Hypoellipticity of Hankel Convolution Equations in $\mathcal{D}_{L^1}$-Type Spaces

Jorge J. Betancor

Abstract. In this paper we analyze the hypoellipticity of Hankel convolution equations in distribution spaces of $L^p$-growth. The spaces that we consider are $\mathcal{D}_{L^p}$-type spaces in the Hankel setting.

Keywords: Hypoelliptic, Hankel transform, convolution equations

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1. Introduction

The space $\mathcal{D}_{L^p}$, $1 \leq p \leq \infty$, were studied by L. Schwartz ([14]). Assume that $n \in \mathbb{N} \setminus \{0\}$. If $1 \leq p < \infty$, a smooth function $\phi$ on $\mathbb{R}^n$ is in $\mathcal{D}_{L^p}$ provided that

$$
\|\phi\|_{p,k} = \left( \int_{\mathbb{R}^n} |D^k \phi(x)|^p \, dx \right)^{\frac{1}{p}} < \infty,
$$

for every $k \in \mathbb{N}^n$. A smooth function $\phi$ on $\mathbb{R}^n$ is in $\mathcal{D}_{L^\infty}$ when

$$
\|\phi\|_{\infty,k} = \sup_{x \in \mathbb{R}^n} |D^k \phi(x)| < \infty,
$$

and $\lim_{|x| \to \infty} D^k \phi(x) = 0$, for every $k \in \mathbb{N}^n$. Here, for each $k = (k_1, k_2, ..., k_n) \in \mathbb{N}^n$ we understand as usual

$$
D^k \phi = \frac{\partial^{k_1+k_2+...+k_n}}{\partial x_1^{k_1} \partial x_2^{k_2} ... \partial x_n^{k_n}} \phi.
$$

The space $\mathcal{D}_{L^p}$, $1 \leq p \leq \infty$, is endowed with the topology associated with the family $\{\| \cdot \|_{p,k}\}_{k \in \mathbb{N}^n}$ of seminorms. Thus, $\mathcal{D}_{L^p}$, $1 \leq p \leq \infty$ is a Fréchet space.

Departamento de Análisis Matemático, Universidad de La Laguna, 38271 La Laguna, Tenerife, Spain; jbetanco@ull.es

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Moreover the space \( D \) of the smooth and compact support functions in \( \mathbb{R}^n \) is dense in \( D_{L^p} \), \( 1 \leq p \leq \infty \). The dual space \( D'_{L^p} \) of \( D_{L^p} \) is hence a normal space of distributions for each \( 1 \leq p \leq \infty \).

In [3] J. J. Betancor and B. González introduced the space \( D_{L^p} \)-type in the setting of Hankel transforms. There, the rule of the derivatives was played by the Bessel operator \( \Delta_{\mu} = x^{-2\mu-1}D_x2^\mu+1D_x \). The spaces \( H_{\mu,p} \) and \( \mathcal{H}_{\mu,p} \), \( \mu > -\frac{1}{2} \) and \( 1 \leq p \leq \infty \), were defined in [3] as follows. Let \( 1 \leq p \leq \infty \) and \( \mu > -\frac{1}{2} \). A Lebesgue measurable function \( f \) defined on \( (0, \infty) \) is in \( H_{\mu,p} \) if, for every \( k \in \mathbb{N} \), \( \Delta_{\mu}^k f \in L^p(x^{2\mu+1} \, dx) \), that is, there exists \( h_k \in L^p(x^{2\mu+1} \, dx) \) such that

\[
\int_0^\infty f(x)\Delta_{\mu}^k(\phi)(x)x^{2\mu+1} \, dx = \int_0^\infty \phi(x)h_k(x)x^{2\mu+1} \, dx, \quad \phi \in S_e.
\]

Here by \( S_e \) we understand the space that consists of all the even functions in the Schwartz space \( S(\mathbb{R}) \). The space \( H_{\mu,p} \) is equipped with the topology generated by the family \( \{\gamma_{k,\mu}^p\}_{k \in \mathbb{N}} \) of seminorms, where

\[
\gamma_{k,\mu}^p(\phi) = \|\Delta_{\mu}^k \phi\|_{\mu,p}, \quad \phi \in H_{\mu,p}, \ k \in \mathbb{N},
\]

and \( \|\cdot\|_{\mu,p} \) being the usual norm in the Lebesgue space \( L^p(x^{2\mu+1} \, dx) \). Note that \( L^\infty(x^{2\mu+1} \, dx) = L^\infty(\, dx) \). Thus \( H_{\mu,p} \) is a Fréchet space ([3, Proposition 2.1]).

It is not hard to see that \( S_e \) is properly contained in \( H_{\mu,p} \). The space \( \mathcal{H}_{\mu,p} \) is defined as the closure of \( S_e \) into \( H_{\mu,p} \).

In [3] the author and B. González investigated the Hankel convolution on the spaces \( H_{\mu,p} \), \( \mathcal{H}_{\mu,p} \) and their duals. Motivated by the paper of D. H. Pahk [13] in [3, open question 3.3] J. J. Betancor and B. González propose the study of the hypoellipticity of Hankel convolution equations on the spaces \( H'_{\mu,1} \), the dual space of \( H_{\mu,1} \). This is our objective in this paper.

We now recall some definitions and properties concerning to Hankel transforms and Hankel convolution that will be useful in the sequel.

The Hankel transform \( h_\mu(f) \) of \( f \in L^1(x^{2\mu+1} \, dx) \) is defined by (see [10] and [11], for instance)

\[
h_\mu(f)(y) = \int_0^\infty (xy)^{-\mu}J_\mu(xy)f(x)x^{2\mu+1} \, dx, \quad y \in (0, \infty),
\]

where \( J_\mu \) denotes the Bessel function of the first kind and order \( \mu \) ([16]). Here we assume that \( \mu > -\frac{1}{2} \). The Hankel transform, that is also called Hankel-Schwartz transform (see [7]), is an automorphism in \( S'_e \) ([1, Satz 5]). \( h_\mu \) is defined on \( S'_e \), the dual space of \( S_e \), by transposition. Each function \( f \in L^p(x^{2\mu+1} \, dx) \), \( 1 \leq p \leq \infty \), defines an element of \( S'_e \), that will be continue denoting by \( f \), as follows:

\[
\langle f, \phi \rangle = \int_0^\infty f(x)\phi(x)\frac{x^{2\mu+1}}{2\pi \Gamma(\mu+1)} \, dx, \quad \phi \in S_e.
\]
Throughout this paper, when we say that a function defines a functional on a suitable function space it must be understood as above.

The convolution equation associated with the Hankel transform $h_{\mu}$ was investigated by D. T. Haimo [9] and I. I. Hirschman [11] on the Lebesgue spaces $L^p(x^{2\mu+1} \, dx)$. If $f, g \in L^1(x^{2\mu+1} \, dx)$, the Hankel convolution $f \#_{\mu} g$ of $f$ and $g$ is given through

$$(f \#_{\mu} g)(x) = \int_0^\infty (\mu \tau_x g)(y) f(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu + 1)} \, dy, \quad x \in (0, \infty),$$

where the Hankel translation operator $\mu \tau_x$, $x \in [0, \infty)$, is defined by

$$(\mu \tau_x g)(y) = \int_0^\infty D_{\mu}(x, y, z) g(z) \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu + 1)} \, dz, \quad x, y \in (0, \infty),$$

and $\mu \tau_0 g = g$, and being, for each $x, y, z \in (0, \infty)$,

$$D_{\mu}(x, y, z) = (2^\mu \Gamma(\mu + 1))^2 \int_0^\infty (xt)^{-\mu} J_\mu(xt)(yt)^{-\mu} J_\mu(yt)(zt)^{-\mu} J_\mu(zt)t^{2\mu+1} \, dt.$$ 

The Hankel convolution and the Hankel translations are related to the Hankel transformation $h_{\mu}$ by the following formulas (see [11]):

$$h_{\mu}(f \#_{\mu} g) = h_{\mu}(f) h_{\mu}(g)$$
$$h_{\mu}(\mu \tau_x g)(y) = 2^\mu \Gamma(\mu + 1)(xy)^{-\mu} J_\mu(xy) h_{\mu}(g)(y),$$

where $x, y \in [0, \infty)$ and $f, g \in L^1(x^{2\mu+1} \, dx)$. The Hankel convolution was studied in distribution spaces in [5], [6] and [12].

In the sequel, to simplify we will write $\#$, $\tau_x$, $x \in [0, \infty)$, and $D$, instead $\#_{\mu}$, $\mu \tau_x$, $x \in [0, \infty)$, and $D_{\mu}$, respectively. Throughout this paper by $C$ we always denote a suitable positive constant that can be changed from the one to the other line.

2. Hankel convolution operators in the space $H_{\mu,1}'$

We introduce the space $H_{\mu,\infty}$ as the closure of $H_{\mu,1}$ in $H_{\mu,\infty}$. In this Section, we characterize $H_{\mu,\infty}'$ as the space of Hankel convolution operators on $H_{\mu,1}'$.

Let $f \in H_{\mu,1}$. As it was proved in [3, Remark 1], $y^k h_{\mu}(f) \in L^1(x^{2\mu+1} \, dx)$ for every $k \in \mathbb{N}$. Hence, according to [11, Corollary 2.e],

$$f(x) = \int_0^\infty (xy)^{-\mu} J_\mu(xy) h_{\mu}(f)(y) y^{2\mu+1} \, dy \quad \text{for a.e. } x \in (0, \infty).$$

(2.1)
Here, a.e. refers to the Lebesgue measure on \((0, \infty)\). Moreover, the right hand side of (2.1) defines a smooth function on \((0, \infty)\) and by [17, (7), Chapter 5], for every \(k \in \mathbb{N}\),
\[
\left(\frac{1}{x} \frac{d}{dx}\right)^k \int_0^\infty (xy)^{-\mu} J_\mu(xy)h_\mu(f)(y)y^{2\mu+1} dy = (-1)^k \int_0^\infty (xy)^{-\mu-k} J_{\mu+k}(xy)h_\mu(f)(y)y^{2\mu+1+2k} dy,
\]
for each \(x \in (0, \infty)\). Thus we can consider \(H_{\mu,1}\) as a subspace of \(C^\infty(0, \infty)\).

Moreover, from (2.1) and [17, (6) and (7), Chapter 5] we can deduce that, for every \(k \in \mathbb{N}\),
\[
\Delta^k_\mu f(x) = \int_0^\infty (xy)^{-\mu} J_\mu(xy)(-y^2)^k h_\mu(f)(y)y^{2\mu+1} dy, \quad f \in H_{\mu,1}. \tag{2.2}
\]
Hence, \(H_{\mu,1}\) is continuously contained in \(H_{\mu,\infty}\). From (2.2), according to the Riemann-Lebesgue Lemma for the Hankel transform, we infer that, for every \(f \in H_{\mu,1}\) and \(k \in \mathbb{N}\),
\[
\lim_{x \to \infty} \Delta^k_\mu f(x) = 0.
\]
Also, (2.2) implies that, for every \(f \in H_{\mu,1}\) and \(k \in \mathbb{N}\),
\[
\lim_{x \to 0} \Delta^k_\mu f(x) = \frac{(-1)^k}{2^\mu \Gamma(\mu + 1)} \int_0^\infty f(y)y^{2k+2\mu+1} dy.
\]
Then, it is not hard to see that, for every \(f \in H_{\mu,\infty}\) and \(k \in \mathbb{N}\),
\[
\lim_{x \to \infty} \Delta^k_\mu f(x) = 0,
\]
and, there exists \(\lim_{x \to 0} \Delta^k_\mu f(x)\).

By \(H'_{\mu,1}\) and \(H'_{\mu,\infty}\) we denote, as usual, the dual spaces of \(H_{\mu,1}\) and \(H_{\mu,\infty}\), respectively. Since \(H_{\mu,1}\) is a dense subspace of \(H_{\mu,\infty}\), \(H'_{\mu,\infty}\) is a subspace of \(H'_{\mu,1}\). We now present characterizations of the elements of \(H'_{\mu,1}\) and \(H'_{\mu,\infty}\). They can be proved by employing standard procedures (see [3, Proposition 2.2]).

**Proposition 1.**

(i) Let \(T\) be a functional on \(H_{\mu,1}\). Then \(T \in H'_{\mu,1}\) if, and only if, there exist \(n \in \mathbb{N}\) and \(f_k \in L^\infty(dx)\), \(k = 0, 1, ..., n\), such that
\[
\langle T, \phi \rangle = \sum_{k=0}^n \int_0^\infty f_k(x)\Delta^k_\mu \phi(x)x^{2\mu+1} dx, \quad \phi \in H_{\mu,1}.
\]
(ii) Let $T$ be a functional on $H_{\mu,\infty}$. Then $T \in H'_{\mu,\infty}$ if, and only if, there exist $n \in \mathbb{N}$ and regular complex Borel measures $\gamma_k$ on $[0, \infty)$, $k = 0, 1, \ldots, n$, such that

$$
\langle T, \phi \rangle = \sum_{k=0}^{n} \int_{0}^{\infty} \Delta_{\mu}^k \phi(x) d\gamma_k(x), \quad \phi \in H_{\mu,\infty}.
$$

To study the Hankel convolution on the space $H'_{\mu,1}$ we need to analyze the behaviour of the Hankel translation $\tau_x$, $x \in [0, \infty)$, on the space $H_{\mu,1}$.

**Proposition 2.** Let $x \in (0, \infty)$. The Hankel translation $\tau_x$ defines a continuous linear mapping from $H_{\mu,1}$ into itself.

**Proof.** Let $f \in H_{\mu,1}$. Since $f \in L^1(x^{2\mu+1} \, dx)$, by [4, (3.1)], we can write

$$
h_{\mu}(\tau_x f)(y) = 2^\mu \Gamma(\mu + 1)(xy)^{-\mu} J_{\mu}(xy) h_{\mu}(f)(y), \quad y \in (0, \infty).
$$

Moreover, since $h_{\mu}(f) \in L^1(x^{2\mu+1} \, dx)$ and $z^{-\mu} J_{\mu}(z)$ is a bounded function on $(0, \infty)$, we get

$$
(\tau_x f)(y) = 2^\mu \Gamma(\mu + 1) h_{\mu}((xt)^{-\mu} J_{\mu}(xt) h_{\mu}(f)(t))(y), \quad y \in (0, \infty).
$$

Then, for every $k \in \mathbb{N}$,

$$
\Delta_{\mu}^k (\tau_x f)(y) = (-1)^k 2^\mu \Gamma(\mu + 1) h_{\mu}((xt)^{-\mu} J_{\mu}(xt) t^{2k} h_{\mu}(f)(t))(y)
= \tau_x (\Delta_{\mu}^k f)(y), \quad y \in (0, \infty).
$$

Note that the differentiation under the integral sign is justified because, for every $k \in \mathbb{N}$, $y^{2k} h_{\mu}(f) \in L^1(x^{2\mu+1} \, dx)$.

By invoking now [15, p. 17], for every $k \in \mathbb{N}$, we get

$$
\|\Delta_{\mu}^k (\tau_x f)\|_{\mu,1} \leq C \|\Delta_{\mu}^k f\|_{\mu,1}.
$$

Hence $\tau_x f \in H_{\mu,1}$. That the mapping $f \to \tau_x f$ is continuous from $H_{\mu,1}$ into itself follows also from the inequality (2.3).

We now study the behaviour of the Hankel convolution on the space $H_{\mu,1}$.

**Proposition 3.** The Hankel convolution defines a continuous bilinear mapping from $H_{\mu,1} \times H_{\mu,1}$ into $H_{\mu,1}$.

**Proof.** Let $f, g \in H_{\mu,1}$, $\phi \in S_{e}$ and $k \in \mathbb{N}$. Then, $f \# \phi \in C^\infty(0, \infty)$ and we can write

$$
\Delta_{\mu}^k (f \# \phi) = (\Delta_{\mu}^k f) \# \phi = f \# (\Delta_{\mu}^k \phi).
$$
Moreover,
\[
\langle \Delta^k_k(f \# g)(x), \phi \rangle = \langle f \# g, \Delta^k_k\phi \rangle \\
= \int_0^\infty (f \# g)(y) \Delta^k_k\phi(y) \frac{y^{2\mu+1}}{2\mu\Gamma(\mu + 1)} \, dy \\
= \int_0^\infty f(y)(g \# \Delta^k_k\phi)(y) \frac{y^{2\mu+1}}{2\mu\Gamma(\mu + 1)} \, dy \\
= \int_0^\infty f(y)((\Delta^k_k g) \# \phi)(y) \frac{y^{2\mu+1}}{2\mu\Gamma(\mu + 1)} \, dy \\
= \int_0^\infty (f \# \Delta^k_k g)(y)\phi(y) \frac{y^{2\mu+1}}{2\mu\Gamma(\mu + 1)} \, dy.
\]

Hence \( \Delta^k_k(f \# g) = f \# \Delta^k_k g \). Then, according to [11, Theorem 2.b], we get
\[
\|\Delta^k_k(f \# g)\|_{\mu,1} \leq \|f\|_{\mu,1}\|\Delta^k_k g\|_{\mu,1}.
\]

Thus we conclude that the \#-convolution defines a bilinear and continuous mapping from \( H_{\mu,1} \times H_{\mu,1} \) into \( H_{\mu,1} \). \( \blacksquare \)

By virtue of Proposition 2 we can define the Hankel convolution \( T \# f \) of \( T \in H'_{\mu,1} \) and \( f \in H_{\mu,1} \) as the function
\[
(T \# f)(x) = \langle T, \tau_x f \rangle, \quad x \in [0, \infty).
\]

**Proposition 4.** Let \( T \in H'_{\mu,1} \) and \( f \in H_{\mu,1} \). Then \( T \# f \) is a continuous and bounded function on \( (0, \infty) \). Hence \( T \# f \in H_{\mu,1} \) and, for every \( k \in \mathbb{N} \), we have that
\[
\Delta^k_k(T \# f) = (\Delta^k_k T) \# f = T \# (\Delta^k_k f).
\]

**Proof.** According to Proposition 1, (i) we can find \( n \in \mathbb{N} \) and \( g_k \in L^\infty(dx) \), \( k = 0, 1, \ldots, n \), such that
\[
\langle T, \phi \rangle = \sum^n_{k=0} \int_0^\infty g_k(x)\Delta^k_k\phi(x)x^{2\mu+1} \, dx, \quad \phi \in H_{\mu,1}.
\]

Then,
\[
(T \# f)(x) = \sum^n_{k=0} \int_0^\infty g_k(y)\Delta^k_k\phi(y)g(y)^{2\mu+1} \, dy \\
= \sum^n_{k=0} \int_0^\infty g_k(y)\phi(\Delta^k_k f)(y)y^{2\mu+1} \, dy, \quad x \in (0, \infty).
\]
Hence, by taking into account [15, p. 17], we can conclude that $T \# f$ is a continuous and bounded function on $(0, \infty)$. Then $T \# f$ defines an element of $H'_{\mu,1}$ by

$$\langle T \# f, \phi \rangle = \int_0^\infty (T \# f)(x) \phi(x) \frac{x^{2\mu+1}}{2\mu \Gamma(\mu + 1)} \, dx, \quad \phi \in H_{\mu,1}. $$

Moreover, we can write

$$\langle T \# f, \phi \rangle = \sum_{k=0}^n \int_0^\infty \phi(x) \int_0^\infty g_k(y) \tau_x (\Delta^k_{\mu} f)(y) \, dy \, 2^{M} \Gamma(\mu + 1) \, dy $$

$$= \sum_{k=0}^n \int_0^\infty \phi(x) \tau_y (\Delta^k_{\mu} f)(x) \, 2^{M} \Gamma(\mu + 1) \, dy $$

$$= \sum_{k=0}^n \int_0^\infty \phi(y) \Delta^k_{\mu} f(y) \, dy $$

$$= \langle T, f \# \phi \rangle, \quad \phi \in H_{\mu,1}. $$

Then, a straightforward manipulation leads, for every $k \in \mathbb{N}$ and $\phi \in H_{\mu,1}$, to

$$\langle \Delta^k_{\mu} (T \# f), \phi \rangle = \langle T \# f, \Delta^k_{\mu} \phi \rangle = \langle T, f \# \Delta^k_{\mu} \phi \rangle = \langle T, \Delta^k_{\mu} (f \# \phi) \rangle $$

$$= \langle \Delta^k_{\mu} f, \phi \rangle = \langle T, (\Delta^k_{\mu} f) \# \phi \rangle = \langle T \# \Delta^k_{\mu} f, \phi \rangle. $$

Our next objective is to characterize $H'_{\mu,\infty}$ as the subspace of $H'_{\mu,1}$ that defines convolution operators on $H_{\mu,1}$.

**Proposition 5.** Let $T \in H'_{\mu,1}$. If $T \in H'_{\mu,\infty}$, then $T \# \phi \in H_{\mu,1}$ for every $\phi \in H_{\mu,1}$.

**Proof.** Assume that $T \in H'_{\mu,\infty}$. Then there exist $n \in \mathbb{N}$ and regular complex Borel measures $\gamma_k$, $k = 0, 1, \ldots, n$, on $[0, \infty)$ such that

$$\langle T, f \rangle = \sum_{k=0}^n \int_0^\infty \Delta^k_{\mu} f(x) \, d\gamma_k(x), \quad f \in H_{\mu,\infty}. $$

Let $\phi \in H_{\mu,1}$. We have that

$$(T \# \phi)(x) = \sum_{k=0}^n \int_0^\infty \tau_x (\Delta^k_{\mu} \phi)(y) \, d\gamma_k(y) $$

$$= \sum_{k=0}^n 2^{\mu} \Gamma(\mu + 1) \int_0^\infty h_{\mu}((xt)^{-\mu} J_{\mu}(xt) h_{\mu}(\Delta^k_{\mu} \phi)(t)) \, d\gamma_k(y).$$
Hence, since \(y^{2k}h_{\mu}(\phi) \in L_1(x^{2\mu+1} \, dx)\), for every \(k \in \mathbb{N}\), \([17, (7), \text{Chapter 5}]\) allows us to prove that \(T\#\phi\) is an smooth function on \((0, \infty)\). Moreover, for every \(l \in \mathbb{N}\), we get

\[
\Delta_{\mu}^l(T\#\phi)(x) = \sum_{k=0}^{n} \int_{0}^{\infty} \tau_x(\Delta_{\mu}^{k+l}\phi)(y)d\gamma_k(y), \quad x \in (0, \infty).
\]

By interchanging the order of integration and according to \([15, \text{p. 17}]\) we can obtain

\[
\|\Delta_{\mu}^l(T\#\phi)\|_{\mu, 1} \leq \sum_{k=0}^{n} \int_{0}^{\infty} \int_{0}^{\infty} |\tau_x(\Delta_{\mu}^{k+l}\phi)(y)|d\mu_k|(y)\cdot x^{2\mu+1} \, dx
\]

\[
\leq C \sum_{k=0}^{n} \|\Delta_{\mu}^{k+l}\phi\|_{\mu, 1}, \quad l \in \mathbb{N}.
\]

Thus we have proved that \(T\#\phi \in H_{\mu, 1}\). □

We now define the Hankel convolution \(T\#S\) of \(T \in H'_{\mu, 1}\) and \(S \in H'_{\mu, \infty}\) as follows:

\[
\langle T\#S, \phi \rangle = \langle T, S\#\phi \rangle, \quad \phi \in H_{\mu, 1}.
\]

Thus \(T\#S \in H'_{\mu, 1}\). Next we prove some properties of the \#-convolution on the spaces under consideration that will be useful in the sequel.

**Proposition 6.** Let \(T \in H'_{\mu, 1}\) and \(R, S \in H'_{\mu, \infty}\). Then:

(i) \(R\#S \in H'_{\mu, \infty}\) and \(R\#S = S\#R\).

(ii) \(\Delta_{\mu}(T\#R) = (\Delta_{\mu}T)\#R = T\#(\Delta_{\mu}R)\).

(iii) \(T\#(R\#S) = (T\#R)\#S\).

**Proof.** (i): According to Proposition 1, (ii), we can write

\[
\langle R, \phi \rangle = \sum_{k=0}^{r} \int_{0}^{\infty} \Delta_{\mu}^k\phi(x)d\gamma_k(x), \quad \phi \in H_{\mu, \infty}
\]

\[
\langle S, \phi \rangle = \sum_{k=0}^{\alpha} \int_{0}^{\infty} \Delta_{\mu}^k\phi(x)d\nu_k(x), \quad \phi \in H_{\mu, \infty}
\]

where \(r, \alpha \in \mathbb{N}\) and \(\gamma_0, \ldots, \gamma_k\) and \(\nu_0, \ldots, \nu_\alpha\) are complex regular Borel measures on \([0, \infty)\). Let \(\phi \in H_{\mu, 1}\). We can write

\[
\langle R\#S, \phi \rangle = \langle R, S\#\phi \rangle
\]

\[
= \sum_{k=0}^{r} \int_{0}^{\infty} \Delta_{\mu}^k(S\#\phi)(x)d\gamma_k(x)
\]

\[
= \sum_{k=0}^{r} \int_{0}^{\infty} \Delta_{\mu}^k, x \sum_{l=0}^{\alpha} \int_{0}^{\infty} \tau_x(\Delta_{\mu}^l\phi)(y)d\nu_l(y)d\gamma_k(x)
\]
and hence, by using \[4, (3.1)\] and Fubini’s theorem,

\[
\langle R\# S, \phi \rangle = \sum_{k=0}^{r} \sum_{l=0}^{\alpha} \int_{0}^{\infty} \int_{0}^{\infty} \tau_x(\Delta_{\mu}^{k+l} \phi)(y) \nu_t(y) d\gamma_k(x) \tag{2.5}
\]

\[
= 2^\mu \Gamma(\mu + 1) \sum_{k=0}^{r} \sum_{l=0}^{\alpha} \int_{0}^{\infty} \int_{0}^{\infty} h_{\mu}(xt)^{-\mu} J_{\mu}(xt) \times h_{\mu}(\Delta_{\mu}^{k+l} \phi)(t) \times t^{2\mu+1} dt d\gamma_k(x) \tag{2.6}
\]

Here, if \(\gamma\) is a complex regular Borel measure on \([0, \infty)\), \(h_{\mu}(\gamma)\) is defined by

\[
h_{\mu}(\gamma)(x) = \int_{0}^{\infty} (xt)^{-\mu} J_{\mu}(xt) d\gamma(t), \quad x \in (0, \infty).
\]

From (2.5) we deduce that,

\[
|\langle R\# S, \phi \rangle| \leq C \sum_{k=0}^{r} \sum_{l=0}^{\alpha} \|\Delta_{\mu}^{k+l} \phi\|_{\mu, \infty}.
\]

Thus, we conclude that \(R\# S\) defines a linear and continuous mapping on \(H_{\mu,1}\) when we consider on \(H_{\mu,1}\) the topology induced by \(H_{\mu,\infty}\). Hence, since \(H_{\mu,1}\) is dense in \(H_{\mu,\infty}\), \(R\# S\) can be extended in a unique way to \(H_{\mu,\infty}\) as an element of \(H'_{\mu,\infty}\). The equality \(R\# S = S\# R\) follows immediately from (2.6).

\(\text{(ii):}\) Suppose that \(R\) admits the representation (2.4) and that

\[
\langle T, \phi \rangle = \sum_{l=0}^{\alpha} \int_{0}^{\infty} f_l(x) \Delta_{\mu}^l \phi(x) dx, \quad \phi \in H_{\mu,1},
\]

where \(\alpha \in \mathbb{N}\) and \(f_l \in L^\infty(dx), l = 0, ..., \alpha\). We can write, for every \(\phi \in H_{\mu,1},\)

\[
\langle \Delta_{\mu}(T\# R), \phi \rangle = \langle T\# R, \Delta_{\mu} \phi \rangle
\]

\[
= \langle T, R\#(\Delta_{\mu} \phi) \rangle
\]

\[
= \sum_{l=0}^{\alpha} \int_{0}^{\infty} f_l(t) \Delta_{\mu}^l \sum_{k=0}^{r} \int_{0}^{\infty} \tau_t(\Delta_{\mu}^{k+l} \phi)(x) d\gamma_k(x) dt
\]

\[
= \sum_{l=0}^{\alpha} \int_{0}^{\infty} f_l(t) \sum_{k=0}^{r} \int_{0}^{\infty} \tau_t(\Delta_{\mu}^{k+l} \phi)(x) d\gamma_k(x) dt,
\]

\[
\tau_t(\Delta_{\mu}^{k+l} \phi)(x) = t^{\mu} \Gamma(\mu + 1) \sum_{k=0}^{r} \sum_{l=0}^{\alpha} \int_{0}^{\infty} \int_{0}^{\infty} h_{\mu}(\Delta_{\mu}^{k+l} \phi)(t) h_{\mu}(\nu_t)(t) h_{\mu}(\gamma_k)(t) \times t^{2\mu+1} dt d\gamma_k(x) \tag{2.6}
\]
which proves (ii).

(iii): By using the procedure developed in the proof of (i) and (ii) and according to Proposition 1, it is sufficient to prove the property when

$$\langle T, \phi \rangle = \int_0^\infty f(t)\Delta^{k}_{\mu} \phi(t) t^{2\mu + 1} dt, \quad \phi \in H_{\mu,1}$$

$$\langle R, \phi \rangle = \int_0^\infty \Delta^{l}_{\mu} \phi(t) d\gamma(t), \quad \phi \in H_{\mu,\infty}$$

$$\langle S, \phi \rangle = \int_0^\infty \Delta^{m}_{\mu} \phi(t) d\nu(t), \quad \phi \in H_{\mu,\infty},$$

where \( f \in L^\infty(dx) \), \( \gamma \) and \( \nu \) are complex regular Borel measures on \([0, \infty)\) and \( k, l, m \in \mathbb{N} \).

Let \( \phi \in H_{\mu,1} \). By (2.6) we have

$$\langle R#S, \phi \rangle = 2^\mu \Gamma(\mu + 1) \int_0^\infty h_\mu(\Delta^{l+m}_{\mu} \phi(t)) h_\mu(\gamma(t)) h_\mu(\nu(t)) t^{2\mu + 1} dt.$$  

Then \( \langle T#(R#S), \phi \rangle = \langle T, (R#S)#\phi \rangle \), and hence

$$\langle T#(R#S), \phi \rangle = \int_0^\infty f(t)\Delta^{k}_{\mu}((R#S)#\phi)(t) t^{2\mu + 1} dt$$

$$= \int_0^\infty f(x)\Delta^{k}_{\mu,x}(2^\mu \Gamma(\mu + 1) \int_0^\infty h_\mu(\Delta^{l+m}_{\mu} \tau_x \phi)(t)$$

$$\times h_\mu(\gamma(t)) h_\mu(\nu(t)) t^{2\mu + 1} dt)x^{2\mu + 1} dx$$

$$= 2^\mu \Gamma(\mu + 1) \int_0^\infty f(x) \int_0^\infty h_\mu(\tau_x(\Delta^{l+m+k}_{\mu} \phi))(t)$$

$$\times h_\mu(\gamma(t)) h_\mu(\nu(t)) t^{2\mu + 1} dt x^{2\mu + 1} dx$$

$$= (2^\mu \Gamma(\mu + 1))^2 \int_0^\infty f(x) \int_0^\infty (xt)^{-\mu} J_\mu(xt) h_\mu(\Delta^{k+l+m}_{\mu} \phi)(t)$$

$$\times h_\mu(\gamma(t)) h_\mu(\nu(t)) t^{2\mu + 1} dt x^{2\mu + 1} dx.$$  

Also we can write

$$\langle T#R, \phi \rangle = \langle T, R#\phi \rangle$$

$$= \int_0^\infty f(x)\Delta^{k}_{\mu,x} \int_0^\infty \Delta^{l}_{\mu,t}(\tau_z \phi)(t)d\gamma(t)x^{2\mu + 1} dx$$

$$= 2^\mu \Gamma(\mu + 1) \int_0^\infty f(x) \int_0^\infty h_\mu((xz)^{-\mu} J_\mu(xz)$$

$$\times h_\mu(\Delta^{l+k}_{\mu} \phi)(z))(t)d\gamma(t)x^{2\mu + 1} dx$$

$$= 2^\mu \Gamma(\mu + 1) \int_0^\infty f(x) \int_0^\infty (xz)^{-\mu} J_\mu(xz)$$

$$\times h_\mu(\Delta^{l+k}_{\mu} \phi)(z) h_\mu(\gamma(z) z^{2\mu + 1} d\gamma(z)) x^{2\mu + 1} dx.$$
Hence, by using (\(ii\)),

\[
\langle (T\#R)\#S, \phi \rangle = \langle T\#R, S\#\phi \rangle
\]

\[
= 2^\mu \Gamma(\mu + 1) \int_0^\infty f(x) \int_0^\infty (xz)^{-\mu} J_\mu(xz) h_\mu(\Delta^{1+k}_\mu(S\#\phi))(z)
\]

\[
\times h_\mu(\gamma)(z)(zx)^{2\mu+1} \, dz \, dx
\]

\[
= 2^\mu \Gamma(\mu + 1) \int_0^\infty f(x) \int_0^\infty (xz)^{-\mu} J_\mu(xz) h_\mu(\gamma)(z)
\]

\[
\int_0^\infty (tz)^{-\mu} J_\mu(tz) \int_0^\infty \tau_\mu(\Delta^{k+l+m}_\mu \phi)(y)d\nu(y) (tx)^{2\mu+1} \, dt \, dz \, dx
\]

\[
= (2^\mu \Gamma(\mu + 1))^2 \int_0^\infty f(x) \int_0^\infty (xz)^{-\mu} J_\mu(xz) h_\mu(\gamma)(z)
\]

\[
\int_0^\infty (yz)^{-\mu} J_\mu(yz) d\nu(y) h_\mu(\Delta^{k+l+m}_\mu \phi)(y)(zx)^{2\mu+1} \, dz \, dx
\]

\[
= (2^\mu \Gamma(\mu + 1))^2 \int_0^\infty f(x) \int_0^\infty (xz)^{-\mu} J_\mu(xz) h_\mu(\gamma)(z) h_\mu(\nu)(z)
\]

\[
\times h_\mu(\Delta^{k+l+m}_\mu \phi)(y)(zx)^{2\mu+1} \, dz \, dx.
\]

Thus we conclude that \(T\#(R\#S) = (T\#R)\#S\). \(\blacksquare\)

**Proposition 7.** Let \(T \in H_{\mu,1}'\). Then, \(T \in H_{\mu,\infty}'\) provided \(T\#\phi \in H_{\mu,1}\), for every \(\phi \in H_{\mu,1}\).

**Proof.** Suppose that \(T\#\phi \in H_{\mu,1}\), for every \(\phi \in H_{\mu,1}\). By [3, Proposition 1.1], for every \(m \in \mathbb{N}\) there exists an \(r \in \mathbb{N}\) such that

\[
\delta = (1 - \Delta_\mu)^r \varphi + \psi,
\]

where \(\delta\) is the Dirac functional, \(\psi \in S_e\) and \(\varphi \in C^{2m}(0, \infty)\), for which \(\psi(x) = \varphi(x) = 0, x \geq 1,\) and

\[
\lim_{x \to 0^+} \left( \frac{1}{x} \right)^k \varphi(x) = 0,
\]

for every \(k = 0, 1, \ldots, 2m\). The equality in (2.7) is understood in [3] in \(S_e'\). It is not hard to see that (2.7) also holds in \(H_{\mu,\infty}'\).

We now choose \(m \in \mathbb{N}\) large enough. According to Proposition 6 we have

\[
T = T\#\delta = (1 - \Delta_\mu)^r (T\#\varphi) + T\#\psi.
\]

By the assumption, \(T\#\psi \in H_{\mu,1}\). Then \(T\#\psi \in H_{\mu,\infty}'\).

Assume now \((k_n)_{n \in \mathbb{N}}\) is a sequence in \(S_e\) such that, for every \(n \in \mathbb{N}\), \(k_n\) satisfies the following properties:
(a) $k_n \geq 0$
(b) $k_n(x) = 0$, $x \notin \left( \frac{1}{n+1}, \frac{1}{n} \right)$
(c) $\int_0^\infty k_n(x)x^{2\mu+1}dx = 2^n\Gamma(\mu + 1)$.

From [4, Proposition 3.5] we deduce that $k_n\#\phi \in S_e$ and $(k_n\#\phi)(x) = 0$, $x \geq 2$, for every $n \in \mathbb{N}$, and

$$\sup_{x \in (0, \infty)} |\Delta_{\mu}^k(k_n\#\phi - \phi)(x)| \to 0, \quad \text{as } n \to \infty,$$

for $k = 0, 1, \ldots, m$. According to Proposition 1 (i), there exist $l \in \mathbb{N}$ and $f_k \in L^\infty(dx)$, $k = 0, \ldots, l$, such that, for every $\phi \in H_{\mu,1}$,

$$(T\#\phi)(x) = \sum_{k=0}^l \int_0^\infty f_k(y)\tau_x(\Delta_{\mu}^k\phi)(y)y^{2\mu+1}dy, \quad x \in (0, \infty).$$

Then, by [15, p. 17] we get

$$|(T\#(k_n\#\phi) - T\#\phi)(x)| \leq C \sum_{k=0}^l \|\Delta_{\mu}^k(\phi\#k_n - \phi)\|_{\mu,1}$$

$$\leq C \sum_{k=0}^l \sup_{x \in (0, \infty)} |\Delta_{\mu}^k(\phi\#k_n - \phi)(x)| \to 0$$

as $n \to \infty$, uniformly in $(0, \infty)$.

On the other hand, $T\#(k_n\#\phi) \in H_{\mu,1}$, for every $n \in \mathbb{N}$. Moreover, we can write, for certain $l \in \mathbb{N}$,

$$\|T\#(k_n\#\phi)\|_{\mu,1} \leq C \sum_{k=0}^l \|\Delta_{\mu}^k\phi\|_{\mu,1} \|k_n\|_{\mu,1} \leq C \sum_{k=0}^l \|\Delta_{\mu}^k\phi\|_{\mu,1}$$

for every $n \in \mathbb{N}$. Hence we conclude that $T\#\phi \in L^1(x^{2\mu+1}dx)$.

Thus, since the operator $\Delta_{\mu}$ defines a continuous linear mapping from $H_{\mu,\infty}$ into itself, we establish that $T \in H_{\mu,\infty}$. \hfill \Box

**Proposition 8.** Let $T \in H'_{\mu,1}$. Then, $T\#\phi \in H_{\mu,1}$, for every $\phi \in H_{\mu,1}$, if, and only if, the mapping $\phi \to T\#\phi$ is continuous from $H_{\mu,1}$ into itself.

**Proof.** Assume that $T\#\phi \in H_{\mu,1}$, for every $\phi \in H_{\mu,1}$. To see that the mapping $\phi \to T\#\phi$ is continuous from $H_{\mu,1}$ into itself, we are going to use the closed graph theorem. Suppose that $(\phi_n)_{n \in \mathbb{N}}$ is a sequence in $H_{\mu,1}$ such that $\phi_n \to \phi$ and $T\#\phi_n \to \psi$, as $n \to \infty$, in $H_{\mu,1}$, where $\phi, \psi \in H_{\mu,1}$. We can write (Proposition 1, (i))

$$\langle T, \varphi \rangle = \sum_{k=0}^l \int_0^\infty f_k(y)\Delta_{\mu}^k\varphi(y)y^{2\mu+1}dy, \quad \varphi \in H_{\mu,1}.$$
for certain \( l \in \mathbb{N} \) and \( f_k \in L^\infty(dx) \), \( k = 0, 1, \ldots, l \).

We assume that, by taking a subsequence if it is necessary, \( T\# \phi_n \to \psi \), a.e. on \((0, \infty)\). Moreover, we have that

\[
| (T\# \phi_n - T\# \phi)(x) | \leq \sum_{k=0}^{l} \int_0^\infty |f_k(y)| \tau_x(\| \Delta_k^\mu (\phi_n - \phi) \|)(y) y^{2n+1} dy \\
\leq C \sum_{k=0}^{l} \| \Delta_k^\mu (\phi_n - \phi) \|_{\mu, 1}, \quad x \in \mathbb{R}.
\]

Hence \( T\# \phi_n \to T\# \phi \), as \( n \to \infty \), uniformly in \((0, \infty)\). Thus we conclude that \( T\# \phi = \psi \), and the proof is finished.

We summarize the above results in the following theorem, where the space \( H'_{\mu, \infty} \) is characterized as the space of convolution operators of \( H_{\mu, 1} \).

**Theorem 1.** Let \( T \in H'_{\mu, 1} \). Then the following assertions are equivalent:

(i) \( T \in H'_{\mu, \infty} \).

(ii) \( T\# \phi \in H_{\mu, 1} \), for every \( \phi \in H_{\mu, 1} \).

(iii) The mapping \( \phi \mapsto T\# \phi \) is continuous from \( H_{\mu, 1} \) into itself.

### 3. The Hankel transformation on the space \( H'_{\mu, 1} \)

We now define the Hankel transformation on the space \( H'_{\mu, 1} \). We denote by \( S_{\mu, 1} \) the space of Hankel transforms of functions in \( H_{\mu, 1} \), that is, \( S_{\mu, 1} = \{ h_\mu(\phi) : \phi \in H_{\mu, 1} \} \).

Since \( h_\mu \) is one to one on \( H_{\mu, 1} \), we endow to \( S_{\mu, 1} \) with the topology induced on it by \( H_{\mu, 1} \) via \( h_\mu \). Thus, \( S_{\mu, 1} \) is a Fréchet space. It is not hard to see that if \( P \) is a polynomial, then the mapping \( \psi \mapsto P(x^2) \psi \) is continuous from \( S_{\mu, 1} \) into itself.

Let \( T \in H'_{\mu, 1} \). The Hankel transform \( h_\mu' T \) is defined as the element of \( S'_{\mu, 1} \), the dual space of \( S_{\mu, 1} \), given by

\[
\langle h_\mu' T, h_\mu \phi \rangle = \langle T, \phi \rangle, \quad \phi \in H_{\mu, 1}.
\]

Then if \( T \in H'_{\mu, \infty} \) is given by

\[
\langle T, \phi \rangle = \sum_{k=0}^{l} \int_0^\infty \Delta_k^\mu \phi(x) d\gamma_k(x), \quad \phi \in H_{\mu, \infty}.
\]
where \( l \in \mathbb{N} \) and \( \gamma_k \) is a regular complex Borel measure on \([0, \infty)\), \( k = 0, 1, \ldots, l \), we can write

\[
\langle h'_\mu T, \phi \rangle = \sum_{k=0}^{l} \int_0^\infty \Delta^k \mu(\phi)(y) d\gamma_k(y)
\]

\[
= \sum_{k=0}^{l} (-1)^k \int_0^\infty \int_0^\infty x^{2k} (xy)^{-\mu} J_\mu(xy) \phi(x) x^{2\mu+1} dx d\gamma_k(y)
\]

\[
= \sum_{k=0}^{l} (-1)^k \int_0^\infty \phi(x) x^{2k} \int_0^\infty (xy)^{-\mu} J_\mu(xy) d\gamma_k(y) x^{2\mu+1} dx, \quad \phi \in S_{\mu,1}.
\]

Hence, we obtain that \( h_\mu(T) \) is a continuous function on \((0, \infty)\) and

\[
h'_\mu(T) = 2^\mu \Gamma(\mu + 1) \sum_{k=0}^{l} (-1)^k x^{2k} h_\mu(\gamma_k).
\]

Hence \( h'_\mu(T) \) is an slow growth function.

We now prove interchange distributional formulas.

**Proposition 9.** Let \( T \in H'_{\mu,1}, S \in H'_{\mu,\infty} \) and \( \phi \in H_{\mu,1} \). Then:

\[
h_\mu(S \# \phi) = h'_\mu(S) h_\mu(\phi) \quad (3.1)
\]

\[
h_\mu(T \# S) = h'_\mu(T) h'_\mu(S). \quad (3.2)
\]

**Proof.** Assume that

\[
\langle S, \psi \rangle = \sum_{k=0}^{l} \int_0^\infty \Delta^k \psi(y) d\gamma_k(y), \quad \psi \in H_{\mu,\infty},
\]

where \( l \in \mathbb{N} \) and \( \gamma_k, k = 0, 1, \ldots, l \) is a regular complex Borel measure on \([0, \infty)\). Then,

\[
(S \# \phi)(x) = \sum_{k=0}^{l} \int_0^\infty \tau_x(\Delta^k \phi)(y) d\gamma_k(y), \quad x \in (0, \infty).
\]

Hence, by interchanging the order of integration, we obtain

\[
h_\mu(S \# \phi)(x) = \sum_{k=0}^{l} \int_0^\infty \int_0^\infty \tau_t(\Delta^k \phi)(y) d\gamma_k(y)(xt)^{-\mu} J_\mu(xt) t^{2\mu+1} dt
\]

\[
= 2^\mu \Gamma(\mu + 1) \sum_{k=0}^{l} (-1)^k \int_0^\infty h_\mu(\phi)(xy)^{-\mu} J_\mu(xy) x^{2k} d\gamma_k(y)
\]

\[
= h_\mu(\phi)(x) h'_\mu(S)(x), \quad x \in (0, \infty).
\]
Let now $\psi \in S_{\mu,1}$. We can write

$$\langle h'_\mu(T \# S), \psi \rangle = \langle T \# S, h_\mu(\psi) \rangle = \langle T, S \# h_\mu(\psi) \rangle = \langle h'_\mu(T), h'_\mu(S) \psi \rangle = (h'_\mu(T)h'_\mu(S), \psi).$$

Thus (3.2) is shown.

\section{Hypoellipticity of Hankel convolution equations.}

Our next objective is to analyze the hypoellipticity of Hankel convolution equations in $H'_{\mu,1}$. Firstly we prove a useful result.

\textbf{Proposition 10.} Assume that $(a_j)_{j \in \mathbb{N}} \subset \mathbb{C}$ and $(\xi_j)_{j \in \mathbb{N}} \subset (0, \infty)$ such that $2^j \leq 2\xi_{j-1} < \xi_j$, $j \in \mathbb{N}$, and that $|a_j| = O(\xi_j^{-m})$, as $j \to \infty$, for some $m \in \mathbb{N}$. We define the functional $S \in S'_e$ as follows

$$S(x) = 2^\mu \Gamma(\mu + 1) \sum_{j=1}^{\infty} (x \xi_j)^{-\mu} J_\mu(x \xi_j) a_j.$$

Then $S \in H'_{\mu,1}$. Moreover, $S \in H_{\mu,\infty}$ if, and only if, $|a_j| = o(\xi_j^{-n})$ as $j \to \infty$, for every $n \in \mathbb{N}$.

\textbf{Proof.} Let $l_1, l_2 \in \mathbb{N}$, $l_1 < l_2$, and $\phi \in H_{\mu,1}$. We can write

$$\left| \sum_{j=l_1}^{l_2} a_j \int_0^\infty (x \xi_j)^{-\mu} J_\mu(x \xi_j) \phi(x) x^{2\mu+1} dx \right| \leq \sum_{j=l_1}^{l_2} |a_j||h_\mu(\phi)(\xi_j)| \leq C \sum_{j=l_1}^{l_2} \xi_j^m |h_\mu(\phi)(\xi_j)|.$$

Since $y^k h_\mu(\phi)$ is bounded on $(0, \infty)$, for every $k \in \mathbb{N}$ (because, for each $k \in \mathbb{N}$, $\Delta^k \phi \in L^1(x^{2\mu+1} dx)$, and since $\xi_j > 2^j$, $j \in \mathbb{N}$, the series $\sum_{j=1}^{\infty} a_j h_\mu(\phi)(\xi_j)$ converges and $S \in H'_{\mu,1}$.

Suppose now that $|a_j| = o(\xi_j^n)$, as $j \to \infty$, for every $n \in \mathbb{N}$. By [17, (6) and (7), Chapter 5] we can see that $S \in C^\infty(0, \infty)$ and that, for every $l \in \mathbb{N}$,

$$\Delta^{\lambda}_\mu S(x) = 2^\mu \Gamma(\mu + 1) \sum_{j=1}^{\infty} (-\xi_j)^{2l} (x \xi_j)^{-\mu} J_\mu(x \xi_j) a_j, \quad x \in (0, \infty).$$

Then, for every $l \in \mathbb{N}$, $\|\Delta^{\lambda}_\mu S\|_{\mu,\infty} \leq C \sum_{j=1}^{\infty} \xi_j^{2l} |a_j| < \infty$. Thus we have proved that $S \in H_{\mu,\infty}$. 

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Assume now that \( S \in H_{\mu,\infty} \). To see that \(|a_j| = o(\xi^n)\), as \( j \to \infty \), for every \( n \in \mathbb{N} \), we proceed as in the proof of [6, Proposition 3.2]. Let \( k \in \mathbb{N} \) and \( \phi \in S_c \). By [4, (3.1)] and integrating by parts we can write
\[
\langle (xt)^{-\mu}J_{\mu}(xt)\Delta^k_{\mu}S(x), \phi \rangle = \langle \Delta^k_{\mu}S(x), (xt)^{-\mu}J_{\mu}(xt)\phi(x) \rangle
\]
\[
= \langle S(x), \Delta^k_{\mu,x}((xt)^{-\mu}J_{\mu}(xt)\phi(x)) \rangle
\]
\[
= 2^{\mu}\Gamma(\mu + 1) \sum_{j=1}^{\infty} \langle (x\xi_j)^{-\mu}J_{\mu}(x\xi_j)a_j; \Delta^k_{\mu,x}((xt)^{-\mu}J_{\mu}(xt)\phi(x)) \rangle
\]
\[
= \sum_{j=1}^{\infty} a_j \int_{0}^{\infty} (x\xi_j)^{-\mu}J_{\mu}(x\xi_j)\Delta^k_{\mu,x}((xt)^{-\mu}J_{\mu}(xt)\phi(x)) x^{2\mu+1} dx
\]
\[
= \frac{1}{2^{\mu}\Gamma(\mu + 1)} \sum_{j=1}^{\infty} a_j (-1)^j \xi_j^{2k}\tau_{\xi_j}(h_{\mu,\phi})(t), \quad t \in (0, \infty). \quad (4.1)
\]

Moreover, we have
\[
\langle (xt)^{-\mu}J_{\mu}(xt)\Delta^k_{\mu}S(x), \phi(x) \rangle
\]
\[
= \int_{0}^{\infty} (xt)^{-\mu}J_{\mu}(xt)\Delta^k_{\mu}S(x)\phi(x) x^{2\mu+1} \frac{dx}{2^{\mu}\Gamma(\mu + 1)}, \quad t \in (0, \infty).
\]

Since \( \Delta^k_{\mu}S(x)\phi \in L^1(x^{2\mu+1} dx) \), the Riemann-Lebesgue theorem for Hankel transforms implies that
\[
\lim_{t \to \infty} \langle (xt)^{-\mu}J_{\mu}(xt)\Delta^k_{\mu}S(x), \phi(x) \rangle = 0. \quad (4.2)
\]

We now choose a function \( \phi \in S_c \) such that \( h_{\mu,\phi}(x) = 0 \), \( x > 1 \), and \( h_{\mu,\phi}(x) > \frac{1}{2} \), \( x \in (0, \frac{1}{2}) \). By (4.1), since \( \tau_{\xi}(h_{\mu,\phi})(x) = 0 \), provided that \(|z - x| > 1\), we have
\[
|\langle (x\xi_j)^{-\mu}J_{\mu}(x\xi_j)\Delta^k_{\mu}S, \phi \rangle| = |a_j (-1)^j \xi_j^{2k}\tau_{\xi_j}(h_{\mu,\phi})(\xi_j)|
\]
\[
\geq |a_j| \xi_j^{2k-2\mu-1}, \quad j \in \mathbb{N}.
\]

To see last inequality we have used ([8, (8.11.31)]). Then, (4.2) allows us to conclude that \(|a_j| = o(\xi^{-2k+2\mu+1})\), as \( j \to \infty \). Thus the proof is finished. \( \blacksquare \)

Let \( S \in H'_{\mu,\infty} \). We say that \( S \) is hypoelliptic in \( H'_{\mu,1} \) when the following property holds: if \( T \in H'_{\mu,1} \) and \( T \# S \in H_{\mu,\infty} \) then \( T \in H_{\mu,\infty} \).

We now characterize the hypoellipticity of \( S \in H'_{\mu,\infty} \) in terms of the growth of the Hankel transform \( h'_{\mu}(S) \) of \( S \).
Proposition 11. Let \( S \in H'_{\mu,\infty} \) such that there exists a polynomial \( p \) for which
\[
\left| \frac{d^l}{dx^l} h'_\mu(S)(x) \right| \leq p(x), \quad x \in (0, \infty), \; l = 0, 1, 2, \ldots, 2s,
\]
where \( s = [\mu + 2] \). Then, \( S \) is hypoelliptic in \( H'_{\mu,1} \) if, and only if, there exist \( a, M > 0 \) such that
\[
|h'_\mu(S)(x)| \geq x^{-a}, \quad x \in (M, \infty).
\] (4.3)

Proof. Suppose that firstly (4.3) does not hold for every \( a, M > 0 \). Then we can find a sequence \( (\xi_j)_{j \in \mathbb{N}} \subset (0, \infty) \) such that, for every \( j \in \mathbb{N} \),
\[
|h'_\mu(S)(\xi_j)| < \xi_j^{-a} \quad \text{and} \quad 2^j < 2\xi_{j-1} < \xi_j.
\]
We now define the functional \( T \in H'_{\mu,1} \) by
\[
T(x) = 2^\mu \Gamma(\mu + 1) \sum_{j=1}^{\infty} (x\xi_j)^{-\mu} J_\mu(x\xi_j).
\]
According to Proposition 10, \( T \in H'_{\mu,1} \). Moreover, for every \( \phi \in H_{\mu,1} \), from (3.1) we infer
\[
\langle T\#S, \phi \rangle = \langle T, S\#\phi \rangle = \sum_{j=1}^{\infty} \int_0^\infty (x\xi_j)^{-\mu} J_\mu(x\xi_j)(S\#\phi)(x)x^{2\mu+1} \, dx
\]
\[
= \sum_{j=1}^{\infty} h'_\mu(S)(\xi_j)\mu(\phi)(\xi_j)
\]
\[
= \sum_{j=1}^{\infty} h'_\mu(S)(\xi_j) \int_0^\infty (x\xi_j)^{-\mu} J_\mu(x\xi_j)\phi(x)x^{2\mu+1} \, dx
\]
\[
= \sum_{j=1}^{\infty} h'_\mu(S)(\xi_j)(x\xi_j)^{-\mu} J_\mu(x\xi_j)\phi(x)x^{2\mu+1} \, dx.
\]
Hence, we can conclude that
\[
T\#S = 2^\mu \Gamma(\mu + 1) \sum_{j=1}^{\infty} h'_\mu(S)(\xi_j)(x\xi_j)^{-\mu} J_\mu(x\xi_j).
\]
By Proposition 10 we deduce that \( T\#S \in H_{\mu,\infty} \) because the function \( z^{-\mu} J_\mu(z) \) is bounded on \( (0, \infty) \). However, \( T \) is not in \( H_{\mu,\infty} \). Thus we prove that \( S \) is not hypoelliptic in \( H'_{\mu,1} \).

Assume that \( |h'_\mu(S)(x)| > x^{-a}, \; x \in (M, \infty) \), for certain \( a, M > 0 \). We choose a function \( \phi \in S_e \) such that \( \phi(x) = 1, \; x \in (0, M) \), and \( \phi(x) = 0, \)
$x \in (M + 1, \infty)$, and we define the function $F$ as follows

\[
F(x) = \begin{cases} 
\frac{1 - \phi(x)}{h_\mu(S)(x)}, & x \in (M, \infty) \\
0, & x \in [0, M].
\end{cases}
\]

By using an iterated Leibniz rule, since there exists a polynomial $p$ such that

\[
\left| \frac{d^l}{dx} h'_\mu(S)(x) \right| \leq p(x), \quad x \in (0, \infty), \quad l = 0, 1, \ldots, 2s,
\]

where $s = \lfloor \mu + 2 \rfloor$, we can find a $k \in \mathbb{N}$ such that by defining $f = h_\mu(\frac{F(x)}{1 + x^2})$, we have

\[
f(x) = \left(1 - \Delta_\mu\right)^s \left(\frac{F(x)}{1 + x^2}k\right)(x), \quad x \in (0, \infty),
\]

where $s = \lfloor \mu + 2 \rfloor$, being $\Delta_\mu^s(\frac{F(x)}{1 + x^2}) \in L^1(x^{2\mu + 1} \, dx)$. Hence $f \in L^1(x^{2\mu + 1} \, dx)$ and $(1 - \Delta_\mu)^k f = h'_\mu(F) \in H'_{\mu, 1}$. Moreover, $Fh'_\mu(S) = 1 - \phi$. Then

\[G \# S = \delta - \psi,
\]

where $G = (1 - \Delta_\mu)^k f$ and $\psi = h_\mu(\phi)$. Indeed, for every $\varphi \in H_{\mu, 1}$, (3.2) implies that

\[
\langle G \# S, \varphi \rangle = \langle h'_\mu(G)h_\mu(S), h_\mu(\varphi) \rangle
\]

\[= \langle 1 - \phi, h_\mu(\varphi) \rangle
\]

\[= \int_0^\infty h_\mu(\varphi) \frac{x^{2\mu + 1}}{2^\mu \Gamma(\mu + 1)} \, dx - \int_0^\infty \phi(x)h_\mu(\varphi)(x) \frac{x^{2\mu + 1}}{2^\mu \Gamma(\mu + 1)} \, dx
\]

\[= \varphi(0) - \int_0^\infty h_\mu(\phi)(x)\varphi(x) \frac{x^{2\mu + 1}}{2^\mu \Gamma(\mu + 1)} \, dx
\]

\[= \langle \delta, \varphi \rangle - \langle \psi, \varphi \rangle.
\]

Suppose now that $T \# S = R \in H_{\mu, \infty}$, where $T \in H'_{\mu, 1}$. Since $T \# \delta = T$, by using Proposition 6, we can write

\[T = T \# \delta = T \# (G \# S) + T \# \psi = (T \# S) \# G + T \# \psi = R \# G + T \# \psi.
\]

There exist $k \in \mathbb{N}$ and $f_j \in L^\infty(\, dx)$, $j = 0, 1, \ldots, k$, such that

\[
\langle T, \varphi \rangle = \sum_{j=0}^k \int_0^\infty f_j(x)\Delta^1_\mu \varphi(x)x^{2\mu + 1} \, dx, \quad \varphi \in H_{\mu, 1}.
\]
In particular,

$$(T\#\psi)(x) = \sum_{j=0}^{k} \int_{0}^{\infty} f_j(y) \tau_x(\Delta^j_{\mu}\psi)(y)y^{2\mu+1} \, dy.$$ 

According to [15, p. 17], we get that, for every $m \in \mathbb{N}$,

$$\| \Delta^m_{\mu}(T\#\psi) \|_{\mu,\infty} \leq C \sum_{j=0}^{k+m} \| \Delta^j_{\mu}\psi \|_{\mu,1}.$$ 

Hence $T\#\psi \in H_{\mu,\infty}$.

On the other hand, we have

$$R\#G = R\#(1 - \Delta_{\mu})^k f = (1 - \Delta_{\mu})^k R\#f.$$ 

Moreover, for every $m \in \mathbb{N}$,

$$\| \Delta^m_{\mu}(R\#G) \|_{\mu,\infty} = \| (\Delta^m_{\mu}(1 - \Delta_{\mu})^k R\#f) \|_{\mu,\infty} \leq C \| \Delta^m_{\mu}(1 - \Delta_{\mu})^k R \|_{\mu,\infty} \| f \|_{\mu,1}.$$ 

Hence $T\#G \in H_{\mu,\infty}$. Thus we conclude that $T \in H_{\mu,\infty}$ and the hypoellipticity of $S$ on $H^{'}_{\mu,1}$ is established.

References


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