Another Version of Maher’s Inequality

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Abstract. Let $H$ be a separable infinite dimensional complex Hilbert space, and let $L(H)$ denote the algebra of bounded linear operators on $H$ into itself. Let $A = (A_1, A_2, ..., A_n)$, $B = (B_1, B_2, ..., B_n)$ be n-tuples of operators in $L(H)$. We define the elementary operator $\Delta_{A,B}: L(H) \mapsto L(H)$ by $\Delta_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i - X$. In this paper we minimize the map $F_p(X) = \|T - \Delta_{A,B}(X)\|_p^p$, where $T \in \ker \Delta_{A,B} \cap C_p$, and we classify its critical points.

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1. Introduction

Let $H$ be a separable infinite dimensional complex Hilbert space, and let $L(H)$ denote the algebra of bounded linear operators on $H$ into itself. Given $A, B \in L(H)$, we define the generalized derivation $\delta_{A,B}: L(H) \mapsto L(H)$ by $\delta_{A,B}(X) = AX - XB$. Let $A = (A_1, A_2, ..., A_n)$, $B = (B_1, B_2, ..., B_n)$ be n-tuples of operators in $L(H)$. We define the elementary operator $\Delta_{A,B}: L(H) \mapsto L(H)$, $\Delta^{*}_{A,B} : L(H) \mapsto L(H)$ by

$$\Delta_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i - X$$

and

$$\Delta^{*}_{A,B}(X) = \sum_{i=1}^{n} A_i^{*} X B_i^{*} - X$$

respectively. Denote $\delta_{A,A}(X) = \delta_{A}(X) = AX - XA$ and $\Delta_{A,A} = \Delta_A = \sum_{i=1}^{n} A_i X A_i - X$. A well-known result of J. Anderson [1: p.136-137 ] says that if $A$ is a normal operator such that $AS = SA$, then for all $X \in L(H)$,

$$\|S - (AX - XA)\| \geq \|S\|. \quad (1.1)$$

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The related inequality (1.1) was obtained by P. J. Maher [9: Theorem 3.2]. It shows that, if \( A \) is a normal operator and \( AS = SA \), where \( S \in C_p, 1 \leq p < \infty \) and \( S \in \ker \delta_{A,B} \cap C_p \), then the map \( F_p \) defined by

\[
F_p(X) = \| S - (AX - XA) \|_p^p
\]

has a global minimizer at \( V \) if, and for \( 1 < p < \infty \) only if, \( AV - VA = 0 \). In other words, we have

\[
\| S - (AX - XA) \|_p^p \geq \| T \|_p^p,
\]

where \( C_p \) is the von Neumann-Schatten class, \( 1 \leq p < \infty \) and \( \| \cdot \|_p \) its norm. In [6] and [3] the authors generalized P. J. Maher’s result, showing that if the pair \((A,B)\) has the property \((FP)_{C_p}\) (i.e. \( AT = TB \), where \( T \in C_p \) implies \( A^*T = TB^* \)), \( 1 \leq p < \infty \), and \( S \in \ker \delta_{A,B} \cap C_p \), then the map \( F_p \) defined by

\[
F_p(X) = \| S - (AX - XB) \|_p^p
\]

has a global minimizer at \( V \) if, and for \( 1 < p < \infty \) only if, \( AV - VB = 0 \). In other words, we have

\[
\| S - (AX - XB) \|_p^p \geq \| T \|_p^p
\]

if, and for \( 1 < p < \infty \) only if, \( AV - VB = 0 \). In this paper we obtain an inequality similar to (1.3), where the operator \( AX - XB \) is replaced by the operator \( \Delta_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i - X \). We prove that if \( \Delta_{A,B}(T) = 0 = \Delta_{A^*,B^*}(T) \) and \( T \in \ker \Delta_{A,B} \cap C_p \), then the map \( F_p \) defined by

\[
F_p(X) = \| T - \Delta_{A,B}(X) \|_p^p
\]

has a global minimizer at \( V \) if, and for \( 1 < p < \infty \) only if, \( \sum_{i=1}^{n} A_i V B_i - V = 0 \). Moreover, we show that if \( \Delta_{A,B}(T) = 0 = \Delta_{A^*,B^*}(T) \) and \( T \in \ker \Delta_{A,B} \cap C_p, 1 < p < \infty \), then the map \( F_p \) has a critical point at \( W \) if and only if \( \sum_{i=1}^{n} A_i W B_i - W = 0 \), i.e. if \( D_W F_p \) is the Frechet derivative at \( W \) of \( F_p \), then the set

\[
\{ W \in L(H): D_W F_p = 0 \}
\]

coincides with \( \ker \Delta_{A,B} \) (the kernel of \( \Delta_{A,B} \)).

2. Preliminaries

Let \( T \in L(H) \) be compact, and let \( s_1(X) \geq s_2(X) \geq \ldots \geq 0 \) denote the singular values of \( T \), i.e. the eigenvalues of \( |T| = (T^*T)^{\frac{1}{2}} \) are arranged in their decreasing order. The operator \( T \) is said to belong to the Schatten \( p \)-class \( C_p \) if

\[
\| T \|_p = \left( \sum_{j=1}^{\infty} s_j(T)^p \right)^{\frac{1}{p}} = [\text{tr}(T^p)]^{\frac{1}{p}}, \quad 1 \leq p < \infty,
\]
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where \( tr \) denotes the trace function. Hence \( C_1 \) is the trace class, \( C_2 \) is the Hilbert-Schmidt class, and \( C_\infty \) is the class of compact operators where

\[
\|T\|_\infty = s_1(T) = \sup_{\|f\| = 1} \|Tf\|
\]

denotes the usual operator norm. For the general theory of the Schatten \( p \)-classes the reader is referred to [11] and [12]. Let \( \Re z \) be the real part of a complex number \( z \), \( X = U \mid X \) be the polar decomposition of the operator \( X \) and let \( tr \) denote trace.

**Theorem 2.1.** [2] If \( 1 < p < \infty \), then the map \( F_p : C_p \rightarrow \mathbb{R}^+ \) defined by \( X \mapsto \|X\|^p_p \) is differentiable at every \( X \in C_p \) with derivative \( D_X F_p \) given by

\[
D_X F_p(T) = p \cdot \Re tr(|X|^{p-1} U^* T),
\]

(2.1)

If \( \dim H < \infty \), then the same result holds for \( 0 < p \leq 1 \) at every invertible \( X \).

**Theorem 2.2.** [9] If \( U \) is a convex subset of \( C_p \) with \( 1 < p < \infty \) and \( X \in U \), then the map \( X \mapsto \|X\|^p_p \) has at most one global minimizer.

**Lemma 2.1.** [13] Let \( C \) denote the n-tuple of operators \((C_1, C_2, \ldots, C_n)\) in \( L(H) \). Suppose that \( \sum_{i=1}^n C_i C_i^* \leq 1 \) and \( \sum_{i=1}^n C_i^* C_i \leq 1 \). If \( \Delta_C(T) = 0 = \Delta_C^*(T) \) for some compact operator \( T \), then the operator \( |T| \) commutes with \( C_i \) for all \( 1 \leq i \leq n \).

**Definition 2.1.** Let \( F \) and \( G \) be two subspaces of a normed linear space \( E \). If \( \|x + y\| \geq \|y\| \) for all \( x \in F \) and for all \( y \in G \), then \( F \) is said to be orthogonal to \( G \).

### 3. Main Results

Let \( \mathcal{U}(A, B) = \{ X \in L(H) : (\sum_{i=1}^n C_i X C_i - X) \in C_p \} \) and \( F_p : \mathcal{U} \rightarrow \mathbb{R}^+ \) be the map defined by \( F_p(X) = \|T - (\sum_{i=1}^n C_i X C_i - X)\|^p_p \) where \( T \in \ker \Delta_C \cap C_p \), \( 1 \leq p < \infty \). We start with the following lemma which will be used in the proof of Theorem 3.1.

**Lemma 3.1.** Let \( C \) denote the n-tuple of operators \((C_1, C_2, \ldots, C_n)\) in \( L(H) \) such that \( \sum_{i=1}^n C_i C_i^* \leq 1 \), \( \sum_{i=1}^n C_i^* C_i \leq 1 \). Let \( S \) be compact and \( \Delta(S) = 0 = \Delta^*(S) \). If

\[
\sum_{i=1}^n C_i |S|^{p-1} U^* C_i = |S|^{p-1} U^*,
\]

where \( p > 1 \) and \( S = U \mid S \) is the polar decomposition of \( S \), then

\[
\sum_{i=1}^n C_i |S| U^* C_i = |S| U^*.
\]
Proof. If $T = |S|^{p-1}$, then
\[ \sum_{i=1}^{n} C_i T U^* C_i = T U^*. \] (3.1)

We prove that
\[ \sum_{i=1}^{n} C_i T^n U^* C_i = T^n U^*. \] (3.2)

It is known that if $\sum_{i=1}^{n} C_i C_i^* \leq 1$, $\sum_{i=1}^{n} C_i^* C_i \leq 1$ and $\Delta_c(T) = \Delta_c^*(T) = 0$, then the eigenspaces corresponding to distinct non-zero eigenvalues of the compact positive operator $|S|^2$ reduce each $C_i$ (see [4: Theorem 8], [13: Lemma 2.3]). In particular, $|S|$ commutes with $C_i$ for all $1 \leq i \leq n$. This implies also that $|S|^{p-1} = T$ commutes with each $C_i$ for all $1 \leq i \leq n$. Hence
\[ C_i |T| = |T| C_i, \]

and $C_i T^2 = T^2 C_i$. Since $C_i$ commutes with the positive operator $T^2$, then $C_i$ commutes with its square root, that is
\[ C_i T = T C_i. \] (3.3)

By (3.3) and (3.1), we obtain (3.2).

By using an argument similar to the proof of Theorem 3.2 in [9], we can consider the map $f$ defined on $\sigma(T) \subset \mathbb{R}^+$ by $f(t) = t^{\frac{1}{p-1}}$, $1 < p < \infty$. Since $f$ is the uniform limit of a sequence $(P_i)$ of polynomials without constant term (since $f(0) = 0$), it follows from (3.2) that $\sum_{i=1}^{n} C_i P_i(T) U^* C_i = P_i(T) U^*$. Therefore
\[ \sum_{i=1}^{n} C_i T^\frac{1}{p-1} U^* C_i = U^* T^\frac{1}{p-1}. \]

Now we are ready to present our first result on the global minimizer.

**Theorem 3.1.** Let $C = (C_1, C_2, ..., C_n)$ be an $n$-tuple of operators in $L(H)$. If
\[ \sum_{i=1}^{n} C_i C_i^* \leq 1, \quad \sum_{i=1}^{n} C_i^* C_i \leq 1, \]
\[ \Delta_c(T) = \Delta_c^*(T) = 0 \]
and $T \in \ker \Delta_{A, B} \cap C_p$, then for $1 \leq p < \infty$, the map $F_p$ has a global minimizer at $W \in L(H)$ if, and for $1 < p < \infty$ only if,
\[ \sum_{i=1}^{n} C_i W C_i - W = 0. \]
Proof. If
\[ \sum_{i=1}^{n} C_i W C_i - W = 0, \]
then \( F_p(W) = \|T\|_p^p \). It follows from [13: Theorem 2.4] that
\( F_p(X) \geq F_p(W) \).

Conversely, if \( F_p \) has a minimum then
\[ \left\| T - \left( \sum_{i=1}^{n} C_i W C_i - W \right) \right\|_p^p = \|T\|_p^p. \]
Since \( U \) is convex, the set \( V = \{ T - \left( \sum_{i=1}^{n} C_i X C_i - X \right); X \in U \} \) is also convex.
Thus, Theorem 2.2 implies that \( T - \left( \sum_{i=1}^{n} C_i W C_i - W \right) = T \).

In the following theorem we will classify the critical points of the map \( F_p (p > 1) \).

**Theorem 3.2.** Let \( C = (C_1, C_2, ..., C_n) \) be an \( n \)-tuple of operators in \( L(H) \). If
\[ \sum_{i=1}^{n} C_i C_i^* \leq 1, \sum_{i=1}^{n} C_i^* C_i \leq 1, \]
\[ \Delta_c(T) = 0 = \Delta^*_c(T) \]
and \( T \in \ker \Delta_{A,B} \cap C_p \), then for \( 1 \leq p < \infty \), the map \( F_p \) has a critical point at \( W \in L(H) \) if, and for \( 1 < p < \infty \) only if,
\[ \sum_{i=1}^{n} C_i W C_i - W = 0. \]

**Proof.** Since the Frechet derivative of \( F_p \) is given by
\[ D_W F_p(T) = \lim_{h \to 0} \frac{F_p(W + hT) - F_p(W)}{h}, \]
it follows that
\[ D_W F_p(T) = \left[ D_{S - (\sum_{i=1}^{n} C_i W C_i - W)} \right] \left( \sum_{i=1}^{n} C_i T C_i - T \right). \]
If \( W \) is a critical point of \( F_p \), then \( D_W F_p(T) = 0 \forall T \in U \). By applying Theorem 2.1 we get
\[ D_W F_p(T) = p \Re \left[ \left| S - \left( \sum_{i=1}^{n} C_i W C_i - W \right) \right|^{p-1} U_1^* \left( \sum_{i=1}^{n} C_i T C_i - T \right) \right] \]
\[ = p \Re \left[ Y \left( \sum_{i=1}^{n} C_i T C_i - T \right) \right] \]
\[ = 0, \]
where \( S - (\sum_{i=1}^{n} C_i WC_i - W) = U_1 | S - (\sum_{i=1}^{n} C_i WC_i - W)| \) is the polar decomposition of the operator \( S - (\sum_{i=1}^{n} C_i WC_i - W) \) and
\[
Y = | S - \left( \sum_{i=1}^{n} C_i WC_i - W \right) |^{p-1} U_1^*.
\]

An easy calculation shows that \( (\sum_{i=1}^{n} C_i Y C_i - Y) = 0 \), that is
\[
\sum_{i=1}^{n} C_i | S - \left( \sum_{i=1}^{n} C_i WC_i - W \right) |^{p-1} U_1^* = | S - \left( \sum_{i=1}^{n} C_i WC_i - W \right) |^{p-1} U_1^*.
\]

It follows from Lemma 3.1 that
\[
\sum_{i=1}^{n} C_i | S - \left( \sum_{i=1}^{n} C_i WC_i - W \right) | U_1^* = | S - \left( \sum_{i=1}^{n} C_i WC_i - W \right) | U_1^*.
\]

By taking adjoints and since \( \Delta_C = 0 = \Delta_C^* \), we get
\[
\sum_{i=1}^{n} C_i \left( T - \left( \sum_{i=1}^{n} C_i WC_i - W \right) \right) C_i = \left( T - \left( \sum_{i=1}^{n} C_i WC_i - W \right) \right).
\]

Then
\[
\sum_{i=1}^{n} C_i \left[ \left( \sum_{i=1}^{n} C_i WC_i - W \right) \right] C_i = \left( \sum_{i=1}^{n} C_i WC_i - W \right).
\]

Hence
\[
\sum_{i=1}^{n} C_i WC_i - W \in R(\Delta_C) \cap \ker \Delta_C,
\]
where \( R(\Delta_C) \) is the range of \( \Delta_C \). It is easy to see that (arguing as in the proof of [13: Theorem 2.4]), \( \Delta_C(T) = 0 = \Delta_C^*(T) \) and \( T \in \ker \Delta_C \), where \( T \in L(H) \). Then
\[
\| T - \Delta_C(X) \| \geq \| T \|
\]
holds for all \( X \in L(H) \) and for all \( T \in \ker \Delta_C \). Hence \( \sum_{i=1}^{n} C_i WC_i - W = 0 \). Conversely, if \( \sum_{i=1}^{n} C_i WC_i = W \), then \( W \) is a minimum of \( F_p \), and since \( F_p \) is differentiable, \( W \) is a critical point.

In the above theorem we classified the critical points of the map \( F_p \) for \( p > 1 \). In the following theorem we consider the case \( 0 < p \leq 1 \).

**Theorem 3.3.** Let \( C = (C_1, C_2, ..., C_n) \) be an \( n \)-tuple of operators in \( L(H) \). If
\[
\sum_{i=1}^{n} C_i C_i^* \leq 1, \quad \sum_{i=1}^{n} C_i^* C_i \leq 1
\]
such that \( \Delta_C(S) = 0 = \Delta_C^*(S) \) and \( S \in \ker \Delta_C \cap \ker_C \), \( 0 < p \leq 1 \), \( \dim H < \infty \) and \( S - (\sum_{i=1}^{n} C_i WC_i - W) \) is invertible, then \( F_p \) has a critical point at \( W \), if \( \sum_{i=1}^{n} C_i WC_i - W = 0 \).
Proof. Let $W, S \in U$ and let $\phi$, be the map defined by

$$\phi : X \mapsto S - \left( \sum_{i=1}^{n} C_i X C_i - X \right).$$

Suppose that $\dim H < \infty$. If $\sum_{i=1}^{n} C_i W C_i - W = 0$, then $S$ is invertible by hypothesis. Also $|S|$ is invertible, hence $|S|^{p-1}$ exists for $0 < p \leq 1$. Taking $Y = |S|^{p-1} U^*$, where $S = U |S|$ is the polar decomposition of $S$. As shown in Lemma 3.1, $|S|$ commutes with $C_i$ for all $1 \leq i \leq n$. Hence

$$C_i |S| = |S| C_i.$$

Since $\sum_{i=1}^{n} C_i S^* C_i = S^*$, i.e.

$$\sum_{i=1}^{n} C_i |S| U^* C_i = |S| U^*,$$

we find

$$|S| \left( \sum_{i=1}^{n} C_i U^* C_i - U^* \right) = 0,$$

and since

$$A |S|^{p-1} = |S|^{p-1} A,$$

we have

$$\sum_{i=1}^{n} C_i Y C_i - Y = \sum_{i=1}^{n} C_i |S|^{p-1} U^* C_i - |S|^{p-1} U^*$$

$$= |S|^{p-1} \left( \sum_{i=1}^{n} C_i U^* C_i - U^* \right),$$

so that $\sum_{i=1}^{n} C_i Y C_i - Y = 0$ and $tr \left[ \left( \sum_{i=1}^{n} C_i Y C_i - Y \right) T \right] = 0$ for all $T \in L(H)$. Since

$$S = S - \left( \sum_{i=1}^{n} C_i W C_i - W \right),$$

we have

$$0 = tr \left[ Y \left( \sum_{i=1}^{n} C_i T C_i - T \right) \right]$$

$$= p \Re tr \left[ Y \left( \sum_{i=1}^{n} C_i T C_i - T \right) \right]$$

$$= p \Re tr \left[ |S|^{p-1} U^* \left( \sum_{i=1}^{n} C_i T C_i - T \right) \right]$$

$$= (D_S \phi) \left( \sum_{i=1}^{n} C_i T C_i - T \right)$$
\[ (D_W F_p)(T), \]
which proofs the assertion.

At the end we use a familiar device of considering 2x2 operator matrices to extend the previous theorems to the elementary operator \( \sum_{i=1}^{n} A_i X B_i - X \).

**Theorem 3.4.** Let \( A = (A_1, A_2, \ldots, A_n) \) and \( B = (B_1, B_2, \ldots, B_n) \) be \( n \)-tuples of operators in \( L(H) \) such that
\[
\sum_{i=1}^{n} A_i A_i^* \leq 1, \sum_{i=1}^{n} A_i^* A_i \leq 1, \sum_{i=1}^{n} B_i B_i^* \leq 1, \sum_{i=1}^{n} B_i^* B_i \leq 1.
\]
If \( \Delta_{A,B}(T) = 0 = \Delta_{A,B}^*(T) \) and \( T \in \ker \Delta_{A,B} \cap C_p \), then it holds for \( 1 \leq p < \infty \):

(i) the map \( F_p \) has a global minimizer at \( W \) if, and for \( 1 < p < \infty \) only if, \( \sum_{i=1}^{n} A_i X B_i - W = 0 \)

(ii) the map \( F_p \) has a critical point at \( W \) if, and for \( 1 < p < \infty \) only if, \( \sum_{i=1}^{n} A_i X B_i - W = 0 \)

(iii) the map \( F_p \), \( 0 < p \leq 1 \), has a critical point at \( W \) if \( \sum_{i=1}^{n} A_i X B_i - W = 0 \)

provided \( \dim H < \infty \) and \( S - (\sum_{i=1}^{n} A_i X B_i - W) \) is invertible.

**Proof.** It suffices to take the Hilbert space \( H \oplus H \), and operators
\[
C_i = \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix}, S = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}, X = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}
\]
and apply Theorem 3.1, Theorem 3.2 and Theorem 3.3. These arguments use operator matrices as in Bouali and Cherki [3] and Mecheri [7].

**Remark.**

1. In Theorem 3.2, the implication
\[ W \text{ is a critical point} \implies \sum_{i=1}^{n} A_i X B_i - W = 0 \]
does not hold in the case \( 0 < p \leq 1 \) (cf. Maher [8]).

2. Theorems 3.1, 3.2, 3.3 and 3.4 hold in particular if \( A \) and \( B \) are contractions. Indeed, it is known from [4] that if \( A \) and \( B \) are contractions and \( \Delta_{A,B}(S) = ASB - S = 0 \), where \( S \in C_p \), then
\[ \Delta_{A^*,B^*}(S) = \delta_{A^*,B^*}(S) = \delta_{A,B^*}(S) = 0. \]

3. If \( A \in C_p \), the conclusions of Theorems 3.1, 3.2, 3.3 and 3.4 hold for all \( X \in L(H) \) (cf. Maher [9]).
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References


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