y})\vec{n}(\tilde{y})\prod_{k=1}^{n-1}D_{\alpha_k,\tilde{y}}f(\tilde{y})\,d\Gamma_{\tilde{y}} \\ &+\int_\Omega \psi^{(1,1,\dots,1,1),\ \ (1)}_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1}),\ (\alpha_n)}(x,\tilde{y})\prod_{k=1}^{n}D_{\alpha_k,\tilde{y}}f(\tilde{y})d\tilde{y}, \end{aligned}$$

where $\psi^{(1,1,\dots,1,1),\ \ (1)}_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1}),\ (\alpha_n)}(x,\tilde{y}) = \int_\Omega K^{(1,1,\dots,1)}_{\alpha_1,\alpha_2,\dots,\alpha_{n-1}}(x-y)K^{(1)}_{\alpha_n}(y-\tilde{y})\,dy$. By inductive hypothesis, applying it for $K^{(1,1,\dots,1,1)}_{\alpha_1,\alpha_2,\dots,\alpha_{n-1},\alpha_n}(x-\tilde{y})$ as well as using the assertion (iii) of Theorem 4.1 step by step, we obtain

$$\begin{aligned} &\psi^{(1,1,\dots,1,1),\ \ (1)}_{(\alpha_1,\alpha_2,\dots,\alpha_{n-1}),(\alpha_n)}(x,\tilde{y}) \\ &\qquad = K^{(1,1,\dots,1,1)}_{\alpha_1,\alpha_2,\dots,\alpha_{n-1},\alpha_n}(x-\tilde{y}) + \sum_{k=1}^{n-1}\tilde{\psi}^{(1,1,\dots,1),\ \ (1,1,\dots,1)}_{(\alpha_1,\alpha_2,\dots,\alpha_k),(\alpha_k,\alpha_{k+1},\dots,\alpha_n)}(x,\tilde{y}), \end{aligned}$$

where

$$\tilde{\psi}^{(1,1,\dots,1),\ \ (1,1,\dots,1)}_{(\alpha_1,\alpha_2,\dots,\alpha_k),(\alpha_k,\alpha_{k+1},\dots,\alpha_n)}(x,\tilde{y}) = \int_\Gamma K^{(1,1,\dots,1)}_{\alpha_1,\alpha_2,\dots,\alpha_k}(x-y)\vec{n}(y)K^{(1,1,\dots,1)}_{\alpha_k,\alpha_{k+1},\dots,\alpha_n}(y-\tilde{y})d\Gamma_y.$$

Hence,

$$\begin{aligned} f(x) = &-\int_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)f(y)\,d\Gamma_y \\ &-\sum_{j=2}^{n-1}\int_\Gamma K^{(1,1,\dots,1,1)}_{\alpha_1,\alpha_2,\dots,\alpha_{j-1},\alpha_j}(x-y)\vec{n}(y)\prod_{k=1}^{j-1}D_{\alpha_k,y}f(y)\,d\Gamma_y \\ &-\int_\Gamma K^{(1,1,\dots,1,1)}_{\alpha_1,\alpha_2,\dots,\alpha_{n-1},\alpha_n}(x-y)\vec{n}(y)\prod_{k=1}^{n-1}D_{\alpha_k,y}f(y)\,d\Gamma_y \\ &+\int_\Omega K^{(1,1,\dots,1,1)}_{\alpha_1,\alpha_2,\dots,\alpha_{n-1},\alpha_n}(x-y)\prod_{k=1}^{n}D_{\alpha_k,y}f(y)\,dy \\ &-\int_\Gamma\left[\sum_{k=1}^{n-1}\tilde{\psi}^{(1,1,\dots,1),\ \ (1,1,\dots,1)}_{(\alpha_1,\alpha_2,\dots,\alpha_k),(\alpha_k,\alpha_{k+1},\dots,\alpha_n)}(x,\tilde{y})\right]\vec{n}(\tilde{y})\prod_{j=1}^{n-1}D_{\alpha_j,\tilde{y}}f(\tilde{y})\,d\Gamma_{\tilde{y}} \\ &+\int_\Omega\left[\sum_{k=1}^{n-1}\tilde{\psi}^{(1,1,\dots,1),\ \ (1,1,\dots,1)}_{(\alpha_1,\alpha_2,\dots,\alpha_k),(\alpha_k,\alpha_{k+1},\dots,\alpha_n)}(x,\tilde{y})\right]\prod_{j=1}^{n}D_{\alpha_j,\tilde{y}}f(\tilde{y})d\tilde{y}. \end{aligned} \tag{4.6}$$

Applying the quaternionic Stokes' formula (2.7) yields

$$\begin{aligned} &\int_\Gamma\left[\sum_{k=1}^{n-1}\tilde{\psi}^{(1,1,\dots,1),\ \ (1,1,\dots,1)}_{(\alpha_1,\alpha_2,\dots,\alpha_k),(\alpha_k,\alpha_{k+1},\dots,\alpha_n)}(x,\tilde{y})\right]\vec{n}(\tilde{y})\prod_{j=1}^{n-1}D_{\alpha_j,\tilde{y}}f(\tilde{y})\,d\Gamma_{\tilde{y}} \\ -&\int_\Omega\left[\sum_{k=1}^{n-1}\tilde{\psi}^{(1,1,\dots,1),\ \ (1,1,\dots,1)}_{(\alpha_1,\alpha_2,\dots,\alpha_k),(\alpha_k,\alpha_{k+1},\dots,\alpha_n)}(x,\tilde{y})\right]\prod_{j=1}^{n}D_{\alpha_j,\tilde{y}}f(\tilde{y})d\tilde{y} \\ =&\int_\Omega\left[\sum_{k=1}^{n-1}D_{-\alpha_n,\tilde{y}}\tilde{\psi}^{(1,1,\dots,1),\ \ (1,1,\dots,1)}_{(\alpha_1,\alpha_2,\dots,\alpha_k),(\alpha_k,\alpha_{k+1},\dots,\alpha_n)}(x,\tilde{y})\right]\prod_{j=1}^{n-1}D_{\alpha_j,\tilde{y}}f(\tilde{y})d\tilde{y}. \end{aligned}$$

Using the assertion (iii) of Lemma 4.1 and Remark 4.2 we get by Lemma 4.5

$$\begin{aligned} &\sum_{k=1}^{n-1}D_{-\alpha_n,\tilde{y}}\tilde{\psi}^{(1,1,\dots,1),\ \ (1,1,\dots,1)}_{(\alpha_1,\alpha_2,\dots,\alpha_k),(\alpha_k,\alpha_{k+1},\dots,\alpha_n)}(x,\tilde{y}) \\ &\qquad = -\sum_{k=1}^{n-1}\int_\Gamma K^{(1,1,\dots,1)}_{\alpha_1,\alpha_2,\dots,\alpha_k}(x-y)\vec{n}(y)K^{(1,1,\dots,1)}_{\alpha_k,\alpha_{k+1},\dots,\alpha_{n-1}}(y-\tilde{y})\,d\Gamma_y \end{aligned}$$

$$= -\sum_{k=1}^{n-1}\Big[\Big(\sum_{i=1}^{k}\prod_{\substack{\nu=1\\ \nu\neq i}}^{k}\frac{1}{(\alpha_\nu-\alpha_i)}\Big)\Big(\sum_{j=k}^{n-1}\prod_{\substack{\mu=k\\ \mu\neq j}}^{n-1}\frac{1}{(\alpha_\mu-\alpha_j)}\Big)$$
$$\times\int_\Gamma K^{(1)}_{\alpha_i}(x-y)\vec{n}(y)K^{(1)}_{\alpha_j}(y-\tilde{y})\,d\Gamma_y\Big] = 0\,.$$

Substituting this into equality (4.6) we obtain equality (4.5), i.e., Theorem 4.6 is proved. ∎

In order to obtain the generalized representation of solution for $\prod_{\nu=1}^{j}(D+\alpha_\nu)^{k_\nu} f = g$, we now as an example construct the representation of solutions for the inhomogeneous equation $D^3_{\alpha_1}D_{\alpha_2}f(x) = g(x)$ in a bounded domain Ω, where $\alpha_1 \neq \alpha_2$. The integral representation formulas for higher order D_α equations in [19], Theorem 4.6 and Lemma 4.1 are used.

Theorem 4.7. *Let* $f \in C^4(\Omega, \mathbb{H}(\mathbb{C}))\cap C^3(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$, $\alpha_1 \neq \alpha_2,\ \alpha_1, \alpha_2 \in \mathbb{C}$. *Then*

$$\begin{aligned} f(x) = &-\sum_{k=1}^{3}\int_\Gamma K^{(k)}_{\alpha_1}(x-y)\vec{n}(y)D^{k-1}_{\alpha_1,y}f(y)\,d\Gamma_y \\ &-\int_\Gamma K^{(3,1)}_{\alpha_1,\alpha_2}(x-y)\vec{n}(y)D^3_{\alpha_1,y}f(y)\,d\Gamma_y \\ &+\int_\Omega K^{(3,1)}_{\alpha_1,\alpha_2}(x-y)D^3_{\alpha_1,y}D_{\alpha_2,y}f(y)\,dy. \end{aligned}$$

Proof. Applying the quaternionic Cauchy–Pompeiu formula for $D^3_{\alpha_1,y}f(y)$ we get

$$\begin{aligned} D^3_{y,\alpha_1}f(y) = &-\int_\Gamma K^{(1)}_{\alpha_2}(y-\tilde{y})\vec{n}(\tilde{y})D^3_{\alpha_1,\tilde{y}}f(\tilde{y})\,d\Gamma_{\tilde{y}} \\ &+\int_\Omega K^{(1)}_{\alpha_2}(y-\tilde{y})D_{\alpha_2,\tilde{y}}D^3_{\alpha_1,\tilde{y}}f(\tilde{y})d\tilde{y}. \end{aligned}$$

It follows

$$\begin{aligned} f(x) = &-\sum_{k=1}^{3}\int_\Gamma K^{(k)}_{\alpha_1}(x-y)\vec{n}(y)D^{k-1}_{\alpha_1,y}f(y)\,d\Gamma_y \\ &-\int_\Gamma \psi^{(3),(0,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y})\vec{n}(\tilde{y})D^3_{\alpha_1,\tilde{y}}f(\tilde{y})\,d\Gamma_{\tilde{y}} \\ &+\int_\Omega \psi^{(3),(0,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y})D_{\alpha_2,\tilde{y}}D^3_{\alpha_1,\tilde{y}}f(\tilde{y})d\tilde{y}, \end{aligned} \tag{4.7}$$

where $\psi^{(3),(0,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}) = \int_\Omega K^{(3)}_{\alpha_1}(x-y)K^{(1)}_{\alpha_2}(y-\tilde{y})\,dy$, and it is easy to see that

$$\begin{aligned}
K^{(1,1)}_{\alpha_1,\alpha_2}(x) &= \frac{1}{\alpha_2-\alpha_1}\left(K^{(1)}_{\alpha_1}(x) - K^{(1)}_{\alpha_2}(x)\right)\\
K^{(2,1)}_{\alpha_1,\alpha_2}(x) &= \frac{1}{\alpha_2-\alpha_1}\left(K^{(2)}_{\alpha_1}(x) - K^{(1,1)}_{\alpha_1,\alpha_2}(x)\right)\\
K^{(3,1)}_{\alpha_1,\alpha_2}(x) &= \frac{1}{\alpha_2-\alpha_1}\left(K^{(3)}_{\alpha_1}(x) - K^{(2,1)}_{\alpha_1,\alpha_2}(x)\right)\\
D^\nu_{\alpha_1}K^{(3,1)}_{\alpha_1,\alpha_2}(x) &= K^{(3-\nu,1)}_{\alpha_1,\alpha_2}(x) \qquad (\nu = 1,2,3)
\end{aligned}$$

in the sense of distribution, because of Lemma 3.1 and Lemma 4.1. Applying the representaion for higher order powers of D_α in Theorem 3.2 to $K^{(3,1)}_{\alpha_1,\alpha_2}(x-\tilde{y})$ gives

$$\psi^{(3),(0,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}) = K^{(3,1)}_{\alpha_1,\alpha_2}(x-\tilde{y}) + \sum_{\nu=1}^{3}\tilde{\psi}^{(\nu),(4-\nu,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}),$$

where $\tilde{\psi}^{(\nu),(4-\nu,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}) = \int_\Gamma K^{(\nu)}_{\alpha_1}(x-y)\vec{n}(y)K^{(4-\nu,1)}_{\alpha_1,\alpha_2}(y-\tilde{y})\,d\Gamma_y$. Substituting this into (4.7), then applying the Stokes' formula again with $x \neq y$, shows that

$$\begin{aligned}
f(x) = &-\sum_{k=1}^{3}\int_\Gamma K^{(k)}_{\alpha_1}(x-y)\vec{n}(y)D^{k-1}_{\alpha_1,y}f(y)\,d\Gamma_y\\
&-\int_\Gamma K^{(3,1)}_{\alpha_1,\alpha_2}(x-y)\vec{n}(y)D^3_{\alpha_1,y}f(y)\,d\Gamma_y\\
&+\int_\Omega K^{(3,1)}_{\alpha_1,\alpha_2}(x-y)D^3_{\alpha_1,y}D_{\alpha_2,y}f(y)\,dy\\
&-\int_\Omega\sum_{\nu=1}^{3}D_{-\alpha_2,\tilde{y}}\tilde{\psi}^{(\nu),(4-\nu,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y})D^3_{r,\alpha_1,\tilde{y}}D_{\alpha_2,\tilde{y}}f(\tilde{y})d\tilde{y}.
\end{aligned}$$

Namely, for arbitrary fixed x and $\tilde{y}$, the functions $K^{(k)}_{\alpha_1}(x-y)$, $K^{(k)}_{\alpha_1}(y-\tilde{y}), k = 1,2$, are C^1-functions in the whole domain Ω except for the two points x and $\tilde{y}$. Therefore, for $\Omega_{x,\varepsilon} = \Omega - \{y \in \Omega, |y-x| \leq \varepsilon\}$, $\Omega_{\tilde{y},\varepsilon} = \Omega - \{y \in \Omega, |y-\tilde{y}| \leq \varepsilon\}$ and $\Omega_\varepsilon = \Omega - \{y \in \Omega \,|\, |y-x| \leq \varepsilon \text{ and } |y-\tilde{y}| \leq \varepsilon\}$ with $\varepsilon > 0$ small enough,

$$\begin{aligned}
\int_\Gamma K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(3)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y &= \int_{\partial\Omega_{x,\varepsilon}} K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(3)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y\\
&\quad+\int_{|y-x|=\varepsilon} K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(3)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y\\
\int_\Gamma K^{(3)}_{\alpha_1}(x-y)\vec{n}(y)K^{(1)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y &= \int_{\partial\Omega_{\tilde{y},\varepsilon}} K^{(3)}_{\alpha_1}(x-y)\vec{n}(y)K^{(1)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y\\
&\quad+\int_{|y-\tilde{y}|=\varepsilon} K^{(3)}_{\alpha_1}(x-y)\vec{n}(y)K^{(1)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y
\end{aligned}$$

and

$$\int_{\Gamma} K^{(2)}_{\alpha_1}(x-y)\vec{n}(y)K^{(2)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y = \int_{\partial\Omega_\varepsilon} K^{(2)}_{\alpha_1}(x-y)\vec{n}(y)K^{(2)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y + \int_{|y-x|=\varepsilon} K^{(2)}_{\alpha_1}(x-y)\vec{n}(y)K^{(2)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y + \int_{|y-\tilde{y}|=\varepsilon} K^{(2)}_{\alpha_1}(x-y)\vec{n}(y)K^{(2)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y.$$

Applying Stokes' formula for $\Omega_{x,\varepsilon}$ and $\Omega_{\tilde{y},\varepsilon}$, respectively, gives

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_{x,\varepsilon}} K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(3)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y = K^{(3)}_{\alpha_1}(x-\tilde{y}) + \lim_{\varepsilon\to 0}\int_{\Omega_{x,\varepsilon}} K^{(1)}_{\alpha_1}(x-y)K^{(2)}_{\alpha_1}(y-\tilde{y})\,dy$$

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_{\tilde{y},\varepsilon}} K^{(3)}_{\alpha_1}(x-y)\vec{n}(y)K^{(1)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y = K^{(3)}_{\alpha_1}(x-\tilde{y}) - \lim_{\varepsilon\to 0}\int_{\Omega_{\tilde{y},\varepsilon}} K^{(2)}_{\alpha_1}(x-y)K^{(1)}_{\alpha_1}(y-\tilde{y})\,dy\,.$$

Applying now Stokes' formula for Ω_ε and observing $D_{r,-\alpha_1,y}K^{(2)}_{\alpha_1}(x-y) = -D_{\alpha_1,x}K^{(2)}_{\alpha_1}(x-y) = K^{(1)}_{\alpha_1}(x-y)$ shows

$$\int_{\partial\Omega_\varepsilon} K^{(2)}_{\alpha_1}(x-y)\vec{n}(y)K^{(2)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y = -\int_{\Omega_\varepsilon} K^{(1)}_{\alpha_1}(x-y)K^{(2)}_{\alpha_1}(y-\tilde{y})\,dy + \int_{\Omega_\varepsilon} K^{(2)}_{\alpha_1}(x-y)K^{(1)}_{\alpha_1}(y-\tilde{y})\,dy.$$

On the orther hand

$$\int_{\Omega_{x,\varepsilon}} K^{(1)}_{\alpha_1}(x-y)K^{(2)}_{\alpha_1}(y-\tilde{y})\,dy = \int_{\Omega_\varepsilon} K^{(1)}_{\alpha_1}(x-y)K^{(2)}_{\alpha_1}(y-\tilde{y})\,dy + \int_{|y-\tilde{y}|<\varepsilon} K^{(1)}_{\alpha_1}(x-y)K^{(2)}_{\alpha_1}(y-\tilde{y})\,dy$$

$$\int_{\Omega_{\tilde{y},\varepsilon}} K^{(2)}_{\alpha_1}(x-y)K^{(1)}_{\alpha_1}(y-\tilde{y})\,dy = \int_{\Omega_\varepsilon} K^{(2)}_{\alpha_1}(x-y)K^{(1)}_{\alpha_1}(y-\tilde{y})\,dy + \int_{|y-x|<\varepsilon} K^{(2)}_{\alpha_1}(x-y)K^{(1)}_{\alpha_1}(y-\tilde{y})\,dy.$$

From equality (3.5) it follows that

$$\lim_{\varepsilon\to 0}\int_{|y-\tilde{y}|=\varepsilon} K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(3)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y = 0$$
$$\lim_{\varepsilon\to 0}\int_{|y-x|=\varepsilon} K^{(2)}_{\alpha_1}(x-y)\vec{n}(y)K^{(2)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y = 0$$
$$\lim_{\varepsilon\to 0}\int_{|y-\tilde{y}|=\varepsilon} K^{(2)}_{\alpha_1}(x-y)\vec{n}(y)K^{(2)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y = 0$$
$$\lim_{\varepsilon\to 0}\int_{|y-x|=\varepsilon} K^{(1)}_{\alpha_1}(x-y)\vec{n}(y)K^{(3)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y = -K^{(3)}_{\alpha_1}(x-\tilde{y})$$
$$\lim_{\varepsilon\to 0}\int_{|y-\tilde{y}|=\varepsilon} K^{(3)}_{\alpha_1}(x-y)\vec{n}(y)K^{(1)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y = \quad K^{(3)}_{\alpha_1}(x-\tilde{y}).$$

Using the above equalities and the Cauchy–Pompeiu representation formulas (3.1) and (3.2) for D^n_α $(n=1,2)$ and $K^{(3)}_{\alpha_1}(x-\tilde{y})$ we obtain

$$\lim_{\varepsilon\to 0}\int_{|y-\tilde{y}|<\varepsilon} K^{(1)}_{\alpha_1}(x-y)K^{(2)}_{\alpha_1}(y-\tilde{y})\,dy = 0$$
$$\lim_{\varepsilon\to 0}\int_{|y-x|<\varepsilon} K^{(2)}_{\alpha_1}(x-y)K^{(1)}_{\alpha_1}(y-\tilde{y})\,dy = 0.$$

This leads to

$$\sum_{\nu=1}^{3} D_{\tilde{y},-\alpha_2}\tilde{\psi}^{(\nu),\ (4-\nu,1)}_{(\alpha_1),(\alpha_1,\alpha_2)}(x,\tilde{y}) = -\sum_{\nu=1}^{3}\int_\Gamma K^{(\nu)}_{\alpha_1}(x-y)\vec{n}(y)K^{(4-\nu)}_{\alpha_1}(y-\tilde{y})\,d\Gamma_y = 0.$$

Hence Theorem 4.7 is proved. ∎

In a similar way as in the proof of Theorem 4.7 and using the representation formulas for higher order D_α operators in [19, Theorem 3.2] as well as Theorem 4.6, by induction we can also prove the next result.

Theorem 4.8. *Let $f \in C^n(\Omega,\mathbb{H}(\mathbb{C}))\cap C^{n-1}(\bar{\Omega},\mathbb{H}(\mathbb{C}))$ and $\sum_{\nu=1}^{j} k_\nu = n$. Then*

$$\begin{aligned} f(x) = &-\sum_{\nu_1=1}^{k_1}\int_\Gamma K^{(\nu_1)}_{\alpha_1}(x-y)\vec{n}(y)D^{\nu_1-1}_{\alpha_1,y}f(y)\,d\Gamma_y \\ &-\sum_{\nu_2=1}^{k_2}\int_\Gamma K^{(k_1,\ \nu_2)}_{\alpha_1,\ \alpha_2}(x-y)\vec{n}(y)D^{\nu_2-1}_{\alpha_2,y}D^{k_1}_{\alpha_1,y}f(y)\,d\Gamma_y - \dots \\ &-\sum_{\nu_j=1}^{k_j}\int_\Gamma K^{(k_1,k_2,\ \dots,\ k_{j-1},\nu_j)}_{\alpha_1,\alpha_2,\dots,\alpha_{j-1},\ \alpha_j}(x-y)\vec{n}(y)D^{\nu_j-1}_{\alpha_{k_j},y}\prod_{\mu=1}^{j-1}D^{k_\mu}_{\alpha_\mu,y}f(y)\,d\Gamma_y \\ &+\int_\Omega K^{(k_1,k_2,\ \dots,\ k_{j-1},k_j)}_{\alpha_1,\alpha_2,\dots,\alpha_{j-1},\ \alpha_j}(x-y)\prod_{\nu=1}^{j}D^{k_\nu}_{\alpha_\nu,y}f(y)\,dy, \end{aligned}$$

where $\alpha_1,\alpha_2,\dots,\alpha_{j-1},\alpha_j$ are mutually different complex constants.

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