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Boundedness in Asymmetric Oscillations at Resonance

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Abstract. The boundedness problem of all solutions for the nonlinear equation

$$(\phi_p(x'))' + (p-1)[\alpha\phi_p(x^+) - \beta\phi_p(x^-)] = f(t)$$

is discussed, where $\phi_p(u) = |u|^{p-2}u, p > 1, \alpha, \beta$ are positive constants satisfying the condition $\alpha^{-\frac{1}{p}} + \beta^{-\frac{1}{p}} = \frac{2}{n}$, where $n \in \mathbb{N}$, $f \in C^\infty(S^1)$ ($S^1 =: \mathbb{R}/2\pi_p\mathbb{Z}$) is $2\pi_p$-periodic, $x^\pm = \max\{\pm x, 0\}$ and $\pi_p = \frac{2\pi}{p\sin(\pi/p)}$.

Keywords: *Boundedness, p-Laplacian, asymmetric oscillations, Moser's twist theorem*
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1. Introduction

In this paper, motivated by the papers [1, 7] and [8], we consider the boundedness problem of all the solutions for the following p-Laplacian like nonlinear equation:

$$(\phi_p(x'))' + (p-1)[\alpha\phi_p(x^+) - \beta\phi_p(x^-)] = f(t) \tag{1}$$

($' = d/dt$), where $\phi_p(u) = |u|^{p-2}u$, $p > 1$ is a constant, $x^\pm = \max\{\pm x, 0\}$, α, β are positive constants satisfying

$$\alpha^{-\frac{1}{p}} + \beta^{-\frac{1}{p}} = \frac{2}{n} \qquad (n \in N), \tag{2}$$

where $f \in C^\infty(S^1)$ is a $2\pi_p$-periodic function ($S^1 =: \mathbb{R}/2\pi_p\mathbb{Z}$), $\pi_p = \frac{2\pi}{p\sin(\pi/p)}$. If $p = 2$, eq. (1) reduces to the linear equation

$$x'' + \alpha x^+ - \beta x^- = f(t) \tag{3}$$

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and $\pi_2 = \pi$ with α, β satisfying $\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2}{n}$. The unboundedness problem of solutions of (3) was recently discussed in [1] in case $\alpha \neq \beta$.

Let $C(t)$ be the solution of the initial value problem

$$\begin{aligned} & x'' + \alpha x^+ - \beta x^- = 0 \\ & x(0) = 1, \quad x'(0) = 0. \end{aligned}$$

Then it is well-known that $C(t) \in C^2(\mathbb{R})$ is τ-periodic with

$$\tau = \frac{\pi}{\sqrt{\alpha}} + \frac{\pi}{\sqrt{\beta}}.$$

Define a 2π-periodic function $\lambda(\theta)$(see also [7]) if $\tau = \frac{2m\pi}{n}$, with $m, n \in \mathbb{N}$,

$$\lambda(\theta) = \int_0^{2\pi} C\Big(\frac{m\theta}{n} + t\Big) f(t)\, dt, \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}\,.$$

Then it is proved in [1] that if the set $\Omega = \{\theta \in \mathbb{R}/2\pi\mathbb{Z},\ \lambda(\theta) = 0\}$ is nonempty and for every $\theta \in \Omega, \lambda'(\theta) \neq 0$, then there exists an $R_0 > 0$ such that every solution $x(t)$ of (3) with initial value $(x(t_0), x'(t_0))$ such that $(x(t_0))^2 + (x'(t_0))^2 > R_0^2$ for some $t_0 \in \mathbb{R}$, goes to infinity in the future or in the past.

If $\Omega = \emptyset$, Liu [7], by applying Ortega's version of Moser's twist theorem [11], proved that all solutions of (3) are bounded provided $f \in C^6$. The author [12] generalized Liu's results to nonlinear equations, which are a little more general than (1), under some additional conditions. For more recent results on the boundedness and unboundedness problem of solutions of equations which are similar to (1), we refer [2–6, 10, 13, 14] and the references therein.

In case $\Omega \neq \emptyset$, no boundedness results for solutions of (1) are available as far as the author knows. As a special case, we assume in this paper $\lambda(\theta) \equiv 0$. In this case, the higher-order terms of perturbation must be considered. After a series of somewhat tedious calculations, we obtain some relations between the higher-order terms. By using a method similar to the one used in [7] and by applying Ortega's version of Moser's twist theorem, we obtain some sufficient conditions for the boundedness and unboundedness of all the solutions of (1) in case $\lambda(\theta) \equiv 0$. The results obtained in this paper are natural generalizations and refinements of the results obtained in [7] for the case $m = 1$.

The method of proving the boundedness of solutions of (1) is as follows. By means of action and angle variables transformations, equation (1) is , outside of a large disc $D_r = \{(x, x') \in \mathbb{R}^2, x^2 + (x')^2 \leq r^2\}$ in the (x, x')-plane, transformed into a perturbation of an integrable Hamiltonian system. The Poincaré map of the transformed system is close to a so-called twist map in $\mathbb{R}^2/D_r$. Then Ortega's version of Moser's twist theorem guarantees the existence of arbitrary large invariant curves diffeomorphic to circles and surrounding the origin in the

(x, x')-plane. Every such curve is the base of a time-periodic and flow-invariant cylinder in the extended phase space $(x, x', t) \in \mathbb{R}^2 \times \mathbb{R}$, which confines the solutions in the interior and which leads to a bound of these solutions.

2. Generalized polar coordinates transformation

Introduce a new variable $y = \phi_p(x')$, then (1) is equivalent to the planar system

$$x' = \phi_q(y), \quad y' = -(p-1)\big[\alpha\phi_p(x^+) - \beta\phi_p(x^-)\big] + f(t)\,, \tag{4}$$

where $q = \frac{p}{p-1}$ is the conjugate exponent of p.

Let $u = \sin_p t$ be the solution of the initial value problem

$$\begin{aligned}&\big(\phi_p(u')\big)' + (p-1)\phi_p(u) = 0\\ &u(0) = 0, \quad u'(0) = 1,\end{aligned}$$

which for $t \in [0, \frac{\pi_p}{2}]$ can be expressed implicitly by (see [8])

$$t = \int_0^{\sin_p t} \frac{ds}{(1-s^p)^{\frac{1}{p}}}.$$

It follows from [9], that $u = \sin_p t$ can be extended to $\mathbb{R}$ as a $2\pi_p$-periodic odd C^2-function which satisfies $\sin_p t : [0, \frac{\pi_p}{2}] \to [0,1]$ and $\sin_p(\pi_p - t) = \sin_p t$ for $t \in [\frac{\pi_p}{2}, \pi_p]$, $\sin_p(2\pi_p - t) = -\sin_p t$ for $t \in [\pi_p, 2\pi_p]$.

Let $S(t)$ be the solution of the initial value problem

$$\begin{aligned}&\big(\phi_p(x')\big)' + (p-1)\big[\alpha\phi_p(x^+) - \beta\phi_p(x^-)\big] = 0\\ &x(0) = 0, \quad x'(0) = 1.\end{aligned}$$

Then it is well-known that $S \in C^2(\mathbb{R})$ is $\frac{2\pi_p}{n}$-periodic which can be given by

$$S(t) = \begin{cases} \alpha^{-\frac{1}{p}} \sin_p \alpha^{\frac{1}{p}} t, & t \in \big[0, \alpha^{-\frac{1}{p}}\pi_p\big]\\ -\beta^{-\frac{1}{p}} \sin_p \beta^{\frac{1}{p}}\big(t - \alpha^{-\frac{1}{p}}\pi_p\big), & t \in \big[\alpha^{-\frac{1}{p}}\pi_p, \frac{2\pi_p}{n}\big]\end{cases}$$

Define $C(t) = \phi_p(S'(t))$, then $C \in C^1(\mathbb{R})$ is $\frac{2\pi_p}{n}$-periodic and the equality holds

$$|C(t)|^q + \alpha(S^+(t))^p + \beta(S^-(t))^p \equiv 1, \quad t \in \mathbb{R} \tag{5}$$

holds. For $\rho > 0, \theta \pmod{2\pi_p}$, we define the canonical transformation $T : (\rho, \theta) \to (x, y)$ as

$$x = d^{\frac{1}{p}}\rho^{\frac{1}{p}} S\Big(\frac{\theta}{n}\Big), \quad y = d^{\frac{1}{q}}\rho^{\frac{1}{q}} C\Big(\frac{\theta}{n}\Big), \quad d = nq.$$

Under this transformation and by using (11), system (4) is changed into the generalized polar coordinates system

$$\rho' = \frac{d^{\frac{1}{p}}}{n}\rho^{\frac{1}{p}}S'\Big(\frac{\theta}{n}\Big)f(t), \qquad \theta' = n - \frac{d^{\frac{1}{p}}}{p}\rho^{-\frac{1}{q}}S\Big(\frac{\theta}{n}\Big)f(t). \tag{6}$$

If we further define $r = \rho^{\frac{1}{q}}$, then system (6) is of the form

$$r' = d^{\frac{-1}{q}}S'\Big(\frac{\theta}{n}\Big)f(t), \qquad \theta' = n - d^{\frac{1}{p}}p^{-1}r^{-1}S\Big(\frac{\theta}{n}\Big)f(t). \tag{7}$$

Since the right side of (7) is only C^1 or C^2 in θ, we cannot apply Moser's twist theorem directly, therefore we change the rule of θ and t as follows:

For $r \gg 1$, we can solve $t = t(\theta)$ from the second equation of (7) and (7) can be rewritten as

$$\frac{dr}{d\theta} = \frac{d_1 S'(\frac{\theta}{n})f(t)}{1 - d_2 r^{-1}S(\frac{\theta}{n})f(t)}, \qquad \frac{dt}{d\theta} = \frac{1}{n\Big(1 - d_2 r^{-1}S(\frac{\theta}{n})f(t)\Big)}, \tag{8}$$

where $d_1 = d^{\frac{-1}{q}}/n$, $d_2 = d^{\frac{1}{p}}/(np)$, $\theta \in [0, 2n\pi_p]$.

For $r_0 \gg 1$, let $(r(\theta), t(\theta)) = (r(\theta; r_0, t_0), t(\theta; r_0, t_0))$ be the solution of (8) with initial value (r_0, t_0). Then for large initial value, i.e., for $r_0 \gg 1$, by the boundedness of f, S and S', for $\theta \in [0, 2n\pi_p]$, we have the expressions

$$\begin{aligned} r(\theta) &= r_0 + \mu_0(\theta, t_0) + \mu_1(\theta, t_0)r_0^{-1} + \mu_2(\theta, t_0)r_0^{-2} + O(r_0^{-3}) \\ t(\theta) &= t_0 + \frac{\theta}{n} + \lambda_1(\theta, t_0)r_0^{-1} + \lambda_2(\theta, t_0)r_0^{-2} + \lambda_3(\theta, t_0)r_0^{-3} + O(r_0^{-4}). \end{aligned} \tag{9}$$

From the first equation of (9), we obtain for $r_0 \gg 1$

$$r^{-1}(\theta) = r_0^{-1}\big[1 - \mu_0(\theta, t_0)r_0^{-2} + (\mu_0^2(\theta, t_0) - \mu_1(\theta, t_0))r_0^{-2} + O(r_0^{-3})\big]. \tag{10}$$

Substituting (9), (10) into (8) and integrating from 0 to θ, we obtain

$$\begin{aligned} r(\theta) = r_0 &+ d_1\int_0^\theta S'\Big(\frac{\tau}{n}\Big)f(t(\tau))\,d\tau \\ &+ d_1 d_2\int_0^\theta r^{-1}(\tau)S\Big(\frac{\tau}{n}\Big)S'\Big(\frac{\tau}{n}\Big)f^2(t(\tau))\,d\tau \\ &+ d_1 d_2^2\int_0^\theta r^{-2}(\tau)S^2\Big(\frac{\tau}{n}\Big)S'\Big(\frac{\tau}{n}\Big)f^3(t(\tau))\,d\tau + O(r_0^{-3}) \end{aligned} \tag{11}$$

$$t(\theta) = t_0 + \frac{\theta}{n} + \frac{d_2}{n}\int_0^\theta r^{-1}(\tau)S\Big(\frac{\tau}{n}\Big)f(t(\tau))\,d\tau + \frac{d_2^2}{n}\int_0^\theta r^{-2}(\tau)S^2\Big(\frac{\tau}{n}\Big)f^2(t(\tau))\,d\tau + \frac{d_2^3}{n}\int_0^\theta r^{-3}(\tau)S^3\Big(\frac{\tau}{n}\Big)f^3(t(\tau))\,d\tau + O(r_0^{-4}). \tag{12}$$

Substituting (9) and (10) into (11) and (12) respectively, we obtain formulas for $\mu_0, \lambda_1, \mu_1, \lambda_2, \mu_2$ and λ_3. Let $\mu_k(t) = \mu_k(2n\pi_p, t)$, $k = 0, 1, 2, \ldots$; $\lambda_m(t) = \lambda_m(2n\pi_p, t)$, $m \in \mathbb{N}$, we obtain by using $S(0) = S(2\pi_p) = 0$,

$$\begin{aligned}
\mu_0(t) &= d_1 \int_0^{2n\pi_p} S'\Big(\frac{\theta}{n}\Big) f\Big(t + \frac{\theta}{n}\Big)\,d\theta = -nd_1 \int_0^{2\pi_p} S(\theta)f'(t+\theta)\,d\theta,\\
\lambda_1(t) &= \frac{d_2}{n}\int_0^{2n\pi_p} S\Big(\frac{\theta}{n}\Big) f\Big(t + \frac{\theta}{n}\Big)\,d\theta = d_2 \int_0^{2\pi_p} S(\theta)f(t+\theta)\,d\theta\\
\mu_1(t) &= nd_1d_2\Big[\int_0^{2\pi_p} S'(\theta)f'(t+\theta)\int_0^\theta S(\tau)f(t+\tau)\,d\tau\,d\theta\\
&\quad + \int_0^{2\pi_p} S(\theta)S'(\theta)f^2(t+\theta)\,d\theta\Big].
\end{aligned} \tag{13}$$

If we simply write $S(\cdot)$, $f(t+\cdot)$ as S and f respectively, we can also obtain the expressions

$$\begin{aligned}
\lambda_2(t) &= d_2^2(2-p)\Big[\int_0^{2\pi_p} Sf'\int_0^\theta Sf + \int_0^{2\pi_p} S^2f^2\Big] + (p-1)\lambda_1(t)\lambda_1'(t)]\\
\mu_2(t) &= nd_1d_2^2\Big[\int_0^{2\pi_p} S'f'\int_0^\theta Sf'\int_0^\tau Sf + \int_0^{2\pi_p} S'f'\int_0^\theta S^2f^2\\
&\quad - (p-1)\int_0^{2\pi_p} S'f'\int_0^\theta Sf\int_0^\tau S'f + \frac{1}{2}\int_0^{2\pi_p} S'f''\Big(\int_0^\theta Sf\Big)^2\\
&\quad + 2\int_0^{2\pi_p} SS'ff'\int^\theta Sf - (p-2)\int_0^{2\pi_p} S^2S'f^3\\
&\quad + (p-1)\int_0^{2\pi_p} SS'f^2\int_0^\theta Sf'\Big]\\
\lambda_3(t) &= d_2^3\Big[\int_0^{2\pi_p} Sf'\int_0^\theta Sf'\int_0^\tau Sf + \int_0^{2\pi_p} Sf'\int_0^\theta S^2f^2\\
&\quad - (p-1)\int_0^{2\pi_p} Sf'\int_0^\theta Sf\int_0^\tau S'f + \frac{1}{2}\int_0^{2\pi_p} Sf''\Big(\int_0^\theta Sf\Big)^2\\
&\quad + 2\int_0^{2\pi_p} S^2ff'\int_0^\theta Sf - 2(p-1)\int_0^{2\pi_p} S^2f^2\int_0^\theta S'f
\end{aligned} \tag{14}$$

$$+\int_0^{2\pi_p} S^3f^3+(p-1)^2\int_0^{2\pi_p} Sf\Big(\int_0^\theta S'f\Big)^2$$
$$-(p-1)\int_0^{2\pi_p} Sf'\int_0^\theta S'f\int_0^\theta Sf-(p-1)\int_0^{2\pi_p} Sf\int_0^\theta S'f'\int_0^\tau Sf$$
$$-(p-1)\int_0^{2\pi_p} Sf\int_0^\theta SS'f^2\Big].$$

Using integration by parts, for $t\in[0,2\pi_p]$, we can verify the equations

$$\mu_0(t)=-(p-1)\lambda_1'(t) \tag{15}$$
$$\lambda_2'(t)=\Big(\frac{p-2}{p-1}\Big)\mu_1(t)+(p-1)\lambda_1(t)\lambda_1''(t)+\frac{p}{2}\Big(\lambda_1'(t)\Big)^2. \tag{16}$$

From above equations, under the assumption $\lambda_1(t)\equiv 0$, after a series of tedious calculations and simplifications, we obtain that

$$\lambda_3'(t)=\frac{(2p-3)}{p-1}\mu_2(t). \tag{17}$$

Especially for $p=2$, we have

$$\lambda_2(t)=\lambda_1(t)\lambda_1'(t), \tag{18}$$

and for $\lambda_1(t)\equiv 0$,

$$\lambda_3'(t)=\mu_2(t). \tag{19}$$

For $p\neq 2,\ \frac{3}{2}$ and $\lambda_1(t)\equiv 0$, we obtain from (16) and (17) the relations

$$\mu_1(t)=\frac{p-1}{p-2}\lambda_2'(t)$$
$$\mu_2(t)=\frac{p-1}{2p-3}\lambda_3'(t).$$

Remark 1. For $k\in\mathbb{N}$, it is conjectured that under the assumption $\lambda_1(t)=\lambda_2(t)=\ldots=\lambda_{k-1}\equiv 0$, for $p\neq\frac{k+1}{k}$, the following equality holds:

$$\mu_k(t)=\frac{p-1}{k(p-1)-1}\lambda_{k+1}'(t). \tag{20}$$

For $k>1$, $p=2$ the conjecture (29) becomes

$$\mu_k(t)=\frac{\lambda_{k+1}'(t)}{k-1}.$$

3. Bounded motions of planar mappings

In this section, we adopt the notations used in [1]. Given $\sigma > 0$, let the set E_σ be the exterior of the open ball B_σ centered at the origin and of radius σ, that is $E_\sigma = \mathbb{R}^2 - B_\sigma$. Then $E_\sigma = \{(\theta, r) | r \geq \sigma,\, \theta \in S^1\}$.

Lemma 1. *Let P be the Poincaré map of* (9) *having the form*

$$\begin{aligned} t_1 &= t_0 + 2\pi_p + \lambda_k(t_0) r_0^{-k} + F_{k+1}(t_0, r_0) \\ r_1 &= r_0 + \mu_{k-1}(t_0) r_0^{-(k-1)} + G_k(t_0, r_0), \end{aligned} \tag{21}$$

where $k \in \mathbb{N}$, $\lambda_k(t) \neq 0$ for all $t \in S^1$, λ_k, μ_{k-1}, F_{k+1}, G_k are C^∞-functions and $F_{k+1} = O(r_0^{-(k+1)})$, $G_k = O(r_0^k)$ for $r_0 \gg 1$ and periodic in t . If $\mu_{k-1}(t) = -c_k \lambda_k'(t)$ for some nonzero constant c_k, then the map P has an invariant curve $\Gamma \subset [0,\, \pi_p] \times [\frac{1}{\Delta},\, \Delta]$ for some $\Delta > 1$.

Proof. In order to prove the above proposition, we need a variant of Moser's twist theorem which is due to Ortega[10]. Let $A = S^1 \times [a,\, b]$ be a finite cylinder with universal cover $\mathbb{R} \times [a,\, b]$. Consider the map $M : A \to S^1 \times \mathbb{R}$. We assume the map M has the intersection property, that is , for every Jordan curve $\Gamma \subset A$ which is homotopic to the circle $u = \mathit{constant}$ satisfies $M(\Gamma) \cap \Gamma \neq \emptyset$. Suppose that a lift of M has the form

$$\begin{aligned} \theta_1 &= \theta + 2m\pi + \delta L_1(\theta, u) + \delta\psi_1(\theta, u) \\ u_1 &= u + \delta L_2(\theta, u) + \delta\psi_2(\theta, u), \end{aligned} \tag{22}$$

where $m \in \mathbb{N}$, $\delta \in (0, 1)$ is a parameter and L_1, L_2, ψ_1 and ψ_2 are functions satisfying

$$L_1 \in C^6(A), \quad L_1(\theta, u) > 0, \quad \frac{\partial L_1}{\partial u}(\theta, u) > 0 \qquad \forall\, (\theta, u) \in A \tag{23}$$

$$L_2,\, \psi_1,\, \psi_2 \in C^5(A). \tag{24}$$

In addition we assume that there exists a function $I : A \to \mathbb{R}$ satisfying $I \in C^6(A)$,

$$\frac{\partial I}{\partial u}(\theta, u) > 0 \qquad \forall\, (\theta, u) \in A \tag{25}$$

$$L_1(\theta, u)\frac{\partial I}{\partial \theta}(\theta, u) + L_2(\theta, u)\frac{\partial I}{\partial u}(\theta, u) = 0 \qquad \forall\, (\theta, u) \in A. \tag{26}$$

Define on $[a, b]$ the functions $\overline{I}(u) = \max_{\theta\in\mathbb{R}} I(\theta, u)$ and $\underline{I}(u) = \min_{\theta\in\mathbb{R}} I(\theta, u)$. Since the function I is periodic in θ, the above two functions are well-defined and finite. ∎

Lemma 2. ([10, Theorem 3.1]) *Let M be such that* (22)–(24) *hold. Assume in addition that there exists a function I satisfying* (25) *and* (26). *If there exist numbers a' and b' with $a < a' < b' < b$ such that*

$$\overline{I}(a) < \underline{I}(a') \le \overline{I}(a') < \underline{I}(b') \le \overline{I}(b') < \underline{I}(b),$$

then there exist ε, $\delta_0 > 0$ such that if $0 < \delta < \delta_0$ and $\|\psi_1\|_{C^5(A)} + \|\psi_2\|_{C^5(A)} < \varepsilon$, the map M has an invariant curve $\Gamma \subset A$. The constant ε is independent of δ. Furthermore, if we denote by $R(\Gamma,\delta) \in S^1$ the rotation number of M, then

$$\lim_{\delta \to 0} R(\Gamma, \delta) = 0.$$

Remark 2. In Lemma 1, it can be proved that the assumptions $L_1 > 0$, $\frac{\partial L_1}{\partial u} > 0$ can be replaced by $L_1 < 0$, $\frac{\partial L_1}{\partial u} < 0$.

Proof. Now taking Remark 2 into consideration, we can assume, without loss of generality, that $\lambda_k(t) > 0$ for all $t \in R$. For fixed $\Delta > 1$, we introduce a new variable u varying in the closed interval $[\frac{1}{\Delta},\ \Delta]$ and a small positive parameter δ by the formula

$$r = u\delta$$

and the positive constant $\Delta > 1$ will be determined later. In the new variables (t, u), eq. (21) has the form

$$\begin{aligned} t_1 &= t + 2\pi_p + \lambda_k(t)\delta^k u^k + H_1(t, \delta, u) \\ u_1 &= u - c_k \lambda_k'(t)\delta^k u^{k+1} + H_2(t, \delta, u), \end{aligned} \tag{27}$$

where H_1, $H_2 = O(\delta^{k+1})$ and C^∞ in t and u. If we introduce a new variable $\varepsilon = \delta^k$, then (27) has the form

$$\begin{aligned} t_1 &= t + 2\pi_p + \lambda_k(t)\varepsilon u^k + J_1(t, \varepsilon, u) \\ u_1 &= u - c_k \lambda_k'(t)\varepsilon u^{k+1} + J_2(t, \varepsilon, u), \end{aligned}$$

where $J_1,\ J_2 = O(\varepsilon^{1+\lambda})$ for some $\lambda > 0$.

Define a function $Q(t)$ as

$$Q(t) = \exp\left[\int_0^t \frac{-c_k \lambda_k'(s)\, ds}{\lambda_k(s)}\right] = \left(\frac{\lambda_k(t)}{\lambda_k(0)}\right)^{-c_k},$$

then Q is positive and $2\pi_p$–periodic. Let $I(t, u) = uQ(t)$, $L_1(t, u) = u^k \lambda_k(t)$, $L_2(t, u) = -c_k u^{k+1} \lambda_k'(t)$. Then $I > 0$, $\frac{\partial I}{\partial u} = Q > 0$. Now we choose the constant Δ with $\Delta = 4\,\frac{Q_{max}}{Q_{min}} \ge 4$, where $Q_{max} = \max_{t \in R} Q(t)$, $Q_{min} = \min_{t \in R} Q(t)$. Then it is easy to see that L_1, L_2, I, H_1, $H_2 \in C^\infty(A)$ and for all $(t, u) \in A$,

$$L_1(t, u) > 0, \quad \frac{\partial L_1}{\partial u}(t, u) > 0$$
$$L_1(t, u)\frac{\partial I}{\partial t}(t, u) + L_2(t, u)\frac{\partial I}{\partial u}(t, u) = 0,$$

and $\overline{I}(\Delta^{-1}) < \underline{I}(\Delta_1) \le \overline{I}(\Delta_1) < \underline{I}(\Delta_2) \le \overline{I}(\Delta_2) < \underline{I}(\Delta)$, where $\Delta_1 = 1 < \Delta_2 = \frac{\Delta}{2}$. Now, by applying Lemma 1, we see for each $\varepsilon \ll 1$, the map M has an invariant closed curve diffeomorphic to $u = const$. ∎

4. Main result

We can now state and prove the main result of this paper.

Theorem. *Consider equation* (1)*. Let $\mu_{k-1}(t)$, $\lambda_k(t)$, $k \in \mathbb{N}$, be given as in Section* 2*. Assume $\lambda_1(t) \equiv 0$.*

(I) *If one of the conditions*

- **(i)** $p = 2$, $\lambda_3(t) \neq 0$ *and* $\mu_1(t) \equiv 0$ *for all* $t \in \mathbb{R}$
- **(ii)** $p \neq 2$, $\lambda_2(t) \neq 0$ *for all* $t \in \mathbb{R}$
- **(iii)** $p \neq 2$, $p \neq \frac{3}{2}$, $\lambda_2(t) \equiv 0$ *and* $\lambda_3(t) \neq 0$ *for all* $t \in \mathbb{R}$

holds, then every solution of (1) *is bounded, i.e., if $x(t)$ is a solution of* (1)*, then*

$$\sup_{t\in\mathbb{R}} \left(x^2(t) + (x'(t))^2\right) < +\infty.$$

(II) *If, however, $p = 2$, $\lambda_1(t) \equiv 0$ and $\mu_1(t) \neq 0$, for all $t \in \mathbb{R}$, then every solution with large initial value, i.e., $(x(0))^2 + (x'(0))^2 \gg 1$, are unbounded in the future if $\mu_1(t) > 0$ for all $t \in \mathbb{R}$, or they are unbounded in the past if $\mu_1(t) < 0$ for all $t \in \mathbb{R}$.*

Proof. `Proof of (I)`: We need only prove the fact that every solution of (7) is bounded if one of the assumptions of (i)–(iii)holds.

For $p > 1$ and $f \in C^\infty(S^1)$, the right side of (7) satisfies a local Lipschitz condition for $r > 0$, hence the existence and uniqueness of solutions to initial value is guaranteed. Therefore we need to consider the case $r_0 \gg 1$, only.

Let $(r(\theta), t(\theta))$ be the solution of (9) satisfying $(r(0), t(0)) = (r_0, t_0)$ with $r_0 \gg 1$. Since $\lambda_1(t) \equiv 0$, we obtain from (15)–(19), $\mu_0(t) \equiv 0$. $\lambda_2(t) \equiv 0$ if $p = 2$ and $\mu_1(t) = \frac{p-1}{p-2}\lambda_2'(t)$ if $p \neq 2$. For $p \neq 2$, $\frac{3}{2}$, $\lambda_2(t) \equiv 0$, then by (17), $\lambda_3'(t) = \frac{2p-3}{p-1}\mu_2(t)$. In all cases, Lemma 1 implies that the map P has an invariant curve for each fixed $r_0 \gg 1$. Every such curve is the base of a time-periodic and flow-invariant cylinder in the extended phase space $(r, \theta, t) \in \mathbb{R}^2 \times \mathbb{R}$, which confines the solutions in the interior and which leads to a bound of these solutions. Going back to the (x, x', t) space, we get the boundedness of solutions with large initial values, by uniqueness results, every solution of (1) is therefore bounded.

`Proof of (II)`. Now we assume $p = 2$, $\lambda(t) \equiv 0$, $\mu_1(t) > 0$ for all $t \in \mathbb{R}$. Let $\mu^* = \min_{t\in\mathbb{R}} \mu_1(t)$, then by the periodicity of μ_1, $\mu^* > 0$. For $r_0 \gg 1$, (which is equivalent to $(x(0))^2 + (x'(0))^2 \gg 1$), we obtain from (21)

$$r_1 = r_0 + \mu_2(t) r_0^{-2} + O(r_0^{-3}) \geq r_0 + \frac{\mu^*}{2} r_0^{-2}. \tag{28}$$

and

$$r_1 \leq r_0 + 2\mu_{max} r_0^{-2}, \tag{29}$$

where $\mu_{max} = \max_{t\in\mathbb{R}} \mu_1(t)$. Replacing r_0 by r_n, r_1 by r_{n+1} and, by induction, we get from (29)

$$r_{n+1} \leq r_0 + 2n\mu_{max} r_0^{-2}, \tag{30}$$

which implies that r_n is defined in the future. Similarly, we obtain from (28)

$$r_{n+1} \geq r_n + \frac{1}{2}\mu^* r_n^{-2} > r_n, \tag{31}$$

which implies that r_n is monotone increasing. We claim $\lim_{n\to+\infty} r_n = +\infty$. Otherwise, let $\lim_{n\to+\infty} r_n = r^* < +\infty$, then by taking limits in both sides of (31), we obtain $r^* \geq r^* + \frac{\mu^*}{2} r^* > r^*$, which is a contradiction. The case $\mu_1(t) < 0$ can be proved similarly. This finishes our proof of the theorem. ∎

Remark 3. We see from (16) that the case $p = 2$ and the case $p \neq 2$ have a great difference. If the conjecture (20) holds, then our theorem can be further generalized by applying Lemma 1. Moreover, the assumption $f \in C^\infty(S^1)$ can be replaced by $f \in C^6(S^1)$.

Example. Assume $p = 2$, $f(t) = \sin t$, $\alpha > 0$, $\beta > 0$, $\alpha \neq \beta$ satisfying (2). Then (1) reduces to

$$x'' + \alpha x^+ - \beta x^- = \sin t.$$

Let the Fourier expression of $S(t)$ be $S(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \big(a_k \cos knt + b_k \sin knt\big)$, where

$$a_k = \frac{1}{\pi}\int_0^{2\pi} S(t)\cos knt\,dt, \qquad b_k = \frac{1}{\pi}\int_0^{2\pi} S(t)\sin knt\,dt$$

for $k = 0, 1, \ldots$. We can get the explicit expressions for a_k, b_k as

$$a_k = \frac{n(1+\cos(kn\pi/\sqrt{\alpha}))(\beta-\alpha)}{2\pi(k^2n^2-\alpha)(k^2n^2-\beta)}, \qquad b_k = \frac{n\sin(kn\pi/\sqrt{\alpha})(\beta-\alpha)}{2\pi(k^2n^2-\alpha)(k^2n^2-\beta)}$$

if $\alpha \neq k^2n^2$, $\beta \neq k^2n^2$ for all $k \in \mathbb{N}$. From the expression of λ_1, it is not difficult to obtain

$$\lambda_1(t) = \begin{cases} \frac{1}{\sqrt{2}}\pi\big(a_1 \sin t + b_1 \cos t\big), & \text{if } n = 1 \\ 0, & \text{if } n \geq 2. \end{cases}$$

For $n \geq 2$, we have $\lambda_2(t) \equiv 0$, and for $n \geq 3$, we have $\mu_0(t) = \mu_1(t) \equiv 0$, $\lambda_3(t) \equiv const > 0$. Since for $n \geq 3$, we have $\lambda_1(t) = \lambda_2(t) \equiv 0$ in our example, the results of previous literature can not be applied in this case. But our theorem implies the boundedness of all the solutions of (1) for $p = 2$ and $n \geq 3$.

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