Nonlinear Boundary Value Problems
Involving the $p$-Laplacian
and $p$-Laplacian-Like Operators

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Abstract. We study nonlinear boundary value problems for systems driven by the vector $p$-Laplacian or $p$-Laplacian-like operators and having a maximal monotone term. We consider periodic problems and problems with nonlinear boundary conditions formulated in terms of maximal monotone operators. This way we achieve a unified treatment of the classical Dirichlet, Neumann and periodic problems. Our hypotheses permit the presence of Hartman and Nagumo-Hartman nonlinearities, partially extending this way some recent works of Mawhin and his coworkers.

Keywords: Ordinary $p$-Laplacian, $p$-Laplacian-like operator, maximal monotone operator, Nagumo-Hartman nonlinearity, fixed point, complete continuity

MSC 2000: 34B15, 34C25

1. Introduction

In this paper we study the following two nonlinear boundary value problems in $\mathbb{R}^N$:

\begin{equation}
\begin{cases}
(\alpha(x'(t)))' \in A(x(t)) + F(t, x(t), x'(t)) \quad \text{a.e. on } T = [0, b] \\
x(0) = x(b), \ x'(0) = x'(b),
\end{cases}
\end{equation}

and

\begin{equation}
\begin{cases}
\|x'(t)\|^{p-2}x'(t) \in A(x(t)) + F(t, x(t), x'(t)) \quad \text{a.e. on } T = [0, b] \\
\phi_p(x'(0)), -\phi_p(x'(b)) \in \xi(x(0), x(b)), \ 1 < p < \infty.
\end{cases}
\end{equation}

Here $\alpha : \mathbb{R}^N \to \mathbb{R}^N$ is a suitable homeomorphism which is not in general homogeneous, $A : D(A) \subseteq \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map, $F : T \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a real-valued function, $\phi_p(t) = \int_0^t |s|^{p-2}s \, ds$, $1 < p < \infty$. The mapping $\xi : \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a given maximal monotone operator.

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$2^{\mathbb{R}^N} \setminus \{\emptyset\}$ is a multivalued in general nonlinearity satisfying Caratheodory type conditions, $\varphi_p : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is the homeomorphism defined by

$$\varphi_p(r) = \begin{cases} \|r\|^{p-2}r & \text{if } r \neq 0 \\ 0 & \text{if } r = 0 \end{cases}$$

and $\xi : D(\xi) \subseteq \mathbb{R}^N \times \mathbb{R}^N \rightarrow 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is a maximal monotone map.

Boundary value problems involving the ordinary $p$-Laplacian have been the focus of attention of many researchers in the last decade. Most of the works deal with the scalar problem. We refer to the works of Boccardo-Drabek-Giacchetti-Kučerka [2], De Coster [4], Del Pino-Manasevich-Murua [5], Fabry-Fayyad [8], Guo [11] and the references therein. We also mention the work of Dang-Oppenheimer [3], where the ordinary scalar $p$-Laplacian is replaced by a one-dimensional possibly nonhomogeneous nonlinear differential operator.

Recently in a series of interesting papers, Mawhin and coworkers studied systems driven by the ordinary vector $p$-Laplacian or $p$-Laplacian like operators and having primarily periodic boundary conditions. We refer to the papers of Manasevich-Mawhin [16] Mawhin [18, 19] and Mawhin-Urena [20]. As the Nagumo-Hartman condition used here is distinct from the one used by Mawhin-Urena [20] we provide a partial extension of the works by Mawhin [18] and Mawhin-Urena [20], where the authors employ nonlinearities of the Hartman and Nagumo-Hartman type. Also in these works the ordinary vector $p$-Laplacian with periodic boundary conditions is used, $A \equiv 0$ and the nonlinearity is single-valued.

The problems that we study here are more general since they involve the maximal monotone operator $A$, which in the case of Problem (1) is not necessarily defined everywhere (see hypotheses $H(A)_1$). This way we incorporate in our framework differential variational inequalities. Moreover, in the case of Problem (2), the nonlinear multivalued boundary conditions used here achieve a unified treatment of the Dirichlet, Neumann and periodic problems and go beyond them (see Section 5). This way we extend the semilinear works (i.e., $p = 2$) of Erbe-Krawcewicz [7], Frigon [9], Kandilakis-Papageorgiou [14] and Halidias-Papageorgiou [12] and the recent nonlinear works of Kyrtsi-Matzakos-Papageorgiou [15] and Papageorgiou-Papageorgiou [21]. Our approach is based on nonlinear operator theory and fixed point arguments.

2. Mathematical background

Let $(\Omega, \Sigma)$ be a measurable space and $X$ a separable Banach space. We introduce the notations

$$P_f(c)(X) = \{A \subseteq X : A \text{ is nonempty, closed (and convex)}\}$$

$$P_{(w)k}(c)(X) = \{A \subseteq X : A \text{ is nonempty, (weakly) compact (and convex)}\}.$$
A multifunction $F : \Omega \to P_f(X)$ is said to be measurable, if for all $x \in X$ \( \omega \to d(x,F(\omega)) = \inf[\|x - u\| : u \in F(\omega)] \) is measurable. Also we say that $F : \Omega \to 2^X \setminus \{\emptyset\}$ is graph measurable, if $\text{Gr}F = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \subseteq \Sigma \times B(X)$, with $B(X)$ being the Borel $\sigma$-field of $X$. For multifunctions with values in $P_f(X)$ measurability implies graph measurability, while the converse holds if $\Sigma$ is complete. Next let $(\Omega, \Sigma, \mu)$ be a finite measure space and $F : \Omega \to 2^X \setminus \{\emptyset\}$ a multifunction. For $1 \leq p \leq \infty$ we introduce the set
\[
S_F^p = \{ f \in L^p(\Omega, X) : f(\omega) \in F(\omega) \; \mu \text{-a.e. on } \Omega \}.
\]
Let $Y, Z$ be Hausdorff topological spaces. A multifunction $G : Y \to 2^Z \setminus \{\emptyset\}$ is said to be upper semicontinuous (usc for short) (respectively lower semicontinuous (lsc for short)) if for every closed set $C \subseteq Z$, the set $G^-(C) = \{y \in Y : G(y) \cap C \neq \emptyset\}$ (respectively the set $G^+(C) = \{y \in Y : G(y) \subseteq C\}$) is closed in $Y$. If $Z$ is regular and $F$ is $P_f(X)$-valued and usc, then it has a closed graph, i.e., $\text{Gr}G = \{(y, z) \in Y \times Z : z \in G(y)\}$ is closed in $Y \times Z$. The converse is true if $G$ is locally compact.

Now let $X$ be a reflexive Banach space and $X^*$ its topological dual. Recall that a monotone, demicontinuous operator $A : X \to X^*$ is maximal monotone. Also a maximal monotone coercive operator, is surjective. When $X = H$ (Hilbert space) and $A : D(A) \subseteq H \to 2^H$ is a maximal monotone operator, then for every $\lambda > 0$ we introduce the well-known operators
\[
J_\lambda = (I + \lambda A)^{-1} \quad \text{(resolvent of } A)\]
\[
A_\lambda = \frac{1}{\lambda}(I - J_\lambda) \quad \text{(Yosida approximation of } A).\]
Both operators are single-valued and defined on all of $H$. Moreover, $J_\lambda$ is nonexpansive, while $A_\lambda$ is Lipschitz continuous with constant $\frac{1}{\lambda}$ (hence $A_\lambda$ is maximal monotone).

We return to the general case of $X$ being a reflexive Banach space. An operator $A : X \to 2^{X^*}$ is said to be pseudomonotone, if
\begin{enumerate}[(a)]
\item for all $x \in X$, $A(x) \in P_{wkc}(X^*)$;
\item $A$ is usc from every finite dimensional subspace $Z$ of $X$ into $X^*_w$;
\item if $x_n \xrightarrow{w} x$ in $X$, $x^*_n \in A(x_n)$ and $\limsup_{n \to \infty} \langle x^*_n, x_n - x \rangle \leq 0$, then for every $y \in X$, there exists $x^*(y) \in A(x)$ such that $\langle x^*(y), x - y \rangle \leq \liminf_{n \to \infty} \langle x^*_n, x_n - y \rangle$.
\end{enumerate}
We say that $A : D(A) \subseteq X \to 2^{X^*}$ is generalized pseudomonotone, if for all $x^*_n \in A(x_n)$ such that $x_n \rightharpoonup x$ in $X$, $x^*_n \rightharpoonup x^*$ in $X^*$ and $\limsup_{n \to \infty} \langle x^*_n, x_n - x \rangle \leq 0$, we have $x^* \in A(x)$ and $\langle x^*_n, x_n \rangle \to \langle x^*, x \rangle$. A maximal monotone operator is generalized pseudomonotone and a pseudomonotone operator is generalized pseudomonotone. A generalized pseudomonotone operator is pseudomonotone,
Proposition 2.1. If $X$, $Y$ are Banach spaces with $X$ reflexive, $W$ is a bounded open subset of $X$ with $0 \in W$, $G : W \to P_{wkc}(Y)$ is usc from $W$ into $Y_w$, bounded, and $K : Y \to X$ is completely continuous, then one of the following alternatives holds:

(a) there exist $x_0 \in \partial W$ and $s \in (0, 1)$ such that $x_0 \in s(K \circ G)(x_0)$; or

(b) $\Phi = G \circ K$ has a fixed point (i.e., there exist $\varpi \in W$ such that $\varpi \in \Phi(\varpi)$).

3. Problems with $p$-Laplacian–like operators

In this section we deal with Problem (1) and we do not require that $D(A) = \mathbb{R}^N$. Our analysis of Problem (1) starts with the study of the auxiliary periodic problem

\[
\begin{cases}
-\left(\alpha(x'(t))\right)' + A_\lambda(x(t)) + \|x(t)\|^{p-2}x(t) = g(t) \quad \text{a.e. on } T = [0, b] \\
x(0) = x(b), \quad x'(0) = x'(b),
\end{cases}
\]

where $1 < p < \infty$, $g \in L^q(T, \mathbb{R}^N)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\lambda > 0$. We introduce the following hypotheses on the maps $a$ and $A$:

- $H(a)_1$: $a : \mathbb{R}^N \to \mathbb{R}^N$ is continuous, strictly monotone and there exists a function $\gamma : [0, +\infty) \to [0, +\infty)$ such that $\gamma(r) \to +\infty$ as $r \to +\infty$ for all $x \in \mathbb{R}^N$ we have $\gamma(\|x\|)\|x\| \leq \langle a(x), x \rangle_{\mathbb{R}^N}$.

- $H(A)_1$: $A : D(A) \subseteq \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map such that $0 \in A(0)$.

Remark 3.1. We emphasize that we do not require that $D(A) = \mathbb{R}^N$.

In what follows we shall use the two spaces $C^1_{per}(T, \mathbb{R}^N) = \{x \in C^1(T, \mathbb{R}^N) : x(0) = x(b), x'(0) = x'(b)\}$ and $W^{1,p}_{per}(T, \mathbb{R}^N) = \{x \in W^{1,p}(T, \mathbb{R}^N) : x(0) = x(b)\}$.

Proposition 3.2. If hypotheses $H(a)_1$ and $H(A)_1$ hold, then Problem (3) has a unique solution $x \in C^1_{per}(T, \mathbb{R}^N)$ such that $a(x') \in W^{1,q}_{per}(T, \mathbb{R}^N)$. 
Hereafter by \( \cdot \) some is maximal monotone. Indeed first note that \( S \) \( \in \mathbb{R}^N \) be defined by \( \eta(x) = x \). Then if \( h(t) = h^1_1(t) \) where \( h^1_1(t) = \sup_{r>0}[-r^p + r^{p-1} + \frac{1}{\lambda}r + \|g(t)||r + \|g(t)||] \), and \( R_0 > \max\{1, \|g\|\} \) where \( g = \frac{1}{\lambda} \int_0^y g(t)dt \), with all the above data we can apply Corollary 3.1 of Manasevich-Mawhin [16] and obtain a solution for (3). The uniqueness follows at once from hypotheses \( \text{H}(\alpha)_1 \) and the monotonicity of \( A_\lambda \) and strict monotonicity of \( \varphi_p \).

Let \( \hat{D} = \{ x \in C^1_{\text{per}}(T, \mathbb{R}^N) : a(x') \in W^{1,q}_\text{per}(T, \mathbb{R}^N) \} \). For \( \lambda > 0 \), let \( S_\lambda : \hat{D} \subseteq L^p(T, \mathbb{R}^N) \rightarrow L^q(T, \mathbb{R}^N) \) be the nonlinear operator defined by \( S_\lambda(x) = -(a(x'))' + \hat{A}_\lambda(x) \), where for every \( x \in \hat{D} \), \( \hat{A}_\lambda(x)(\cdot) = A_\lambda(x(\cdot)) \). Note that if \( x \in \hat{D} \), then \( A_\lambda(x(\cdot)) \in C(T, \mathbb{R}^N) \).

**Proposition 3.3.** If the hypothesis \( \text{H}(\alpha)_1 \) holds and \( \lambda > 0 \), then \( S_\lambda : \hat{D} \subseteq L^p(T, \mathbb{R}^N) \rightarrow L^q(T, \mathbb{R}^N) \) is maximal monotone.

**Proof.** Let \( J : L^p(T, \mathbb{R}^N) \rightarrow L^q(T, \mathbb{R}^N) \) be the continuous, strictly monotone (thus maximal monotone) operator defined by \( J(x)(\cdot) = \|x(\cdot)\|^{p-2}x(\cdot) \). From Proposition 3.2 we know that \( R(S_\lambda + J) = L^q(T, \mathbb{R}^N) \). We will show that \( S_\lambda \) is maximal monotone. Indeed first note that \( S_\lambda \) is monotone. Suppose that for some \( y \in L^p(T, \mathbb{R}^N) \) and some \( v \in L^q(T, \mathbb{R}^N) \), we have

\[
(S_\lambda(x) - v, x - y)_{qp} \geq 0 \quad \text{for all } x \in \hat{D}. \tag{4}
\]

Hereafter by \( (\cdot, \cdot)_{qp} \) we denote the duality brackets for the pair \( (L^q(T, \mathbb{R}^N), L^p(T, \mathbb{R}^N)) \). Since \( S_\lambda + J \) is surjective, we can find \( x_1 \in \hat{D} \) such that \( S_\lambda(x_1) + J(x_1) = v + J(y) \). Using this in (4) with \( x = x_1 \in \hat{D} \), we obtain \( y = x_1 \in \hat{D} \) since \( J \) is strictly monotone and \( v = S_\lambda(x_1) \).

Next we study of the following regular approximation of Problem (1):

\[
\begin{aligned}
&\left\{ \begin{array}{l}
(\alpha(x'(t)))' \in A_\lambda(x(t)) + F(t, x(t), x'(t)) \quad \text{a.e. on } T = [0, b] \\
x(0) = x(b), \quad x'(0) = x'(b),
\end{array} \right.
\end{aligned} \tag{5}
\]

where \( \lambda > 0 \). Our hypotheses on the data of (5) are the following:

**H(a)_2:** \( a : \mathbb{R}^N \rightarrow \mathbb{R}^N \) is a monotone map such that \( a(y) = c(y)y \) or \( a(y) = (c_k(y_k)y_k)_{k=1}^N \) for all \( y = (y)^N_{k=1} \in \mathbb{R}^N \), with \( c : \mathbb{R}^N \rightarrow \mathbb{R}_+ \) and \( c_k : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \), \( k \in \{1, ..., N\} \), continuous maps and for all \( y \in \mathbb{R}^N \) we have \( (a(y), y)_{\mathbb{R}^N} \geq c_0\|y\|^p \) for some \( c_0 > 0 \).

**H(F)_1:** \( F : T \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow P_{\text{cm}}(\mathbb{R}^N) \) is a multifunction such that

(i) for all \( x, y \in \mathbb{R}^N \), \( t \rightarrow F(t, x, y) \) is graph measurable;

(ii) for almost all \( t \in T \), \( (x, y) \rightarrow F(t, x, y) \) has closed graph;
(iii) for almost all \( t \in T \), all \( x, y \in \mathbb{R}^N \) and all \( v \in F(t, x, y) \) we have
\[
(v, x)_{\mathbb{R}^N} \geq -c_1\|x\|^p - c_2\|x\|^r\|y\|^{p-r} - c_3(t)\|x\|^s
\]
with \( c_1, c_2 > 0, c_3 \in L^1(T)_+, 1 \leq r, s < p; \)
(iv) there exists \( M > 0 \) such that if \( \|x_0\| = M \) and \((x_0, y_0)_{\mathbb{R}^N} = 0\), we can find a \( \delta > 0 \) such that for almost all \( t \in T \) we have
\[
\inf [(v, x)_{\mathbb{R}^N} + c_0\|y\|^p : \|x - x_0\| + \|y - y_0\| < \delta, v \in F(t, x, y)] \geq 0;
\]
(v) for almost all \( t \in T \), all \( \|x\| \leq M \), all \( y \in \mathbb{R}^N \) and all \( v \in F(t, x, y) \), we have
\[
\|v\| \leq c_4(t) + c_5\|y\|^{p-1}
\]
with \( c_4(t) \in L^p(T)_+, \eta = \max\{2, q\}, c_5 > 0 \).

**Remark 3.4.** Hypothesis \( H(F)_1(iv) \) is a suitable extension to the present setting of the so-called “Hartman condition” (see Mawhin [19]).

**Proposition 3.5.** If hypotheses \( H(a)_2, H(A)_1 \) and \( H(F)_1 \) hold, then Problem (5) has a solution \( x \in C^1_{\text{per}}(T, \mathbb{R}^N) \) with \( a(x') \in W^{1,q}_{\text{per}}(T, \mathbb{R}^N) \).

**Proof.** First we do the proof by assuming the following stronger version of hypothesis \( H(F)_1(iv) \):

“(iv)” there exists an \( M > 0 \) such that if \( \|x_0\| = M \) and \((x_0, y_0)_{\mathbb{R}^N} = 0\), we can find a \( \delta > 0 \) and \( c_6 > 0 \) such that for almost all \( t \in T \) we have
\[
\inf [(v, x)_{\mathbb{R}^N} + c_0\|y\|^p : \|x - x_0\| + \|y - y_0\| < \delta, v \in F(t, x, y)] \geq c_6 > 0. \tag{6}
\]

Let \( S_\lambda : \hat{D} \subseteq L^p(T, \mathbb{R}^N) \rightarrow L^q(T, \mathbb{R}^N) \) be the maximal monotone operator introduced earlier in this section (see Proposition 3.3). Also as before let \( J : L^p(T, \mathbb{R}^N) \rightarrow L^q(T, \mathbb{R}^N) \) be defined by \( J(x)(\cdot) = \|x(\cdot)\|^{p-2}x(\cdot) \). This operator is maximal monotone. Set \( V_\lambda = S_\lambda + J \). Then \( V_\lambda \) is maximal monotone. Also let \( U : \hat{D} \subseteq L^p(T, \mathbb{R}^N) \rightarrow L^q(T, \mathbb{R}^N) \) be the nonlinear differential operator defined by \( U(x) = -\lambda(a(x'))', x \in \hat{D}. \) From Proposition 3.3 we have that \( U \) is maximal monotone. Clearly \( V_\lambda \) is coercive. So \( R(V_\lambda) = L^q(T, \mathbb{R}^N) \). Moreover, \( V_\lambda \) is also injective. So we can define the map
\[
K_\lambda = V_\lambda^{-1} : L^q(T, \mathbb{R}^N) \rightarrow \hat{D} \subseteq W^{1,p}_{\text{per}}(T, \mathbb{R}^N).
\]

**Claim 1:** \( K_\lambda : L^q(T, \mathbb{R}^N) \rightarrow W^{1,p}_{\text{per}}(T, \mathbb{R}^N) \) is completely continuous.

Suppose that \( u_n \rightharpoonup u \) in \( L^q(T, \mathbb{R}^N) \). Set \( x_n = K_\lambda(u_n), n \geq 1 \). We have
\[
\|x_n\|_{1,p}^{p-1} \leq c_8\|u_n\|_q \quad \text{with} \quad c_8 > 0,
\]
hence \( \{x_n\}_{n \geq 1} \subseteq W^{1,p}_{\text{per}}(T, \mathbb{R}^N) \) is bounded. Therefore we may assume that \( x_n \rightharpoonup x \) in \( W^{1,p}_{\text{per}}(T, \mathbb{R}^N) \) and \( x_n \rightarrow x \) in \( L^p(T, \mathbb{R}^N) \). Because \( u_n = V_\lambda(x_n), n \geq 1, \)
it follows that $u = V_\lambda(x) = S_\lambda(x) + J(x) = U(x) + \tilde{A}_\lambda(x) + J(x)$. For every $n \geq 1$, $x_n \in \mathcal{D}$ and so $a(x'_n) \in W_{\text{per}}^{1,q}(T, \mathbb{R}^N)$. Hence $a(x'_n) = \bar{a}_n + \hat{a}_n$, with $\bar{a}_n \in \mathbb{R}^N$ and $\hat{a}_n \in V = \{ v \in W_{\text{per}}^{1,q}(T, \mathbb{R}^N) : \int_0^b v(t)dt = 0 \}$. From the equation 

$$U(x_n) + \tilde{A}_\lambda(x_n) + J(x_n) = u_n,$$

it follows that $\{(a(x'_n))'_n\}_{n \geq 1} \subseteq L^q(T, \mathbb{R}^N)$ is bounded, hence it follows that $\{\hat{a}_n\}_{n \geq 1} \subseteq C(T, \mathbb{R}^N)$ is relatively compact. For every $n \geq 1$ and every $t \in T$, we have

$$x'_n(t) = a^{-1}(\bar{a}_n + \hat{a}_n(t)).$$

Integrating this equation over $T = [0, b]$ and since $x_n(0) = x_n(b)$, we obtain

$$\int_0^b a^{-1}(\bar{a}_n + \hat{a}_n(t))dt = 0.$$ 

Invoking Proposition 2.2 of Manasevich-Mawhin [16], we infer that $\{\bar{a}_n\}_{n \geq 1} \subseteq \mathbb{R}^N$ is bounded. So we conclude that $\{(a(x'_n))'_n\}_{n \geq 1} \subseteq C(T, \mathbb{R}^N)$ is relatively compact. Hence $\{(a(x'_n))'_n\}_{n \geq 1} \subseteq W_{\text{per}}^{1,q}(T, \mathbb{R}^N)$ is bounded and so we may assume that $a(x'_n) \rightharpoonup \beta$ in $W_{\text{per}}^{1,q}(T, \mathbb{R}^N)$. Because $x_n \rightharpoonup x$ in $L^p(T, \mathbb{R}^N)$ and $U$ is maximal monotone, it follows that $\beta = U(x)$, hence $a(x'_n) \rightharpoonup a(x')$ in $W_{\text{per}}^{1,q}(T, \mathbb{R}^N)$ and so $a(x'_n) \to a(x')$ in $C(T, \mathbb{R}^N)$. So we have that $x_n \to x'$ in $C(T, \mathbb{R}^N)$. Therefore finally we can say that $x_n \to x$ in $W_{\text{per}}^{1,p}(T, \mathbb{R}^N)$ (in fact we have shown that $x_n \to x$ in $C^1(T, \mathbb{R}^N)$). We conclude that the whole sequence $\{x_n = K_\lambda(u_n)\}_{n \geq 1}$ strongly converges to $x = K_\lambda(u)$. This proves the claim.

Next let $N : C = \{ x \in W_{\text{per}}^{1,p}(T, \mathbb{R}^N) : \| x(t) \| \leq M \text{ for all } t \in T \} \to L^q(T, \mathbb{R}^N)$ be the multivalued operator defined by $N(x) = S_q^{\|F(x(t),x'(t))\|}$. From Hu-Papageorgiou [13, p. 236] we know that $N$ has values in $P_{\text{wk}}(L^q(T, \mathbb{R}^N))$ and it is use from $C$ with the relative $W_{\text{per}}^{1,p}(T, \mathbb{R}^N)$-norm topology into $L^q(T, \mathbb{R}^N)_w$. Set $N_1(x) = -N(x) + J(x)$. Then Problem (5) is equivalent to the abstract multivalued fixed point problem

$$x \in K_\lambda N_1(x).$$

(7)

Let $M_1 > 0$ be such that $M_1^p > \frac{p}{r_{\tau_0}} \left[ c_1 M^p + \frac{p^q}{c_\lambda} M^p b^q + \| c_3 \|_1 M^s \right]$. We consider the following set in $W_{\text{per}}^{1,p}(T, \mathbb{R}^N)$:

$$W = \{ x \in W_{\text{per}}^{1,p}(T, \mathbb{R}^N) : \| x(t) \| < M \text{ for all } t \in T \text{ and } \| x' \|_p < M_1 \}.$$ 

Set $W_1 = \{ x \in W_{\text{per}}^{1,p}(T, \mathbb{R}^N) : \| x(t) \| < M \text{ for all } t \in T \}$ and $W_2 = \{ x \in W_{\text{per}}^{1,p}(T, \mathbb{R}^N) : \| x' \|_p < M_1 \}$. We have $W = W_1 \cap W_2$ and $W_1, W_2$ are open. So $W = W_1 \cap W_2$ is an open and of course bounded subset of $W_{\text{per}}^{1,p}(T, \mathbb{R}^N)$ with $0 \in W$. Note that $\overline{W} = \{ x \in W_{\text{per}}^{1,p}(T, \mathbb{R}^N) : \| x(t) \| \leq M \text{ for all } t \in T \text{ and } \| x' \|_p \leq M_1 \}$. 


Claim 2: For every \( x \in \partial W \) and every \( \xi \in (0,1) \), we have \( x \notin \xi(K_\lambda \circ N_1)(x) \).

Let \( x \in \overline{W} \) and suppose that for some \( \xi \in (0,1) \), we have \( x \in \xi(K_\lambda \circ N_1)(x) \).

Then \( U(\frac{1}{\xi}x) + \tilde{A}_\lambda(\frac{1}{\xi}x) + J(\frac{1}{\xi}x) = -f + J(x) \) with \( f \in N(x) \), and hence

\[
c_0\|x'\|^p \leq -\xi^{p-1}(f,x)_{qp} + (\xi^{p-1} - 1)\|x\|^p \leq -\xi^{p-1}(f,x)_{qp}
\]

(since \( 0 < \xi < 1 \)). Using hypothesis H(F)(iii), we obtain

\[
-\xi^{p-1}(f,x)_{qp} \leq \xi^{p-1}c_1\|x\|^p + \xi^{p-1}c_2 \int_0^b \|x(t)\|^r\|x'(t)\|^{p-r}dt + \xi^{p-1}c_3\|x\|_\infty^s.
\]

Set \( \tau = p - r \), \( \theta = \frac{p}{r} \) and \( \theta' = \frac{p}{r} \left( \frac{1}{\theta} + \frac{1}{\theta'} \right) = 1 \). From Hölder’s inequality, we have

\[
-\xi^{p-1}(f,x)_{qp} \leq \xi^{p-1}c_1\|x\|^p + \xi^{p-1}c_2\|x\|^p\|x'\|^p + \xi^{p-1}c_3\|x\|_\infty^s.
\]

Using this in (8) and because \( 0 < \xi < 1 \), we obtain (recall the choice of \( M_1 \))

\[
\|x'\|^p \leq \frac{p}{r c_0} \left[ c_1 M_1 b + \frac{r c_0^p}{c_0 p} + c_3\|x\|_\infty^s \right] < M_1^p.
\]

To conclude that \( x \in W \) it remains to show that \( \|x(t)\| < M \) for all \( t \in T \). We argue by contradiction. So suppose that for some \( t_0 \in T \) we have \( \|x(t_0)\| = M \). Since \( x \in \overline{W} \), we must have that \( \|x(t_0)\| = \max_{t \in T} \|x(t)\| \). Let \( \theta(t) = \frac{1}{p}\|x(t)\|^p \). We see that \( \theta(t) \) attains its maximum on \( T = [0,b] \) at the point \( t_0 \in T \). If \( t_0 \in (0,b) \), then \( \theta'(t_0) = 0 \) and so \( \|x(t_0)\|^{p-2}(x(t_0),x'(t_0))_{\mathbb{R}^N} = 0 \), hence \( (x(t_0),x'(t_0))_{\mathbb{R}^N} = 0 \). By virtue of (6), for almost all \( t \in T \) we have

\[
\inf \left[ (v,z)_{\mathbb{R}^N} + c_0\|y\|^p : \|z - x(t_0)\| + \|y - x'(t_0)\| < \delta, (v,z) \in F(t,x,y) \right] \geq c_6 > 0.
\]

We can find a \( \delta_1 > 0 \) such that if \( t \in (t_0,t_0 + \delta_1) \) we have \( \|x(t) - x(t_0)\| + \|x'(t) - x'(t_0)\| < \delta \) and \( x(t) \neq 0 \). Then for almost all \( t \in (t_0,t_0 + \delta_1) \),

\[
(f(t),x(t))_{\mathbb{R}^N} + c_0\|x'(t)\|^p \geq c_6 > 0.
\]

We know that a.e. on \( T \)

\[
(f(t),x(t))_{\mathbb{R}^N} = \left( \left( a\left( \frac{1}{\xi}x'(t) \right) \right)', x(t) \right)_{\mathbb{R}^N} - \left( A_\lambda\left( \frac{1}{\xi}x(t) \right), x(t) \right)_{\mathbb{R}^N} + \left( 1 - \frac{1}{\xi^{p-1}} \right)\|x(t)\|^p
\]

and hence (see 9)

\[
\left( \left( a\left( \frac{1}{\xi}x'(t) \right) \right)', x(t) \right)_{\mathbb{R}^N} + c_0\|x'(t)\|^p \geq c_6 > 0 \quad \text{a.e. on } (t_0,t_0 + \delta_1].
\]
Integrating this inequality on \([t_0, t]\) with \(t \in (t_0, t_0 + \delta_1]\), after integration by parts, we obtain

\[
\left( \left( a \left( \frac{1}{\xi} x'(t) \right) \right), x(t) \right)_{\mathbb{R}^N} - \left( \left( a \left( \frac{1}{\xi} x'(t_0) \right) \right), x'(t_0) \right)_{\mathbb{R}^N} - \int_{t_0}^{t} \left( a \left( \frac{1}{\xi} x'(s) \right), x'(s) \right)_{\mathbb{R}^N} \, ds + c_0 \int_{t_0}^{t} \| x'(s) \|^p \, ds \geq c_0 (t - t_0) > 0.
\]

Suppose that the first version of hypothesis \(H(a)_2\) holds, namely that \(a(y) = c(y)y\). The reasoning is similar if the other version is valid. We have

\[
\left( a \left( \frac{1}{\xi} x'(t_0) \right), x(t_0) \right)_{\mathbb{R}^N} = c \left( \frac{1}{\xi} x'(t_0) \right) \frac{1}{\xi} (x(t_0), x(t_0))_{\mathbb{R}^N} = 0.
\]

Therefore for \(t \in (t_0, t_0 + \delta_1]\) we have \(\| x(t) \|_{\mathbb{R}^N} > 0\) (since \(0 < \xi < 1\), i.e., \(\theta'(t) > 0\) for \(t \in (t_0, t_0 + \delta_1]\). So \(\theta\) is strictly increasing on \((t_0, t_0 + \delta_1]\), which contradicts the choice of \(t_0\). Therefore we infer that \(\| x(t) \| < M\) for all \(t \in T\).

If \(t_0 = 0\), then \(\theta'_+(t_0) = \theta'_+(0) \leq 0\) and \(\theta'_-(b) \geq 0\) (because \(\theta(0) = \theta(b)\), from the periodic boundary conditions). So we have \((x(0), x'(0))_{\mathbb{R}^N} = 0\) (since \(x(0) = x(b), x'(0) = x'(b)\), recall that \(x \in \mathcal{D}\)). So we proceed as before. Similarly if \(t_0 = b\). Therefore we conclude that \(\| x(t) \| < M\) for all \(t \in T\) and so \(x \in W\), which proves the claim.

Now we can apply Proposition 2.1 and obtain \(x \in \mathcal{D} \cap \overline{W}\) which solves the fixed point Problem (7). Clearly \(x \in \mathcal{D} \cap \overline{W}\) is a solution of (5).

Finally it remains to remove the stronger version of hypothesis \(H(F)_1(iv)\) (see (6)). To this end let \(\varepsilon_n \downarrow 0\) and set \(F_n(t, x, y) = F(t, x, y) + \varepsilon_n x\). Then Problem (5) with \(F\) replaced by \(F_n\), has a solution \(x_n \in \mathcal{D} \cap \overline{W}, n \geq 1\). Evidently we may assume that \(x_n \mathcal{w} x \in W^1_{\text{per}}(T, \mathbb{R}^N)\). As in the proof of Claim 1, we have \(x_n \to x \in W^1_{\text{per}}(T, \mathbb{R}^N)\) and in the limit as \(n \to \infty\) we obtain \(U(x) + \widehat{A}_\lambda(x) \in \mathcal{N}(x)\). Therefore \(x \in \mathcal{D} \cap \overline{W}\) is a solution of (5).}

Now that we have solved the auxiliary Problem (5), by passing to the limit as \(\lambda \downarrow 0\), we shall obtain a solution for the original Problem (1).

**Theorem 3.6.** If hypotheses \(H(a)_2, H(A)_1\) and \(H(F)_1\) hold, then Problem (1) has a solution \(x \in C^1_{\text{per}}(T, \mathbb{R}^N)\) with \(a(x') \in W^1_{\text{per}}(T, \mathbb{R}^N)\).

**Proof.** Let \(\lambda_n \downarrow 0\) and let \(x_n \in \mathcal{D} \cap \overline{W}\) be solutions of the corresponding auxiliary problems (5). Evidently \(\{x_n\}_{n \geq 1} \subseteq W^1_{\text{per}}(T, \mathbb{R}^N)\) is bounded and so we may assume that \(x_n \mathcal{w} x \in W^1_{\text{per}}(T, \mathbb{R}^N)\). For every \(n \geq 1\), we have

\[
(U(x_n), \widehat{A}_\lambda(x_n))_{qp} + \| \widehat{A}_\lambda(x_n) \|_{qp}^2 = -(f_n, \widehat{A}_\lambda(x_n))_{qp}
\]
From integration by parts and since $x_n(0) = x_n(b)$, $x'_n(0) = x'_n(b)$, we have

$$
(U(x_n), \hat{A}_{\lambda_n}(x_n))_{qp} = \int_0^b \left( - (a(x_n'(t)))', A_{\lambda_n}(x_n(t)) \right)_{\mathbb{R}^N} dt
$$

$$
= \int_0^b \left( a(x_n'(t)), \frac{d}{dt} A_{\lambda_n}(x_n(t)) \right)_{\mathbb{R}^N} dt.
$$

From the chain rule of Marcus-Mizel [17], we have that $\frac{d}{dt} A_{\lambda_n}(x_n(t)) = A'_{\lambda_n}(x_n(t)) x'_n(t)$ a.e. on $T$. So (see $H(A)_1$)

$$
(U(x_n), \hat{A}_{\lambda_n}(x_n))_{qp} = \int_0^b c(x_n'(t))(x'_n(t), A_{\lambda_n}(x_n(t)) x'_n(t))_{\mathbb{R}^N} dt \geq 0.
$$

Using this inequality in (10), we obtain that $\{\hat{A}_{\lambda_n}(x_n)\}_{n \geq 1} \subseteq L^2(T, \mathbb{R}^N)$ is bounded. So we may assume that $\hat{A}_{\lambda_n}(x_n) \overset{w}{\rightharpoonup} u$ in $L^2(T, \mathbb{R}^N)$. If $\hat{J}_{\lambda_n}(x_n(\cdot)) \in A(J_{\lambda_n}(x_n))$ for all $n \geq 1$ and all $t \in T$, we have $\hat{A}_{\lambda_n}(x_n) \in \hat{A}(\hat{J}_{\lambda_n}(x_n))$. Because $A_{\lambda_n}(x_n) \in \hat{A}(\hat{J}_{\lambda_n}(x_n))$, $\hat{J}_{\lambda_n}(x_n) \in L^2(T, \mathbb{R}^N)$ and $\hat{A}_{\lambda_n}(x_n) \overset{w}{\rightharpoonup} u$ in $L^2(T, \mathbb{R}^N)$, we infer that $u \in \hat{A}(x)$, i.e., $u(t) \in A(x(t))$ a.e. on $T$. Moreover, we may assume that $f_n \overset{w}{\rightharpoonup} f$ in $L^p(T, \mathbb{R}^N)$. Arguing as in the proof of Proposition 3.5 (see Claim 1), we obtain $x_n \rightarrow x$ in $C^1(T, \mathbb{R}^N)$. Then in the limit as $n \rightarrow \infty$, we have $f \in N_1(x)$ and $(a(x'(t)))' = u(t) + f(t) \in A(x(t)) + F(t, x(t), x'(t))$ a.e. on $T$, $x(0) = b$, $x'(0) = x'(b)$.

4. Problems with the $p$-Laplacian and nonlinear boundary conditions

In this section we deal with Problem (2). Now, in contrast to the situation of Section 3, we assume that $D(A) = \mathbb{R}^N$. This permits the improvement of the growth condition on $F$ and so we can have multivalued nonlinearities of the Nagumo-Hartman type (see also Mawhin-Urena [20]). More precisely our hypotheses on the data of (2) are the following:

$H(A)_2 : A : \mathbb{R}^N \rightarrow 2^{\mathbb{R}^N}$ is a maximal monotone map with $D(A) = \mathbb{R}^N$ and $0 \in A(0)$.

$H(F)_2 : F : T \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow P_{kc}(\mathbb{R}^N)$ is a multifunction such that $H(F)_1(i), (ii)$ hold and

(iii) for almost all $t \in T$, all $\|x\| \leq M$ and all $\|y\|^{p-1} \geq M_1 > 0$ we have

$$
\sup \left[ \|v\| : v \in F(t, x, y) \right] \leq \eta(\|y\|^{p-1})
$$

where $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \setminus \{0\}$ is a locally bounded Borel measurable function such that $\int_{M_1}^{\infty} \frac{ds}{\eta(s)} = +\infty$.
Proof. We have proved that \( \sigma \) Caratheodory function, thus jointly measurable. Note that \( \in \) is maximal monotone. Clearly

\[
\partial \varphi \in \text{uniformly convex spaces. Let} \quad L \rightarrow \frac{q}{p} \in \mathcal{L}(T, X^*)
\]

Proposition 4.1. If \( X \) is a separable reflexive Banach space and \( A : D(A) \subseteq X \rightarrow 2^{X^*} \) is a maximal monotone operator with \( 0 \in A(0) \), then \( \hat{A} : D(\hat{A}) \subseteq L^p(T, X) \rightarrow 2^{L^q(T, X^*)} \) is maximal monotone too.

Proof. By Troyanski’s renaming theorem (see Hu-Papageorgiou [13, p. 316]), without any loss of generality we may assume that both \( X \) and \( X^* \) are locally uniformly convex spaces. Let \( \mathcal{F} : X \rightarrow X^* \) be the duality map of \( (i.e., \mathcal{F}(x) = \partial \varphi(x) \) with \( \varphi(x) = \frac{1}{2}\|x\|^2 \), see Hu-Papageorgiou [13, p. 30] and Zeidler [22, p. 860]). We know that \( \mathcal{F} \) is a homeomorphism (see Zeidler [22, p. 861]).

We introduce the operator \( J_0 : L^p(T, X) \rightarrow L^q(T, X^*) \) defined by \( J_0(x) = \|\mathcal{F}(x)\|^{p-2}\mathcal{F}(x) \). It is easy to see that \( J_0 \) is continuous, strictly monotone, thus maximal monotone. Clearly \( \hat{A} \) is monotone. We show that \( R(\hat{A} + J_0) = L^q(T, X^*) \) (i.e., surjectivity of \( \hat{A} + J_0 \)). For this purpose let \( h \in L^q(T, X^*) \) and consider the multifunction \( \Gamma : T \rightarrow 2^{X^*} \) defined by \( \Gamma(t) = \{x \in X : A(x) + \varphi(x) \ni h(t)\} \), where \( \varphi : X \rightarrow X^* \) is the monotone continuous map defined by \( \varphi(x) = \|\mathcal{F}(x)\|^{p-2}\mathcal{F}(x) \). Note that \( A + \varphi : D(A) \subseteq X \rightarrow 2^{X^*} \) is maximal monotone. Moreover, because \( 0 \in A(0) \), we have that \( A + \varphi \) is coercive. Therefore \( R(A + \varphi) = X^* \) and so we infer that for all \( t \in T, \Gamma(t) \neq \emptyset \).

Remark that \( \mathcal{F} = \xi^{-1}(\mathcal{F}) \) and since \( \mathcal{F} \) is sequentially closed in \( X \times X^*_w \), we have \( \mathcal{F} \in B(X \times X^*_w) \) (the Borel \( \sigma \)-field). But \( X^*_w \) is a Souslin space and so \( B(X \times X^*_w) = B(X) \times B(X^*_w) \) (see Hu-Papageorgiou [13, p. 153]). Also \( B(X^*_w) = B(X^*) \). Therefore \( \mathcal{F} \) is sequentially closed in \( X \times X^*_w \) and so \( \mathcal{F} \in \mathcal{L} \times B(X) \) with \( \mathcal{L} \) being the Lebesgue \( \sigma \)-field of \( T \). We can apply the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [13, p. 158]) to obtain a measurable map \( x : T \rightarrow X \) such that \( x(t) \in \Gamma(t) \) a.e. on \( T \). We have \( h(t) \in A(x(t)) + \varphi(x(t)) \) a.e. on \( T \). Taking duality brackets with \( x(t) \), we obtain

\[
|h(t), x(t)|_X \leq \|x(t)\|^{p-1} \leq \|h(t)\| \quad \text{a.e. on} \quad T, \text{i.e.,} \quad x \in L^p(T, X). \quad \text{So we have proved that} \quad R(\hat{A} + J_0) = L^q(T, X^*). \quad \text{Then arguing as in the proof} \]
of Proposition 3.3 and exploiting the strict monotonicity of $J_0$, we obtain the maximality of $A$.

The second auxiliary result concerns the periodic problem

\[
\begin{aligned}
&-((|x'(t)|^{p-2}x'(t))' + |x(t)|^{p-2}x(t) = g(t) \quad \text{a.e. on } T = [0, b] \\
&\quad (\varphi_p(x'(0)), -\varphi_p(x'(b))) \in \xi(x(0), x(b)), \quad 1 < p < \infty.
\end{aligned}
\]

From Gasinski-Papageorgiou [10] we have the following result:

**Proposition 4.2.** If $\xi : D(\xi) \subseteq \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is a maximal monotone map with $(0, 0) \in \xi(0, 0)$ and $g \in L^q(T, \mathbb{R}^N)$ ($\frac{1}{p} + \frac{1}{q} = 1$), then Problem (11) has a unique solution $x \in C^1(T, \mathbb{R}^N)$ with $\|x'\|^{p-2}x' \in W^{1,q}(T, \mathbb{R}^N)$.

Let $D_0 = \{ x \in C^1(T, \mathbb{R}^N) : \|x'\|^{p-2}x' \in W^{1,q}(T, \mathbb{R}^N), (\varphi_p(x'(0)), -\varphi_p(x'(b))) \in \xi(x(0), x(b)) \}$ and let $V : D_0 \subseteq L^p(T, \mathbb{R}^N) \to L^q(T, \mathbb{R}^N)$ be defined by

\[
V(x) = -(\|x'\|^{p-2}x')', \quad x \in D_0.
\]

Arguing as in the proof of Proposition 3.3, using this time Proposition 4.2, we obtain

**Proposition 4.3.** If $\xi : D(\xi) \subseteq \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is a maximal monotone map with $(0, 0) \in \xi(0, 0)$, then $V : D_0 \subseteq L^p(T, \mathbb{R}^N) \to L^q(T, \mathbb{R}^N)$ is maximal monotone.

For the existence theorem for Problem (2) we will use the following hypotheses on $\xi$:

\[
\mathbf{H}(\xi) : \quad \xi : D(\xi) \subseteq \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N} \text{ is a maximal monotone map with (0, 0) \in \xi(0, 0)} \quad \text{and one of the following holds:}
\]

(i) for every $(a', d') \in \xi(a, d)$, we have $(a', a)_{\mathbb{R}^N} \geq 0$ and $(d', d)_{\mathbb{R}^N} \geq 0$; or

(ii) $D(\xi) = \{(a, d) \in \mathbb{R}^N \times \mathbb{R}^N : a = d \}$.

**Proposition 4.4.** If the hypotheses $\mathbf{H}(A)_2$, $\mathbf{H}(F)_2$, $\mathbf{H}(\xi)$ and $\mathbf{H}_0$ hold, then Problem (2) has a solution $x \in C^1(T, \mathbb{R}^N)$ with $\|x'\|^{p-2}x' \in W^{1,q}(T, \mathbb{R}^N)$.

**Proof.** Because $A$ is maximal monotone with $D(A) = \mathbb{R}^N$, we have that $\theta = \sup \{ \|u\| : u \in A(x), \|u\| \leq M \} < +\infty$ (see Hu-Papageorgiou [13, p. 308]). Without any loss of generality we may assume that for almost all $r \geq 0$, $0 < \beta \leq \eta(r)$. Set $\eta_1(r) = \theta + \eta(r)$. If $\tilde{\gamma} \leq \frac{\theta}{\beta} + 1$, then we have $\eta_1(r) \leq \tilde{\gamma} \eta(r)$ for all $r \geq 0$ and so $\int_{M_1}^{\infty} \frac{\text{d}t}{\eta_1(t)} = +\infty$.

As we did with Problem (1) (see Section 3), first we assume that the multivalued nonlinearity $F$ satisfies (6) (with $c_0 = 1$) instead of $\mathbf{H}(F)_2$(iv). Let

\[
M_1' = \max \left\{ b^\beta \left( \frac{p}{r c_0} \left[ c_1 M_p b + \frac{r c_2^p M_p b^p}{c_0 p} + \|c_3\|_1 M_s^* \right] \right)^{\frac{p-1}{p}}, M_1 \right\}
\]
and then take $M_2 > 0$ such that $M_2^{p-1} > M_1'$ and $\int_{M_1}^{M_2} \frac{\text{d}x}{\eta(x)} = M_1'$. Also let $W \subseteq C^1(T, \mathbb{R}^N)$ be defined by

$$W = \{ x \in C^1(T, \mathbb{R}^N) : ||x(t)|| < M, ||x'(t)|| < M_2 \text{ for all } t \in T \}.$$  

The set $W$ is open, bounded in $C^1(T, \mathbb{R}^N)$ and $0 \in W$. Moreover, we have

$$\partial W = \{ x \in C^1(T, \mathbb{R}^N) : ||x||_\infty = M, ||x'||_\infty = M_2 \}.$$  

Let $N : \overline{W} \to P_{wkc}(L^q(T, \mathbb{R}^N))$ be defined by $N(x) = S_{F,(.,x(.),x'(.))}^\eta$. We know that $N$ is usc from $\overline{W}$ with the $C^1(T, \mathbb{R}^N)$-norm topology into $L^q(T, \mathbb{R}^N)$ with the weak topology. For each $g \in L^q(T, \mathbb{R}^N)$, we consider Problem (11). By Proposition 4.2 we know that this problem has a unique solution $x = K(g) \in C^1(T, \mathbb{R}^N)$. So we can define the map $K : L^q(T, \mathbb{R}^N) \to C^1(T, \mathbb{R}^N)$ which to each $g \in L^q(T, \mathbb{R}^N)$ assigns the unique solution of (11). It is easy to check that $K$ is completely continuous.

Let $J : C^1(T, \mathbb{R}^N) \to L^2(T, \mathbb{R}^N)$ be the bounded continuous map defined by $J(x)(\cdot) = ||x(\cdot)||^{p-2}x(\cdot)$. Also let $\hat{A} : C^1(T, \mathbb{R}^N) \to 2L^q(T, \mathbb{R}^N)$ be defined by $\hat{A}(x) = S_{A(x(\cdot))}^\eta$. We have that $\hat{A}$ is usc from $C^1(T, \mathbb{R}^N)$ into $L^q(T, \mathbb{R}^N)_w$. Set $N_1(x) = -N(x) + J(x) - \hat{A}(x)$. Evidently $N_1 : \overline{W} \to P_{wkc}(L^q(T, \mathbb{R}^N))$ is usc from $\overline{W}$ with the $C^1(T, \mathbb{R}^N)$-norm topology into $L^q(T, \mathbb{R}^N)_w$. Problem (2) is equivalent to the fixed point problem

$$x \in (K \circ N_1)(x). \quad (12)$$

Claim: For every $x \in \partial W$ and every $\xi \in (0, 1)$, we have $x \notin \xi(K \circ N_1)(x)$.

Let $x \in \overline{W}$ and suppose that for some $\xi \in (0, 1)$ we have $x \in \xi(K \circ N_1)(x)$. Arguing as in the proof of Proposition 3.5 (claim 2), we obtain

$$||x'||_p \leq \frac{p}{r c_0} \left[ c_1 M^p b + \frac{r c_2^p M^p b^p}{c_0 p} + ||c_3||_1 M^p \right],$$

and hence $||x'||_p^{p-1} < \frac{1}{b^p} M_1'$. The function $\vartheta(u) = u^{p-1} - \frac{u^{p-1}}{p} \cdot u \geq 0$, is concave. So using Jensen’s inequality, we have

$$\frac{1}{b^p} ||x'||_p^{p-1} \geq \frac{1}{b} \int_0^b ||x'(t)||^{p-1} dt.$$  

Therefore it follows (since $\frac{1}{p} + \frac{1}{q} = 1$) that

$$\int_0^b ||x'(t)||^{p-1} dt < M_1'.$$
We claim that \( \|x'(t)\| < M_2 \) for all \( t \in T \). Suppose that this is not the case. Then we can find \( t_0 \in T \) such that \( \|x'(t_0)\| = M_2 \), hence \( \|x'(t_0)\|^{p-1} > M_1' \). So from (13) we infer that there exists a \( t_1 \in T \) such that \( \|x(t_1)\|^{p-1} = M_1' \) (take the \( t_1 \in T \) which is closest to \( t_0 \)). Let \( \chi : [M_1', +\infty) \to \mathbb{R}_+ \) be the function defined by \( \chi(r) = \int_{M_1'}^r \frac{s}{\eta_1(s)}\,ds \). Clearly \( \chi \) is continuous, strictly increasing, \( \chi(M_1') = 0 \) and \( \chi(M_2^{p-1}) = M_1' \). We have

\[
M_1' = \chi(M_2^{p-1})
= \left| \chi(\|x'(t_0)\|^{p-1}) \right|
= \left| \int_{M_1'}^{\|x'(t_0)\|^{p-1}} \frac{s}{\eta_1(s)}\,ds \right|
= \left| \int_{\|x'(t_0)\|^{p-1}}^{\|x'(t_1)\|^{p-1}} \frac{s}{\eta_1(s)}\,ds \right|
\leq \int_{t_0}^{t_1} \frac{\|\|x'(t)\|^{p-2}x'(t)\|'}{\eta_1(\|\|x'(t)\|^{p-2}x'(t)\|)} \|x'(t)\|^{p-1}dt.
\]

We also have

\[
\|\|x'(t)\|^{p-2}x'(t)\|' \leq \theta + \eta(\|x'(t)\|^{p-1})
= \eta(\|x'(t)\|^{p-1})
= \eta(\|\|x'(t)\|^{p-2}x'(t)\|).
\]

Using this in (14), we obtain (see (13))

\[
M_1' \leq \int_{t_0}^{t_1} \|x'(t)\|^{p-1}dt
= \int_{\min\{t_0, t_1\}}^{\max\{t_0, t_1\}} \|x'(t)\|^{p-1}dt < M_1',
\]
a contradiction. Therefore \( \|x'(t)\| < M_2 \) for all \( t \in T \). Moreover, following the argument in the proof of Proposition 3.5 and using hypotheses \( H(\xi) \), we can show that \( \|x(t)\| < M \) for all \( t \in T \). Therefore \( x \in W \) and we have proved the claim.

Apply Proposition 2.1 to obtain \( x \in D_0 \cap \overline{W} \) which solves (12). Evidently this is a solution of (2) when (6) (with \( c_0 = 1 \)) is in effect. As in the proof of Proposition 3.5 we remove this extra restriction.

**Remark 4.5.** It will be interesting to have this existence result when \( D(A) \neq \mathbb{R}^N \).

### 5. Special cases and examples

We show that our general formulation of Problem (2) unifies the classical Dirichlet, Neumann and periodic problems and goes beyond them:
(a) Let $K_1, K_2 \in P_{fc}(\mathbb{R}^N)$ with $0 \in K_1 \cap K_2$. By $\delta_{K_1 \times K_2}$ we denote the indicator function of the set $K_1 \times K_2$, i.e.,

$$\delta_{K_1 \times K_2}(x, y) = \begin{cases} 0 & \text{if } (x, y) \in K_1 \times K_2 \\ +\infty & \text{otherwise.} \end{cases}$$

Evidently $\delta_{K_1 \times K_2}$ is proper, lower semicontinuous and convex, i.e., $\delta_{K_1 \times K_2} \in \Gamma_0(\mathbb{R}^N \times \mathbb{R}^N)$. Set $\xi = \partial \delta_{K_1 \times K_2} = N_{K_1 \times K_2} = N_{K_1} \times N_{K_2}$ (given $C \in P_{fc}(\mathbb{R}^N)$ by $N_C(x)$ we denote the normal cone to the set $C$ at $x \in C$, see Hu-Papageorgiou [13, p. 624]). Then Problem (2) becomes

$$\begin{cases} (\|x'(t)\|^{p-2}x'(t))' \in A(x(t)) + F(t, x(t), x'(t)) & \text{a.e. on } T \\ x(0) \in K_1, \ x(b) \in K_2 \\ (x'(0), x(0))_{\mathbb{R}^N} = \sigma(x'(0), K_1), \ (-x'(b), x(b))_{\mathbb{R}^N} = \sigma(-x'(b), K_2). \end{cases} \tag{15}$$

Note that $\xi = \partial \delta_{K_1 \times K_2}$ is maximal monotone, $(0, 0) \in \xi(0, 0)$ and hypothesis $H(\xi)$ is valid (the first option).

(b) In the previous case, let $K_1 = K_2 = \{0\}$. Then Problem (15) becomes the usual Dirichlet problem.

(c) Again in the first example let $K_1 = K_2 = \mathbb{R}^N$. Then $\xi = N_{K_1} \times N_{K_2} = \{(0, 0)\}$ and so we have Neumann problem. The Neumann problem was not examined before in the presence of Nagumo-Hartman nonlinearities (compare with Mawhin-Urena [20]).

(d) Let $K = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x = y\}$ and let $\xi = \partial \delta_K$. Then $\xi(x, y) = K^\perp = \{(v, w) \in \mathbb{R}^N \times \mathbb{R}^N : v = -w\}$. So Problem (2) becomes the usual periodic problem.

(e) Let $\xi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N$ be defined by

$$\xi(x, y) = \left(\frac{1}{\theta + \eta} \varphi_p(x), \frac{1}{\eta + \theta} \varphi_p(y)\right) \quad \text{with } \theta, \eta > 0.$$ 

Evidently, $\xi$ is continuous, monotone (hence maximal monotone) and $\xi(0, 0) = (0, 0)$. With this choice of $\xi$, Problem (2) becomes a Sturm-Liouville type problem

$$\begin{cases} (\|x'(t)\|^{p-2}x'(t))' \in A(x(t)) + F(t, x(t), x'(t)) & \text{a.e. on } T \\ x(0) - \theta x'(0) = 0, \ x(b) + \eta x'(b) = 0. \end{cases} \tag{16}$$

Hypothesis $H(\xi)$ is satisfied.

(f) Let $\xi_1, \xi_2 : \mathbb{R}^N \to \mathbb{R}^N$ be two monotone, continuous maps such that $\xi_1(0) = \xi_2(0) = 0$. Let $\xi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N$ be defined by $\xi(x, y) = (\xi_1(x), \xi_2(y))$. Evidently $\xi$ satisfies hypothesis $H(\xi)$. Then Problem (2) becomes

$$\begin{cases} (\|x'(t)\|^{p-2}x'(t))' \in A(x(t)) + F(t, x(t), x'(t)) & \text{a.e. on } T \\ x'(0) = \varphi_q(\xi_1(x(0))), \ -x'(b) = \varphi_q(\xi_2(x(b))). \end{cases} \tag{17}$$
Next let $\psi = \delta_{\mathbb{R}_+^N}$, $A = \partial \psi$, $K = \{(x, y) \in \mathbb{R}_+^N \times \mathbb{R}^N : x = y\}$ and $\xi = \partial \delta_K = K^\perp$. We have

$$A(x) = \partial \psi(x) = N_{\mathbb{R}_+^N}(x) = \begin{cases} \{0\} & \text{if } x_k > 0 \text{ for all } k \in \{1, \ldots, N\} \\
-\mathbb{R}_+^N \cap \{x\}^\perp & \text{if } x_k = 0 \text{ for some } k \in \{1, \ldots, N\} \end{cases}$$

Then Problem (2) becomes the following differential variational inequality:

$$\begin{cases}
\left(\|x'(t)\|^{p-2}x'(t)\right)' \in F(t, x(t), x'(t)) \\
\text{a.e. on } \{t \in T : x_k(t) > 0 \text{ for all } k = 1, \ldots, N\}
\end{cases}$$

\[ (\|x'(t)\|^{p-2}x'(t))' \in F(t, x(t), x'(t)) - u(t) \]

\[ \text{a.e. on } \{t \in T : x_k(t) = 0 \text{ for some } k = 1, \ldots, N\} \]

\[ x(t) = \left(x_k(t)\right)_{k=1}^N \in \mathbb{R}_+^N \text{ for all } t \in T, \; u \in L^q(T, \mathbb{R}_+^N) \]

\[ x(0) = x(b), \; x'(0) = x'(b). \]

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References


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