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# Existence, Uniqueness and Data Dependence for the Solutions of some Integro-Differential Equations of Mixed Type in Banach Space

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**Abstract.** In this paper we study existence, uniqueness and data dependence for the solutions of some integro-differential equations of mixed type in Banach space by using Picard and weakly Picard operators' technique and suitable Bielecki norms.

**Keywords:** *Integro-differential equations, fixed points, Picard operators, weakly Picard operators*

**AMS subject classification: 34K05, 47H10**

## 1. Introduction

Ordinary differential equations, functional differential equations with or without deviating argument and equations in abstract spaces have been studied in many papers. In the papers [3, 6] theorems about the existence and uniqueness of solutions of some abstract nonlinear non-local Cauchy problems in Banach spaces were considered and in the paper [4] a theorem about the existence of an approximate solution to an abstract nonlinear non-local Cauchy problem in a Banach space was given, too. We remark in the same field the monographs [5, 9, 11 - 13].

Integro-differential equations of mixed type in Banach spaces have been studied in the papers [7, 10], and integro-differential equations of mixed type with impulses in Banach spaces were considered in the paper [14], too. Fredholm-Volterra integral equations in relationship with Maia's theorem were considered in the paper [16].

The aim of the present paper is to obtain existence, uniqueness and data dependence results for the solutions of some integro-differential equations of

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mixed type in Banach space. To do this we use Picard and weakly Picard operators' technique due to I. A. Rus (see [18 - 22]). So, our technique is different from those used in the papers quoted above.

Let $(X, \|\cdot\|)$ be a Banach space. Consider the problem

$$\left.\begin{aligned} &x'(t) = f\Big(t, x(t), \int_0^t K_1(t,s)x(s)\,ds, \int_0^T K_2(t,s)x(s)\,ds\Big) \\ &x(0) = x_0 \end{aligned}\right\} \tag{1}$$

on $[0,T]$, where $f \in C([0,T] \times X^3, X)$, $K_i \in C(D_i, \mathbb{R})$ $(i = 1, 2)$ and $x_0 \in X$. Here

$$D_1 = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t \le T\}$$
$$D_2 = [0,T] \times [0,T].$$

It is well known that $x \in C^1([0,T], X)$ is a solution of problem (1) if and only if $x$ is a solution in $C([0,T], X)$ of the integro-differential equation

$$x(t) = x_0 + \int_0^t f\Big(\xi, x(\xi), \int_0^\xi K_1(\xi,s)x(s)\,ds, \int_0^T K_2(\xi,s)x(s)\,ds\Big)d\xi \tag{2}$$

on $[0,T]$.

In [10] the author combines topological degree theory and monotone iterative technique given in [12] to investigate the existence of solutions and also minimal and maximal solutions of problem (1). In the present paper we consider suitable Bielecki norms in a convenient space and obtain existence, uniqueness and data dependence results for the solutions of equation (2) which is equivalent to problem (1).

In [7] the authors study the existence of solutions of the abstract non-local integro-differential Cauchy problem in arbitrary Banach spaces

$$\left.\begin{aligned} &x'(t) = f\Big(t, x(t), \int_0^t K_1(t,s)x(s)\,ds, \int_0^T K_2(t,s)x(s)\,ds\Big) \\ &x(0) = x_0 - \sum_{i=1}^p c_i x(t_i) \end{aligned}\right\}$$

on $[0,T]$, where $f \in C([0,T] \times X^3, X)$, $0 < t_1 < t_2 < \ldots < t_p \le T$, $c_i \neq 0$, $p \in \mathbb{N}$ and $x_0 \in X$. This problem is equivalent to the integro-differential equation

$$\begin{aligned} x(t) = x_0 &- \sum_{i=1}^p c_i x(t_i) \\ &+ \int_0^t f\Big(\xi, x(\xi), \int_0^\xi K_1(\xi,s)x(s)\,ds, \int_0^T K_2(\xi,s)x(s)\,ds\Big)d\xi \end{aligned} \tag{3}$$

on $[0, T]$. For this purpose, the Kuratowski measure of non-compactness, fixed point principles and a monotone iterative technique were applied. We remark that the weakly Picard operators technique can be used to prove existence of solutions to equation (3).

## 2. Preliminaries

Let $(X, d)$ be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$P(X) = \{Y \subseteq X | \, Y \neq \emptyset\}$

$F_A = \{x \in X | \, A(x) = x\}$ – the fixed point set of $A$

$I(A) = \{Y \in P(X) | \, A(Y) \subseteq Y\}$

$O_A(x) = \{x, A(x), A^2(x), ..., A^n(x), ...\}$ – the $A$-orbit of $x \in X$

$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$

$H(Y, Z) = \max\left(\sup_{a \in Y} \inf_{b \in Z} d(a, b), \sup_{b \in Z} \inf_{a \in Y} d(a, b)\right)$
– the Pompeiu-Hausdorff functional on $P(X)$.

**Definition 2.1** (Rus [18]). Let $(X, d)$ be a metric space. An operator $A : X \to X$ is a *Picard operator* if there exists $x^* \in X$ such that $F_A = \{x^*\}$ and the sequence $(A^n(x_0))_{n \in \mathbb{N}}$ converges to $x^*$ for all $x_0 \in X$.

**Definition 2.2** (Rus [19]). Let $(X, d)$ be a metric space. An operator $A : X \to X$ is a *weakly Picard operator* if the sequence $(A^n(x_0))_{n \in \mathbb{N}}$ converges for all $x_0 \in X$ and its limit (which may depend on $x_0$) is a fixed point of $A$.

If $A$ is a weakly Picard operator, then we consider the operator

$$A^\infty : X \to X, \qquad A^\infty(x) = \lim_{n \to \infty} A^n(x).$$

The following results are useful in what follows:

**Theorem 2.1** [17]. *Let $(Y, d)$ be a complete metric space and $A, B : Y \to Y$ two operators. We suppose the following:*

**(i)** *$A$ is a contraction with contraction constant $\alpha$ and $F_A = \{x_A^*\}$.*

**(ii)** *$B$ has fixed points and $x_B^* \in F_B$.*

**(iii)** *There exists $\eta > 0$ such that $d(A(x), B(x)) \leq \eta$, for all $x \in Y$.*

*Then $d(x_A^*, x_B^*) \leq \frac{\eta}{1-\alpha}$.*

**Theorem 2.2** [22]. *Let $(X, d)$ be a complete metric space and $A, B : X \to X$ two orbitally continuous operators. We suppose the following:*

**(i)** *There exists* $\alpha \in [0,1)$ *such that*

$$\begin{matrix} d(A^2(x), A(x)) \leq \alpha d(x, A(x)) \\ d(B^2(x), B(x)) \leq \alpha d(x, B(x)) \end{matrix} \qquad (x \in X).$$

**(ii)** *There exists* $\eta > 0$ *such that* $d(A(x), B(x)) \leq \eta$ *for all* $x \in X$.
*Then* $H(F_A, F_B) \leq \frac{\eta}{1-\alpha}$ *where* $H$ *denotes the Pompeiu-Hausdorff functional.*

**Theorem 2.3** [19]. *Let* $(X, d)$ *be a metric space. Then* $A : X \to X$ *is a weakly Picard operator if and only if there exists a partition* $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ *of* $X$ *such that*

**(a)** $X_\lambda \in I(A)$

**(b)** $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator, for all* $\lambda \in \Lambda$.

Consider a Banach space $(X, \|\cdot\|)$, let $\|\cdot\|_B$ and $\|\cdot\|_C$ be the Bielecki and Chebyshev norms on $C([0,T], X)$ defined by

$$\|x\|_B = \max_{t \in [0,T]} \|x(t)\| e^{-\tau t} \quad (\tau > 0) \qquad \text{and} \qquad \|x\|_C = \max_{t \in [0,T]} \|x(t)\|$$

and denote by $d_B$ and $d_C$ their corresponding metrics. We consider the set

$$C_L([0,T], X) = \left\{ x \in C([0,T], X) \,\middle|\, \begin{matrix} \|x(t_1) - x(t_2)\| \leq L|t_1 - t_2| \\ \text{for all } t_1, t_2 \in [0,T] \end{matrix} \right\}$$

where $L > 0$ and $B_R = \{x \in X : \|x\| \leq R\}$ with $R > 0$. If $d \in \{d_C, d_B\}$, then $(C([0,T], X), d)$ and $(C_L([0,T], X), d)$ are complete metric spaces.

## 3. A integro-differential equation of mixed type

Consider equation (2). Denote $k_i = \max_{(t,s) \in D_i} |K_i(t,s)| \ \ (i = 1,2)$. We have

**Theorem 3.1.** *Suppose the following:*

**(i)** $f \in C([0,T] \times X^3, X)$.

**(ii)** *There exists a constant* $M > 0$ *such that* $\|f(s, u, v, w)\| \leq M$ *for all* $u, v, w \in X$ *and all* $s \in [0,T]$.

**(iii)** $M \leq L$.

**(iv)** *There exists a constant* $L_0 > 0$ *such that*

$$\|f(s, u_1, v_1, w_1) - f(s, u_2, v_2, w_2)\| \\ \leq L_0 \big( \|u_1 - u_2\| + \|v_1 - v_2\| + \|w_1 - w_2\| \big)$$

*for all* $u_i, v_i, w_i \in X \quad (i = 1, 2)$ *and all* $s \in [0, T]$.

**(v)** *There exists a constant* $\tau > 0$ *such that* $\frac{L_0}{\tau}\big(1 + \frac{k_1}{\tau} + k_2 T e^{\tau T}\big) < 1$.

*Then equation* (2) *has a unique solution* $x^*$ *in* $C_L([0,T],X)$*, and this solution can be obtained by the successive approximation method, starting from any element of* $C_L([0,T],X)$.

**Proof.** Consider the continuous operator

$$A:\big(C_L([0,T],X), \|\cdot\|_B\big) \to \big(C_L([0,T],X), \|\cdot\|_B\big)$$

defined by

$$\begin{aligned}
&A(x)(t)\\
&= x_0 + \int_0^t f\bigg(\xi, x(\xi), \int_0^\xi K_1(\xi,s)x(s)\,ds, \int_0^T K_2(\xi,s)x(s)\,ds\bigg)d\xi.
\end{aligned}$$

We have

$$\begin{aligned}
&\big\|A(x)(t) - A(z)(t)\big\|\\
&\quad \le \int_0^t \bigg\| f\bigg(\xi, x(\xi), \int_0^\xi K_1(\xi,s)x(s)\,ds, \int_0^T K_2(\xi,s)x(s)\,ds\bigg)\\
&\qquad - f\bigg(\xi, z(\xi), \int_0^\xi K_1(\xi,s)z(s)\,ds, \int_0^T K_2(\xi,s)z(s)\,ds\bigg)\bigg\| d\xi\\
&\quad \le L_0 \int_0^t \bigg[\|x(\xi) - z(\xi)\| + \bigg\|\int_0^\xi K_1(\xi,s)(x(s) - z(s))\,ds\bigg\|\\
&\qquad + \bigg\|\int_0^T K_2(\xi,s)(x(s) - z(s))\,ds\bigg\|\bigg] d\xi\\
&\quad \le L_0 \bigg[\int_0^t \|x(\xi) - z(\xi)\| d\xi + k_1 \int_0^t \bigg(\int_0^\xi \|x(s) - z(s)\|\,ds\bigg) d\xi\\
&\qquad + k_2 \int_0^t \bigg(\int_0^T \|x(s) - z(s)\|\,ds\bigg) d\xi\bigg]\\
&\quad \le L_0 \bigg[\int_0^t \|x(\xi) - z(\xi)\| e^{-\tau\xi} e^{\tau\xi} d\xi\\
&\qquad + k_1 \int_0^t \bigg(\int_0^\xi \|x(s) - z(s)\| e^{-\tau s} e^{\tau s} ds\bigg) d\xi\\
&\qquad + k_2 \int_0^t \bigg(\int_0^T \|x(s) - z(s)\| e^{-\tau s} e^{\tau s} ds\bigg) d\xi\bigg]
\end{aligned}$$

$$
\begin{aligned}
&\le L_0\|x-z\|_B\Big[\int_0^t e^{\tau\xi}d\xi + k_1\int_0^t\Big(\int_0^\xi e^{\tau s}ds\Big)d\xi \\
&\quad + k_2\int_0^t\Big(\int_0^T e^{\tau s}ds\Big)d\xi\Big] \\
&= L_0\|x-z\|_B\Big[\Big(\frac{e^{\tau t}}{\tau}-\frac{1}{\tau}\Big) + k_1\int_0^t\Big(\frac{e^{\tau\xi}}{\tau}-\frac{1}{\tau}\Big)d\xi \\
&\quad + k_2\int_0^t\Big(\frac{e^{\tau T}}{\tau}-\frac{1}{\tau}\Big)d\xi\Big] \\
&= L_0\|x-z\|_B\Big[\Big(\frac{e^{\tau t}}{\tau}-\frac{1}{\tau}\Big) + \frac{k_1}{\tau}\Big(\frac{e^{\tau t}}{\tau}-\frac{1}{\tau}-t\Big) + \frac{k_2}{\tau}(e^{\tau T}-1)t\Big] \\
&\le L_0\|x-z\|_B\Big[\frac{e^{\tau t}}{\tau} + \frac{k_1}{\tau}\frac{e^{\tau t}}{\tau} + k_2\frac{e^{\tau t}}{\tau}e^{\tau(T-t)}T\Big] \\
&\le L_0\frac{1}{\tau}e^{\tau t}\Big(1+\frac{k_1}{\tau}+k_2Te^{\tau T}\Big)\|x-z\|_B
\end{aligned}
$$

for all $x, z \in C_L([0,T],X)$. It follows that

$$
\big\|A(x)(t)-A(z)(t)\big\|e^{-\tau t} \le \frac{L_0}{\tau}\Big(1+\frac{k_1}{\tau}+k_2Te^{\tau T}\Big)\|x-z\|_B
$$

for all $t \in [0,T]$. So

$$
\|A(x)-A(z)\|_B \le \frac{L_0}{\tau}\Big(1+\frac{k_1}{\tau}+k_2Te^{\tau T}\Big)\|x-z\|_B
$$

for all $x, z \in C_L([0,T],X)$. The operator $A$ is of Lipschitz type with constant

$$
L_A = \frac{L_0\big(1+\frac{k_1}{\tau}+k_2Te^{\tau T}\big)}{\tau} \tag{4}
$$

and $0 < L_A < 1$. By applying the Contraction Principle to this operator we obtain that $A$ is a Picard operator ∎

Similarly as above, we can prove

**Theorem 3.2.** *Suppose the following:*

**(i)** $f \in C([0,T]\times B_R^3, X)$ *with* $\|f(s,u,v,w)\| \le M(R)$ *for all* $s \in [0,T]$ *and* $u, v, w \in B_R$.

**(ii)** $M(R) \le L$.

**(iii)** $k_iT \le 1 \quad (i = 1,2)$.

**(iv)** $\|x_0\| + M(R)T \le R$.

**(v)** *There exists a constant* $L_0 > 0$ *such that*

$$\begin{aligned}\big\|f(s,u_1,v_1,w_1) - f(s,u_2,v_2,w_2)\big\| \\ \le L_0\big(\|u_1-u_2\| + \|v_1-v_2\| + \|w_1-w_2\|\big)\end{aligned}$$

*for all* $u_i, v_i, w_i \in B_R \quad (i=1,2)$ *and all* $s \in [0,T]$.

**(vi)** *There exists a constant* $\tau > 0$ *such that* $\frac{L_0}{\tau}\big(1 + \frac{k_1}{\tau} + k_2 T e^{\tau T}\big) < 1$.
*Then equation* (2) *has a unique solution in* $C_L([0,T],B_R)$*, and this solution can be obtained by the successive approximation method, starting from any element of* $C_L([0,T],B_R)$.

**Remark 3.1.** If we consider the problem

$$\left.\begin{aligned} x'(t) &= \frac{1}{10}\int_0^t \sin(t+s)x(s)\,ds + \frac{1}{18}\int_0^{\frac{1}{3}} \cos(ts)x(s)\,ds \\ x(0) &= 0 \end{aligned}\right\}$$

on $[0,T]$, then $L_0 = 1$, $k_1 = \frac{1}{10}$, $k_2 = \frac{1}{18}$, and for $\tau = 2$ we have condition (vi) in Theorem 3.2.

Now, we consider both equation (2) and

$$x(t) = y_0 + \int_0^t g\bigg(\xi, x(\xi), \int_0^\xi K_1(\xi,s)x(s)\,ds, \int_0^T K_2(\xi,s)x(s)\,ds\bigg)d\xi \tag{5}$$

on $[0,T]$, where $g \in C([0,T]\times X^3, X)$ and $K_i \in C(D_i,\mathbb{R}) \quad (i=1,2)$ are the same as in equation (2) and $y_0 \in X$. We have

**Theorem 3.3.** *Suppose the following:*

**(i)** *All conditions in Theorem* 3.1 *are satisfied and* $x^* \in C_L([0,T],X)$ *is the unique solution of equation* (2).

**(ii)** *There exists a constant* $M_1 > 0$ *such that* $\|g(s,u,v,w)\| \le M_1$ *for all* $u,v,w \in X$ *and all* $s \in [0,T]$.

**(iii)** *With the same Lipschitz constant* $L_0$ *as in Theorem* 3.1,

$$\begin{aligned}\big\|g(s,u_1,v_1,w_1) - g(s,u_2,v_2,w_2)\big\| \\ \le L_0\big(\|u_1-u_2\| + \|v_1-v_2\| + \|w_1-w_2\|\big)\end{aligned}$$

*for all* $u_i, v_i, w_i \in X \quad (i=1,2)$ *and all* $s \in [0,T]$.

**(iv)** $M_1 \le L$.

**(v)** *There exists a constant* $\eta > 0$ *such that*

$$\big\|f(s,u,v,w) - g(s,u,v,w)\big\| \le \eta$$

*for all* $u, v, w \in X$ *and* $s \in [0, T]$.

*Then, if* $y^*$ *is the solution of equation* (5),

$$\|x^* - y^*\|_B \le \frac{\|x_0 - y_0\| + \eta T}{1 - L_A}$$

*where* $L_A$ *is given by* (4) *with* $\tau = \tau_0 > 0$ *such that* $0 < L_A < 1$.

**Proof.** Consider the operators

$$A, B : C_L([0, T], X) \to C_L([0, T], X)$$

defined by

$$A(x)(t) = x_0 + \int_0^t f\left(\xi, x(\xi), \int_0^\xi K_1(\xi, s)x(s)\, ds, \int_0^T K_2(\xi, s)x(s)\, ds\right) d\xi$$
$$B(x)(t) = y_0 + \int_0^t g\left(\xi, x(\xi), \int_0^\xi K_1(\xi, s)x(s)\, ds, \int_0^T K_2(\xi, s)x(s)\, ds\right) d\xi$$

on $[0, T]$, in which $K_i \in C(D_i, \mathbb{R})$ $(i = 1, 2)$ are the same. We have

$$\big\|A(x)(t) - B(x)(t)\big\| \le \|x_0 - y_0\| + \eta T \qquad (t \in [0, T]).$$

It follows that

$$\|A(x) - B(x)\|_B \le \|x_0 - y_0\| + \eta T.$$

So we can apply Theorem 2.1 ∎

**Remark 3.2.** The results obtained in this section can be generalized to study existence, uniqueness and data dependence for the solutions of the problem with linear modification of the argument

$$\left.\begin{aligned} &x'(t) = f\left(t, x(t), x(\lambda t), \int_0^t K_1(t, s)x(\lambda s)\, ds, \int_0^T K_2(t, s)x(\lambda s)\, ds\right) \\ &x(0) = x_0 \end{aligned}\right\}$$

on $[0, T]$, where $0 < \lambda < 1$, $f \in C([0, T] \times X^4, X)$, $K_i \in C(D_i, R)$ $(i = 1, 2)$ and $x_0 \in X$. This problem is more general than those considered in [15].

## 4. Another integro-differential equation of mixed type

Now, we consider the integral equation of mixed type

$$x(t) = x(0) + \int_0^t f\Big(\xi, x(\xi), \int_0^\xi K_1(\xi,s)x(s)\,ds, \int_0^T K_2(\xi,s)x(s)\,ds\Big)d\xi \quad (6)$$

on $[0,T]$, where $f \in C([0,T]\times X^3, X)$, $K_i \in C(D_i,\mathbb{R})$ and $D_i \ \ (i=1,2)$ are as in problem (1). We have

**Theorem 4.1.** *Suppose that for equation* (6) *the same conditions as in Theorem* 3.1 *are satisfied. Then this equation has solutions in* $C_L([0,T],X)$*. If* $S \subset C_L([0,T],X)$ *is its solutions set, then* $\operatorname{card} S = \operatorname{card} X$*.*

**Proof.** Consider the operator

$$A_* : C_L([0,T],X) \to C_L([0,T],X)$$

defined by

$$A_*(x)(t) = x(0) + \int_0^t f\Big(\xi, x(\xi), \int_0^\xi K_1(\xi,s)x(s)\,ds, \int_0^T K_2(\xi,s)x(s)\,ds\Big)d\xi.$$

This is a continuous operator, but not a Lipschitz one. We can write

$$C_L([0,T],X) = \bigcup_{\alpha\in X} X_\alpha, \quad X_\alpha = \big\{x \in C_L([0,T],X) : x(0)=\alpha\big\}.$$

We have that $X_\alpha$ is an invariant set of $A_*$ and we apply Theorem 3.1 to $A_*|_{X_\alpha}$. By using Theorem 2.3 we obtain that $A_*$ is a weakly Picard operator. Consider the operator

$$A_*^\infty : C_L([0,T],X) \to C_L([0,T],X), \qquad A_*^\infty(x) = \lim_{n\to\infty} A_*^n(x).$$

From $A_*^{n+1}(x) = A_*(A_*^n(x))$ and the continuity of $A_*$, $A_*^\infty(x) \in F_{A_*}$. Then $A_*^\infty(C_L([0,T],X)) = F_{A_*} = S$, and $S \neq \emptyset$. So, $\operatorname{card} S = \operatorname{card} X$ ∎

**Remark 4.1.** Similarly as above we can prove the existence of solutions of equation (3) that corresponds to a problem considered in [7].

In order to study data dependence for the solutions set of equation (6) we consider both (6) and the equation

$$x(t) = x(0) + \int_0^t g\Big(\xi, x(\xi), \int_0^\xi K_1(\xi,s)x(s)\,ds, \int_0^T K_2(\xi,s)x(s)\,ds\Big)d\xi$$

on $[0,T]$ where $K_1, K_2$ are the same as in (6) and $g \in C([0,T]\times X^3, X)$. Let $S_1$ be the solutions set of this equation.

**Theorem 4.2.** *Suppose the following:*

**(i)** *There exists a constant $L_* > 0$ such that*

$$\begin{aligned}&\big\|f(s,u_1,v_1,w_1)-f(s,u_2,v_2,w_2)\big\|\\&\qquad\qquad\le L_*\big(\|u_1-u_2\|+\|v_1-v_2\|+\|w_1-w_2\|\big)\\&\big\|g(s,u_1,v_1,w_1)-g(s,u_2,v_2,w_2)\big\|\\&\qquad\qquad\le L_*\big(\|u_1-u_2\|+\|v_1-v_2\|+\|w_1-w_2\|\big)\end{aligned}$$

*for all $u_i, v_i, w_i \in X \quad (i=1,2)$ and all $s \in [0,T]$.*

**(ii)** *There exists a constant $M_* > 0$ such that*

$$\begin{aligned}\|f(s,u,v,w)\| &\le M_*\\ \|g(s,u,v,w)\| &\le M_*\end{aligned}$$

*for all $u, v, w \in X$ and all $s \in [0,T]$.*

**(iii)** $M_* \le L_*$.

**(iv)** *There exists a constant $\eta_1 > 0$ such that*

$$\big\|f(s,u,v,w)-g(s,u,v,w)\big\| \le \eta_1$$

*for all $u, v, w \in X$ and all $s \in [0,T]$.*

**(v)** $3L_*Tk_0 < 1$, *where $k_0 = max(1, k_1T, k_2T)$.*

*Then*

$$H_{\|\cdot\|_C}(S,S_1) \le \frac{\eta_1 T}{1-3L_*Tk_0}$$

*where by $H_{\|\cdot\|_C}$ we denote the Pompeiu-Hausdorff functional with respect to $\|\cdot\|_C$ on $C_L([0,T],X)$.*

**Proof.** Consider the operator

$$B_*:\, C_L([0,T],X) \to C_L([0,T],X)$$

defined by

$$B_*(x)(t) = x(0) + \int_0^t g\bigg(\xi, x(\xi), \int_0^\xi K_1(\xi,s)x(s)\,ds, \int_0^T K_2(\xi,s)x(s)\,ds\bigg)d\xi$$

on $[0,T]$. We have

$$\begin{aligned}&\big\|A_*^2(x)(t)-A_*(x)(t)\big\|\\&\qquad\le L_*\int_0^t \bigg[\|A_*(x)(\xi)-x(\xi)\|\end{aligned}$$

$$
\begin{aligned}
&+ \left\| \int_0^{\xi} K_1(\xi, s)(A_*(x)(s) - x(s))\, ds \right\| \\
&+ \left\| \int_0^{T} K_2(\xi, s)(A_*(x)(s) - x(s))\, ds \right\| \Bigg] d\xi \\
&\le 3L_* T \max(1, k_1 T, k_2 T) \|A_*(x) - x\|_C \\
&= 3L_* T k_0 \|A_*(x) - x\|_C
\end{aligned}
$$

for all $x \in C_L([0,T], X)$. Similarly,

$$
\left\| B_*^2(x)(t) - B_*(x)(t) \right\| \le 3L_* T k_0 \|B_*(x) - x\|_C
$$

for all $x \in C_L([0,T], X)$. It follows that

$$
\begin{aligned}
\|A_*^2(x) - A_*(x)\|_C &\le 3L_* T k_0 \|A_*(x) - x\|_C \\
\|B_*^2(x) - B_*(x)\|_C &\le 3L_* T k_0 \|B_*(x) - x\|_C.
\end{aligned}
$$

Because of assumption (iv), $\|A_*(x) - B_*(x)\|_C \le \eta_1 T$ for all $x \in C_L([0,T], X)$. By applying Theorem 2.2 we obtain $H_{\|\cdot\|_C}(F_{A_*}, F_{B_*}) \le \frac{\eta_1 T}{1 - 3L_* T k_0}$ and the theorem is proved ∎

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