Abstract. We present fixed point theorems for weakly sequentially upper semicontinuous decomposable non-convex-valued maps. They are based on an extension of the Arino-Gautier-Penot Fixed Point Theorem for weakly sequentially upper semicontinuous maps with convex values. Applications are given to abstract operator inclusions in $L^p$ spaces. An example is included to illustrate the theory.

Keywords: Multi-valued map, operator inclusion, functional-differential inclusion, fixed point, continuation principle, measure of non-compactness, weak topology

AMS subject classification: Primary 54H25, secondary 47H10, 47J35

1. Introduction

Various types of boundary value problems for differential inclusions, integrodifferential inclusions or, more generally, functional-differential inclusions can be equivalently reformulated as operator inclusions of the form

$$u \in \Psi \Phi u$$  \hspace{1cm} (1.1)

in an appropriate space of functions, where by $\Psi \Phi$ we mean the composition $\Psi \circ \Phi$. Most frequently $\Psi$ is an “integral type” map, the inverse of a differential operator, while $\Phi$ is a multi-valued map associated with the right-hand side of the functional-differential inclusion.

For the theory of differential inclusions and its applications we refer the reader to the books of Deimling [9], Gö­rniewicz [12], Hu and Papageorgiou [14, 15] and Kamenskii, Obukhovskii and Zecca [16].

Radu Precup: Babeş-Bolyai University, Fac. Math. & Comp. Sci., 3400 Cluj, Romania; r.precup@math.ubbcluj.ro
Using a fixed point approach to problem (1.1), we may first try to apply fixed point theorems to the composite multi-valued map $F = \Psi \Phi$. Several difficulties arise when treating such multi-valued compositions this way. One of them consists in guaranteeing continuity properties for the maps; another one concerns the geometric properties of their values. For example, even if the values of $\Phi$ are convex and $\Psi$ is single-valued (but nonlinear), the values of $F = \Psi \Phi$ can be non-convex. In this connection we may think to use fixed point theorems for non–convex-valued maps, for example, the Eilenberg-Montgomery Theorem (see Couchouron and Precup [5, 6]). However, it is expectable that one can take advantage from the representation of $F$ as $\Psi \Phi$. Several authors have done this under various aspects (see Andres and Bader [1], Bader [3] and Górniewicz [12]). The main purpose of the present paper is to develop a fixed point theory for maps which are decomposable into $\Psi \Phi$, with both $\Phi$ and $\Psi$ convex-valued maps between Banach spaces. We shall succeed this by considering the Cartezian product map

$$\Pi(x, y) = \Psi y \times \Phi x$$

whose values are convex in the corresponding product space $X \times Y$ endowed with the weak topologies on $X$ and $Y$.

The abstract results established in this paper can be used to prove elementarily that the hypothesis of contractibility asked in Couchouron and Kamenskii [4] and that one of acyclicity from Couchouron and Precup [5, 6] are not necessary (for [4] this was previously shown by Bader [3] by means of a topological fixed-point index theory for decomposable maps, under a stronger compactness condition on $\Psi$). In Section 3 the abstract continuation principle established in Section 2 is applied to discuss operator inclusions in $L^p$ spaces, under general assumptions which were inspired by those in Couchouron and Kamenskii [4] and in Couchouron and Precup [5]. Finally, we present a simple example concerning functional-differential inclusions.

The main contributions of this paper are as follows:

1) A fixed point theory for non–convex-valued maps which can be represented as compositions of two convex-valued maps. This theory improves and extends the results from Couchouron and Kamenskii [4] and from Couchouron and Precup [5, 6]. Also, our theory represents a fixed point alternative to the index theory presented in Bader [3] under some more restrictive conditions (for example, in [3] only $\Phi$ is multi-valued).


3) Theorems of Mönch type for set-valued maps with conditions expressed with respect to the strong or the weak topology. These results complement those in Mönch [17], O’Regan [18] and in O’Regan and Precup [19].
For the remainder of this section we gather together some definitions and results which we will need in what follows.

For any Hausdorff topological space $X$ we define

\[ P_f(X) = \{ A \subset X : A \text{ is non-empty, closed} \} \]

\[ P_k(X) = \{ A \subset X : A \text{ is non-empty, compact} \}. \]

If $X$ is a closed convex subset of a Banach space, then we define

\[ P_{fc}(X) = \{ A \subset X : A \text{ is non-empty, closed, convex} \} \]

\[ P_{kwc}(X) = \{ A \subset X : A \text{ is non-empty, weakly compact, convex} \}. \]

A multi-valued map $\Phi : X \to 2^Y$, where $X$ and $Y$ are Hausdorff topological spaces, is said to be upper semicontinuous if for every closed subset $A$ of $Y$ the set

\[ \Phi^{-1}(A) = \{ x \in X : A \cap \Phi x \neq \emptyset \} \]

is closed in $X$.

Throughout this paper we shall consider multi-valued maps $\Phi : X \to 2^Y$ where $X$ and $Y$ are subsets of two Banach spaces. We shall use the following terminology:

- $\Phi$ is u.s.c. if $\Phi$ is upper semicontinuous with respect to the strong topologies of $X$ and $Y$.
- $\Phi$ is w-u.s.c. if $\Phi$ is upper semicontinuous with respect to the weak topologies of $X$ and $Y$.
- $\Phi$ is sequentially w-u.s.c. if for every weakly closed subset $A \subset Y$ the set $\Phi^{-1}(A)$ is sequentially closed for the weak topology on $X$.

We recall the following two known fixed point theorems: **Theorem 1.1** (Bohnenblust-Karlin). If $X$ is a Banach space, $C$ is a non-empty compact convex subset of $X$ and $\Phi : C \to P_{fc}(C)$ is u.s.c., then there exists an $x \in C$ with $x \in \Phi x$.

**Theorem 1.2** (Arino-Gautier-Penot). If $X$ is a Banach space (or, more generally, a metrizable locally convex linear topological space), $C$ is a non-empty weakly compact convex subset of $X$ and $\Phi : C \to P_{fc}(C)$ is sequentially w-u.s.c., then there exists an $x \in C$ with $x \in \Phi x$.

Notice that Theorem 1.2 is an immediate consequence of Ky Fan’s Fixed Point Theorem (see Deimling [8: pp. 310 – 315]) and of the following lemma (Arino, Gautier and Penot [2], O’Regan [18]) whose proof is based upon
the Eberlein-Šmulian Theorem (see Dunford and Schwartz [11: pp. 430]).

**Lemma 1.1.** Let $X, Y$ be Banach spaces (or, more generally, locally convex linear topological spaces, and $X$ metrizable) and let $C$ be a weakly compact subset of $X$. Then any sequentially w-u.s.c. map $\Phi : C \to 2^Y$ is w-u.s.c.

**Remark 1.1.** For a map $\Phi : C \to 2^C$ with $C$ a compact subset of a Banach space, the notions of u.s.c., w-u.s.c. and sequentially w-u.s.c. are identical. Thus in Theorem 1.1 $\Phi$ can be equivalently assumed to be sequentially w-u.s.c. So Theorem 1.2 appears as a generalization of Theorem 1.1.

Next we recall the definitions of measures of non-compactness and weak non-compactness. By a *measure of non-compactness* in a closed convex subset $C$ of a Banach space $X$ we mean a real function $\mu$ defined on the collection of all non-empty bounded subsets of $C$, such that

\[ \mu(A) = \mu(\overline{A}) \]

\[ \mu(A) = 0 \iff A \text{ is relatively compact} \]

\[ A \subset B \implies \mu(A) \leq \mu(B). \]

We shall denote by $\beta_X$ the *ball measure of non-compactness* in $X$,

\[ \beta_X(A) = \inf \left\{ \varepsilon > 0 : A \text{ admits a finite cover by balls of radius } \varepsilon \right\}. \]

By a *measure of weak non-compactness* in a closed convex subset $C$ of a Banach space we mean a real function $\chi$ defined on the collection of all non-empty bounded subsets of $C$, such that

\[ \chi(A) = \chi(\overline{A}) \]

\[ \chi(A) = 0 \iff A \text{ is relatively weakly compact} \]

\[ A \subset B \implies \chi(A) \leq \chi(B). \]

For an example of a measure of weak non-compactness see De Blasi [7].

We conclude this section with two well-known compactness criteria in $L^p(0, T; E)$ (see Guo, Lakshmikantham and Liu [13: pp. 15 – 18] and Diestel, Ruess and Schachermayer [10], respectively). Here $0 < T < \infty$, $p \in [1, \infty]$ and $E$ is a Banach space with norm $| \cdot |_E$. For a function $u : [0, T] \to E$ we define the *translation* by $h$ ($0 < h < T$) to be the function $\tau_h u : [0, T-h] \to E$ given by $\tau_h u(t) = u(t+h)$. **Theorem 1.3.** Let $p \in [1, \infty]$. Let $M \subset L^p(0, T; E)$ be countable and assume that there exists a function $\nu \in L^p(0, T; \mathbb{R}_+)$ with $|u(t)|_E \leq \nu(t)$ a.e. on $[0, T]$, for all $u \in M$. In addition, assume that $M \subset C([0, T]; E)$ if $p = \infty$. Then $M$ is relatively compact in $L^p(0, T; E)$ if and only if
(i) \( \sup_{u \in M} |\tau_h u - u|_{L^p(0,T-h;E)} \to 0 \) as \( h \downarrow 0 \)

(ii) \( M(t) = \{ u(t) : u \in M \} \) is relatively compact in \( E \) for a.e. \( t \in [0,T] \).

**Theorem 1.4.** Let \( p \in [1,\infty) \). Let \( M \subset L^p(0,T;E) \) be countable and assume that there exists a function \( \nu \in L^p(0,T;\mathbb{R}_+) \) with \( |u(t)|_E \leq \nu(t) \) a.e. on \( [0,T] \), for all \( u \in M \). If \( M(t) \) is relatively compact in \( E \) for a.e. \( t \in [0,T] \), then \( M \) is weakly relatively compact in \( L^p(0,T;E) \).

### 2. Fixed point theory

First we give an extension of the Arino-Gautier-Penot Fixed Point Theorem [2] to decomposable non-convex-valued maps. **Theorem 2.1.** Let \( X \) and \( Y \) be Banach spaces (or, more generally, metrizable locally convex linear topological spaces), let \( A \) and \( B \) be non-empty weakly compact convex subsets of \( X \) and \( Y \), respectively, and let

\[
\Phi : A \to P_{fc}(B) \\
\Psi : B \to P_{fc}(A)
\]

be two multi-valued maps. Assume \( \Phi \) and \( \Psi \) are sequentially w-u.s.c. Then there exists at least one \( x \in A \) with \( x \in \Psi \Phi x \) and, equivalently, there exists at least one \( y \in B \) with \( y \in \Phi \Psi y \).

**Proof.** Let \( X \times Y \) be endowed with the product topology. In this way, \( X \times Y \) is a Banach space (respectively, a metrizable locally convex linear topological space). Consider the multi-valued map acting in \( X \times Y \), \( \Pi : A \times B \to P_{fc}(A \times B) \), given by

\[
\Pi(x,y) = \Psi y \times \Phi x.
\]

We have that \( A \times B \) is a weakly compact convex subset of \( X \times Y \). In addition, \( \Pi \) is sequentially w-u.s.c. (see Kamenskii, Obukhovskii and Zecca [16: Theorem 1.2.12]). Thus we may apply the Arino-Gautier-Penot Fixed Point Theorem. Therefore, there exists a \( (x,y) \in A \times B \) with \( (x,y) \in \Pi(x,y) \). We have \( x \in \Psi y \) and \( y \in \Phi x \). Consequently, \( x \in \Psi \Phi x \) and \( y \in \Phi \Psi y \).
Remark 2.1. The Arino–Gautier–Penot Theorem appears as a particular case of Theorem 2.1, when \( X = Y \), \( A = B \) and \( \Phi \) or \( \Psi \) is the identity map of \( A \).

Theorem 2.2. Let \( X, Y \) be Banach spaces, let \( C \) be a closed convex subset of \( X \), and let

\[
\Phi : C \to P_{k^w_c}(Y) \\
\Psi : \overline{co}\Phi(C) \to P_{fc}(C)
\]

be two multi-valued maps. Assume that, for every weakly compact convex subset \( A \) of \( C \), \( \Phi \) and \( \Psi \) are sequentially w-u.s.c. on \( A \) and on \( \overline{co}\Phi(A) \), respectively. In addition, assume that there exists an \( x_0 \in C \) such that the condition

\[
\begin{aligned}
A &\subset C \\
A = \overline{co}\left(\{x_0\} \cup \Psi(\overline{co}\Phi(A))\right)
\end{aligned} \quad \Rightarrow \quad A \text{ is weakly compact}
\]

(2.1)

is satisfied. Then there exists at least one \( x \in C \) with \( x \in \Psi\Phi x \).

Proof. Let \( \mathcal{M} \) be the collection of all non-empty closed convex subsets \( M \) of \( C \) with

\[
\overline{co}\left(\{x_0\} \cup \Psi(\overline{co}\Phi(M))\right) \subset M.
\]

Clearly, \( C \in \mathcal{M} \) and \( x_0 \in M \) for every \( M \in \mathcal{M} \). Moreover, it is easy to see that

\[
M \in \mathcal{M} \implies \overline{co}\left(\{x_0\} \cup \Psi(\overline{co}\Phi(M))\right) \in \mathcal{M}. \quad (2.2)
\]

Define the set

\[
A = \cap\{M : M \in \mathcal{M}\}.
\]

We have \( A \in \mathcal{M} \). Also, (2.2) implies

\[
A = \overline{co}\left(\{x_0\} \cup \Psi(\overline{co}\Phi(A))\right).
\]
Then (2.1) guarantees that $A$ is weakly compact. Now Theorem 2.1 applies to $A$ and $B = \overline{\text{co}} \Phi(A)$. Notice (see Kamenskii, Obukhovskii and Zecca [16: Theorem 1.1.7]) that $\Phi(A)$ is weakly compact since $\Phi$ is w-u.s.c. on $A$ (from Lemma 1.1) and has weakly compact values. Then the Krein-Šmulian Theorem (Dunford and Schwartz [11: pp. 434]) implies that $\overline{\text{co}} \Phi(A)$ is weakly compact. 

**Remark 2.2.** If in addition $C$ is weakly compact, then condition (2.1) trivially holds and Theorem 2.2 becomes Theorem 2.1.

Theorem 2.2 yields in particular the following result for convex-valued self-maps of a closed convex subset of a Banach space (compare Theorem 4.3 in O’Regan [18] and Theorem 2.1 for single-valued maps in Mönch [17]), an alternative result to Theorem 3.1 in O’Regan and Precup [19].

**Corollary 2.1.** Let $X$ be a Banach space, $C$ a closed convex subset of $X$ and $\Phi : C \to P_{k^{w\text{c}}}(C)$. Assume $\Phi$ is sequentially w-u.s.c. and that there is an $x_0 \in C$ such that

$$A \subset C \quad A = \overline{\text{co}}(\{x_0\} \cup \Phi(A)) \implies A \text{ is weakly compact.}$$

Then there exists at least one $x \in C$ with $x \in \Phi x$.

**Proof.** We apply Theorem 2.2 to $Y = X$ and $\Psi = I_X$, the identity map of $X$. Note that

$$\overline{\text{co}}(\{x_0\} \cup \Psi(\overline{\text{co}} \Phi(A))) = \overline{\text{co}}(\{x_0\} \cup \overline{\text{co}} \Phi(A)) = \overline{\text{co}}(\{x_0\} \cup \Phi(A))$$

and the assertion is proved.

**Remark 2.3.** If in addition $C$ is weakly compact, then condition (2.3) trivially holds and Corollary 2.1 becomes the Arino-Gautier-Penot Theorem.

Under a stronger condition than (2.1) and a weaker one on $\Phi$, we have the following result. **Theorem 2.3.** Let
\( X \) and \( Y \) be Banach spaces, let \( C \) be a closed convex subset of \( X \), and let

\[
\Phi : C \to P_{k^\infty c}(Y) \\
\Psi : \overline{\text{co}} \Phi(C) \to P_{f c}(C)
\]

be two multi-valued maps. Assume that, for every compact convex subset \( A \) of \( C \), \( \Phi \) and \( \Psi \) are sequentially \( w-u.s.c. \) on \( A \) and \( \overline{\text{co}} \Phi(A) \), respectively. In addition, assume that there exists an \( x_0 \in C \) such that the condition

\[
A \subset C \\
A = \overline{\text{co}} (\{x_0\} \cup \Psi(\overline{\text{co}} \Phi(A))) \\
\implies \ A \text{ is compact}
\]

(2.4)

is satisfied. Then there exists at least one \( x \in C \) with \( x \in \Psi \Phi x \).

Next we present a fixed point theorem of Leray-Schauder type (a continuation principle) for decomposable non-convex-valued maps. Theorem 2.4. Let \( X \) and \( Y \) be Banach spaces, \( K \) a closed convex subset of \( X \), \( U \) a convex relatively open subset of \( K \), \( x_0 \in U \) and let

\[
\Phi : U \to P_{k^\infty c}(Y) \\
\Psi : \overline{\text{co}} \Phi(U) \to P_{f c}(K)
\]

be two multi-valued maps. Assume that, for every compact convex subset \( A \) of \( U \), \( \Phi \) and \( \Psi \) are sequentially \( w-u.s.c. \) on \( A \) and \( \overline{\text{co}} \Phi(A) \), respectively. In addition, assume that the two conditions

\[
A \subset U \\
A \text{ closed convex} \\
A \subset \overline{\text{co}} (\{x_0\} \cup \Psi(\overline{\text{co}} \Phi(A))) \\
\implies A \text{ is compact}
\]

(2.5)
and
\[ x \notin (1 - \lambda)x_0 + \lambda \Psi \Phi x \quad \forall \ x \in U \setminus U, \lambda \in (0, 1) \quad (2.6) \]
are satisfied. Then there exists at least one \( x \in U \) with \( x \in \Psi \Phi x \).

Proof. If \( U = K \), then \( U \setminus U = \emptyset \), so (2.6) is superfluous and the result follows from Theorem 2.3, where \( C = K \). Assume \( U \neq K \). Let
\[ C = \overline{co}(\{x_0\} \cup \Psi(\overline{co} \Phi(U))). \]
It is clear that \( x_0 \in C \subset K \) and \( C \) is closed convex. Since \( U \) is open in \( K \), convex, and \( x_0 \in U \), we can define a single-valued operator \( P : K \to \overline{U} \) by
\[ Px = \begin{cases} x & \text{if } x \in \overline{U} \\ (1 - \lambda)x_0 + \lambda x & \text{if } x \notin \overline{U} \end{cases} \]
where \( \lambda \in (0, 1) \) is such that \( (1 - \lambda)x_0 + \lambda x \in \overline{U} \setminus U \). Clearly, \( P \) is continuous.

Consider
\[ \hat{\Phi} : C \to P_{k^{w}_c}(Y), \quad \hat{\Phi}x = \Phi Px \ (x \in C) \]
\[ \hat{\Psi} : \overline{co} \hat{\Phi}(C) \to P_{f_c}(C), \quad \hat{\Psi}y = \Psi y \ (y \in \overline{co} \hat{\Phi}(C)). \]
We first check that \( \hat{\Phi} \) is sequentially \( w\)-u.s.c. on any compact convex subset \( A \) of \( C \). Indeed, we can see that it suffices to prove this for compact convex sets \( A \) with \( x_0 \in A \). In this situation, \( P(A) = A \cap \overline{U} \), so \( P(A) \) is compact and convex. Now let \( B \subset Y \) be weakly closed. We have to show that the set
\[ M = \{ x \in A : \hat{\Phi}x \cap B \neq \emptyset \} \]
is weakly sequentially closed. Assume \( x_k \in A, \hat{\Phi}x_k \cap B \neq \emptyset \) and \( x_k \to x \) weakly. Since \( A \) is compact, there is a
subsequence \((x_{k'})\) of \((x_k)\) with \(x_{k'} \to x\) strongly. Then 
\(Px_{k'} \to Px\) strongly. Since \(P(A)\) is compact convex, \(\Phi\) is sequentially w-u.s.c. on \(P(A)\). Consequently, the set 
\[
N = \{ y \in P(A) : \Phi y \cap B \neq \emptyset \}
\]
is weakly sequentially closed. Since \(Px_{k'}\) belongs to \(N\) for all \(k'\), we have \(Px \in N\), too. Thus \(\Phi Px \cap B \neq \emptyset\) with 
\(x \in A\). Therefore, \(x \in M\) as desired. It is easy to see that \(\hat{\Psi}\) is sequentially w-u.s.c. on \(\overline{co} \hat{\Phi}(A)\).

Next we show that (2.4) holds for the couple \((\hat{\Phi}, \hat{\Psi})\). Let \(A \subset C\) be such that 
\[
A = \overline{co}(\{x_0\} \cup \hat{\Psi}(\overline{co} \hat{\Phi}(A))).
\]
Clearly, 
\[
A = \overline{co}(\{x_0\} \cup \hat{\Psi}(\overline{co} \Phi P(A))).
\]
We have 
\[
P(A) = A \cap \overline{U} \subset \overline{co}(\{x_0\} \cup \Psi(\overline{co} \Phi P(A)))
\]
where \(P(A)\) is a closed convex subset of \(\overline{U}\). Then (2.5) guarantees that \(P(A)\) is compact. Let \((x_k)\) be any sequence in \(A\). Since \(P(A)\) is compact, there exists a subsequence \((x_{k'})\) of \((x_k)\) with \(Px_{k'} \to y\) strongly for some \(y \in P(A)\). We have \(Px_{k'} = (1 - \lambda_{k'})x_0 + \lambda_{k'}x_{k'}\) for some \(\lambda_{k'} \in [0, 1]\). Passing eventually to a new subsequence we may assume that \(\lambda_{k'} \to \lambda\) for some \(\lambda \in [0, 1]\). If \(\lambda > 0\), we immediately find that \((x_{k'})\) is strongly convergent. Assume \(\lambda = 0\). Then \(y = x_0\) and so \(Px_{k'} = x_{k'}\) for all \(k' \geq k_0\). Hence \((x_{k'})\) is strongly convergent as well. Hence \(A\) is compact.

Thus all the assumptions of Theorem 2.3 are satisfied for the couple \((\hat{\Phi}, \hat{\Psi})\). Therefore, there exists \(x \in C\) with \(x \in \hat{\Psi} \hat{\Phi} x\). Clearly, \(x \in \Psi \Phi Px\). We claim that
$x \in U$. Assume the contrary, that is $x \notin U$. Then $Px = (1 - \lambda)x_0 + \lambda x$ for some $\lambda \in (0, 1)$ and $Px \in \overline{U} \setminus U$. From $x \in \Psi \Phi Px$ we deduce

$$ Px = (1 - \lambda)x_0 + \lambda x \in (1 - \lambda)x_0 + \lambda \Psi \Phi Px $$

which contradicts (2.6). Hence $x \in \overline{U}$, so $Px = x$ and $x \in \Psi \Phi x$.

Theorem 2.4 yields in particular the following continuation principle for convex-valued maps (compare Theorem 2.2 for single-valued maps in Mönch [17]), an alternative result to Theorem 3.2 in O’Regan and Precup [19]. Corollary 2.2. Let $X$ be a Banach space, $K$ a closed convex subset of $X$, $U$ a convex relatively open subset of $K$, $x_0 \in U$ and let

$$ \Phi : \overline{U} \to \mathcal{P}_{k^w_c}(K) $$

be a multi-valued map. Assume that $\Phi$ is sequentially w-u.s.c. on each compact convex subset of $U$. In addition, assume that the two conditions

$$ A \subset \overline{U} \quad A \text{ closed convex} \quad A \subset \overline{\text{co}}(\{x_0\} \cup \Phi(A)) $$

and

$$ x \notin (1 - \lambda)x_0 + \lambda \Phi x \quad \forall \ x \in \overline{U} \setminus U, \lambda \in (0, 1) $$

are satisfied. Then there exists at least one $x \in \overline{U}$ with $x \in \Phi x$.

Remark 2.4. Let $U$ be bounded, and let $\Phi$ and $\Psi$ send bounded sets into bounded sets. If $\mu$ is a measure of strong non-compactness in $K$, $\chi$ is a measure of weak
non-compactness on $\overline{co} \Phi(U)$, and there are functions $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi$ non-decreasing such that

$$
\psi \phi(\tau) < \tau \quad (\tau > 0) \quad (2.7)
$$

$$
\chi(\Phi(M)) \leq \phi(\mu(M)) \quad (M \subset U)
$$

$$
\mu(\Psi(M)) \leq \psi(\chi(M)) \quad (M \subset \overline{co} \Phi(U)),
$$

then condition (2.5) holds. Indeed, if $A \subset \overline{U}$ and $A \subset \overline{co}(\{x_0\} \cup \Psi(\overline{co} \Phi(A)))$, then

$$
\mu(A) \leq \mu(\Psi(\overline{co} \Phi(A))) \leq \psi(\chi(\overline{co} \Phi(A))) = \psi(\chi(\Phi(A))) \leq \psi(\mu(A)).
$$

Then (2.7) implies $\mu(A) = 0$, i.e. $A$ is compact.

3. Operator inclusions in $L^p$ spaces

In this section we are concerned with the abstract operator inclusion

$$
w \in \Psi \Phi w \quad (w \in K) \quad (3.1)
$$

in a closed convex subset $K$ of $L^p(0, T; F)$, where

$\Phi : K \to 2^{L^q(0, T; E)}$ is a multi-valued map

$\Psi : L^q(0, T; E) \to K$ is a single-valued operator.

Here $0 < T < \infty$, $p \in [1, \infty]$, $q \in [1, \infty)$, and $E$ and $F$ are Banach spaces. We shall denote by $r$ the conjugate exponent of $q$, i.e. $\frac{1}{q} + \frac{1}{r} = 1$. By $|\cdot|_q$ we shall denote the norm of $L^q(0, T; E)$ and by $\|\cdot\|$ an equivalent norm on the closed subspace of $L^p(0, T; F)$ generated by $K$.

We now state our assumptions:

$(\Psi 1)$ There exists a function $\eta : [0, T] \times L^q(0, T; \mathbb{R}_+) \to \mathbb{R}_+$, non-decreasing in its second variable such that, for every $t \in [0, T]$,

$$
\sup_{g \in L^q(0, T; \mathbb{R}_+)} |\eta(t, g + h) - \eta(t, g)| \to 0 \quad (|h|_q \to 0) \quad (3.2)
$$

and

$$
| (\Psi f_1 - \Psi f_2)(t) |_F \leq \eta(t, |(f_1 - f_2)(\cdot)|_E)
$$
a.e. on $[0, T]$, for all $f_1, f_2 \in L^q(0, T; E)$.

(Ψ2) There exists a constant $L > 0$ with $\|\Psi f_1 - \Psi f_2\| \leq L|f_1 - f_2|_q$ for all $f_1, f_2 \in L^q(0, T; E)$.

(Ψ3) For any compact $C \subset E$ and any sequence $(f_k)$ of $L^q(0, T; E)$ with $\{f_k(t)\}_{k \geq 1} \subset C$ for a.e. $t \in [0, T]$, the weak convergence $f_k \rightharpoonup f$ implies $\Psi f_k \rightharpoonup \Psi f$ strongly in $L^p(0, T; F)$.

(Φ1) The values of $\Phi$ are non-empty, weakly compact, convex, and $\Phi$ is sequentially w-u.s.c. on any compact convex subset $A$ of $K$.

(Φ2) For every $a > 0$ there exists a $\nu_a \in L^q(0, T; \mathbb{R}_+)$ such that $|f(t)|_E \leq \nu_a(t)$ a.e. on $[0, T]$, for all $f \in \Phi w$ and all $w \in K$ satisfying $\|w\| \leq a$.

(Φ3) For every separable closed subspaces $E_0$ and $F_0$ of $E$ and $F$, respectively, there exists a map $\zeta : L^p(0, T; \mathbb{R}_+) \to L^q(0, T; \mathbb{R}_+)$ such that $\zeta(0) = 0$ and

$$\beta_{E_0}(\Phi(M)(t) \cap E_0) \leq \zeta(\beta_{F_0}(M(\cdot)))(t) \quad (3.3)$$

a.e. on $[0, T]$, for every countable set $M \subset K$ with $M(t) \subset F_0$ a.e. on $[0, T]$, for which there exists $\nu \in L^p(0, T; \mathbb{R}_+)$ with $|w(t)|_F \leq \nu(t)$ a.e. on $[0, T]$ for any $w \in M$. In addition, $\varphi = 0$ is the unique solution in $L^p(0, T; \mathbb{R}_+)$ to the inequality

$$\varphi(t) \leq \eta(t, \zeta(\varphi)) \quad \text{a.e. on } [0, T]. \quad (3.4)$$

(L-S) There exists a bounded convex subset $U$ of $K$, open in $K$, and a $w_0 \in U$ such that $w \notin (1 - \lambda)w_0 + \lambda \Psi \Phi w$ for all $w \in \overline{U} \setminus U$ and $\lambda \in (0, 1)$. **Theorem 3.1.** Let assumptions $(Ψ1)$–$(Ψ3)$, $(Φ1)$–$(Φ3)$ and (L-S) hold. Then inclusion problem (3.1) has at least one solution in $U$.

For the proof we need the following Lemmas 3.1 and 3.2. **Lemma 3.1.** Let assumptions $(Ψ1)$ and $(Ψ3)$ hold. Further, let $B \subset L^q(0, T; E)$ be countable with

$$|f(t)|_E \leq \nu(t) \quad (3.5)$$

a.e. on $[0, T]$ for all $f \in B$, where $\nu \in L^q(0, T; \mathbb{R}_+)$. At last, let $E_0$ and $F_0$ be separable closed subspaces of $E$ and $F$, respectively, with $f(t) \in E_0$ and $\Psi f(t) \in F_0$ a.e. on $[0, T]$ for every $f \in B$. Then the function $\varphi$ defined by $\varphi(t) = \beta_{E_0}(B(t))$ belongs to $L^q(0, T; \mathbb{R}_+)$ and satisfies

$$\beta_{F_0}(\Psi(B)(t)) \leq \eta(t, \varphi) \quad (3.6)$$

a.e. on $[0, T]$. 
Proof. Let \( B = \{f_n\}_{n \geq 1} \). The space \( E_0 \) being separable, we may represent it as \( \bigcup_{k \geq 1} E_k \) where, for each \( k \), \( E_k \) is a \( k \)-dimensional subspace of \( E_0 \) with \( E_k \subset E_{k+1} \). The fact that \( \varphi \) is measurable follows from the formula of representation of the ball measure of non-compactness for separable spaces which yields

\[
\varphi(t) = \lim_{k \to \infty} \sup_{n \geq 1} d(f_n(t), E_k). \tag{3.7}
\]

From \( d(f_n(t), E_k) \leq |f_n(t)|_E \), (3.5) and (3.7) we have \( \varphi(t) \leq \nu(t) \) a.e. on \([0,T]\). Consequently, \( \varphi \in L^q(0,T;\mathbb{R}_+) \).

Since \( B \) is countable, we may suppose that (3.5) holds for all \( t \in [0,T] \) and \( f \in B \). To prove (3.6), let \( \varepsilon > 0 \) and choose \( \delta > 0 \) such that

\[
|\Theta| \leq \delta \implies \int_{\Theta} \nu(t)^q dt \leq \varepsilon^q. \tag{3.8}
\]

Here \( |\Theta| \) is the Lebesgue measure of \( \Theta \). Also, choose a constant \( \rho > 0 \) such that \( |\Theta_1| < \frac{\delta}{4} \) for \( \Theta_1 = \{t \in [0,T] : \nu(t) > \rho\} \). So we have \( d(f_n(t), E_k) \leq |f_n(t)|_E \leq \rho \) for \( t \in I \setminus \Theta_1 \) and \( n,k \geq 1 \). Consequently, \( d(f_n(t), E_k) = d(f_n(t), C_k) \) with \( C_k = \{x \in E_k : |x|_E \leq \rho\} \).

From (3.7) and Egoroff’s Theorem (see Dunford and Schwartz [11: pp. 149]) there is a set \( \Theta_2 \subset [0,T] \setminus \Theta_1 \) with \( |\Theta_2| \leq \frac{\delta}{2} \) and an integer \( k_0 \) such that

\[
\sup_{n \geq 1} d(f_n(t), C_k) \leq \varphi(t) + \varepsilon \tag{3.9}
\]

for \( t \in [0,T] \setminus (\Theta_1 \cup \Theta_2) \) and \( k \geq k_0 \). Since \( B \) is a countable set of strongly measurable functions, we may find a set \( \Theta_3 \subset [0,T] \) with \( |\Theta_3| = 0 \) and a countable set \( \tilde{B} = \{\tilde{f}_n\}_{n \geq 1} \) of finitely-valued functions from \([0,T]\) to \( E \) with

\[
|f_n(t) - \tilde{f}_n(t)|_E \leq \varepsilon \tag{3.10}
\]

for \( t \in [0,T] \setminus \Theta_3 \) and \( n \geq 1 \). From (3.9) and (3.10) we obtain

\[
d(\tilde{f}_n(t), C_k) \leq \varphi(t) + 2\varepsilon
\]

for \( n \geq 1, k \geq k_0 \) and \( t \in [0,T] \setminus \Theta \) with \( \Theta = \Theta_1 \cup \Theta_2 \cup \Theta_3 \). Then there exists a finitely-valued function \( \hat{f}_{n,k} \) from \([0,T]\) to \( C_k \) with

\[
|f_n(t) - \hat{f}_{n,k}(t)|_E \leq \varphi(t) + 3\varepsilon \tag{3.11}
\]

for \( n \geq 1, k \geq k_0 \) and \( t \in [0,T] \setminus \Theta \). We put \( \hat{f}_{n,k}(t) = 0 \) for \( t \in \Theta \). Notice that \( |\Theta| \leq \delta \).
For each fixed $k \geq k_0$, Theorem 1.4 guarantees that the set $\{\hat{f}_{n,k}\}_{n \geq 1}$ is weakly relatively compact in $L^q(0,T; E)$. Then, from assumption $(\Psi 3)$, the set $\{\Psi \hat{f}_{n,k}\}_{n \geq 1}$ is relatively compact in $L^p(0,T; F)$. Therefore, by Theorem 1.3, the set $\{\Psi \hat{f}_{n,k}(t)\}_{n \geq 1}$ is relatively compact in $F$ for all $t \in [0,T]$ except a subset of measure zero. Since an at most countable union of sets of measure zero also has measure zero, we may assume that $\{\Psi \hat{f}_{n,k}(t)\}_{n \geq 1}$ is relatively compact for all $k \geq k_0$ and $t \in [0,T] \setminus \Theta_0$, where $|\Theta_0| = 0$. Let $t_0 \in [0,T] \setminus \Theta_0$ be arbitrary. Using assumption $(\Psi 1)$ and (3.11), we obtain

$$
|\Psi f_n(t_0) - \Psi \hat{f}_{n,k}(t_0)|_F \leq \eta(t_0, |f_n(\cdot) - \hat{f}_{n,k}(\cdot)||E)
\leq \eta(t_0, \varphi) + |\eta(t_0, \varphi + h) - \eta(t_0, \varphi)|
$$

where

$$
h(t) = \begin{cases} 
3\varepsilon & \text{for } t \in [0,T] \setminus \Theta \\
\nu(t) & \text{for } t \in \Theta
\end{cases}
$$

Writing

$$
h = h_1 + h_2
$$

with

$$
h_1(t) = \begin{cases} 
3\varepsilon & \text{for } t \in [0,T] \setminus \Theta \\
0 & \text{for } t \in \Theta
\end{cases}
$$

$$
h_2(t) = \begin{cases} 
0 & \text{for } t \in [0,T] \setminus \Theta \\
\nu(t) & \text{for } t \in \Theta
\end{cases}
$$

and using (3.8), we find that

$$
|h|_q \leq |h_1|_q + |h_2|_q \leq 3\varepsilon T^{\frac{1}{3}} + \varepsilon.
$$

Now (3.12) and (3.2) shows that the set $\{\Psi f_n(t_0)\}_{n \geq 1}$ admits a relatively compact $\varepsilon$-net of the form $\{\Psi \hat{f}_{n,k}(t_0)\}_{n \geq 1}$ for every $\varepsilon > \eta(t_0, \varphi)$. Letting $\varepsilon \downarrow \eta(t_0, \varphi)$ we obtain (3.6) \(\blacksquare\)

**Lemma 3.2.** Let assumptions $(\Psi 2)$ and $(\Psi 3)$ hold. Further, let $B$ be a countable subset of $L^q(0,T; E)$ such that $B(t)$ is relatively compact for a.e. $t \in [0,T]$ and there exists a function $\nu \in L^q(0,T; \mathbb{R}^+)$ with $|f(t)|_E \leq \nu(t)$ a.e. on $[0,T]$, for all $f \in B$. Then the set $\Psi(B)$ is relatively compact in $L^p(0,T; F)$. In addition, $\Psi$ is continuous from $B$ equipped with the relative weak topology of $L^q(0,T; E)$ to $L^p(0,T; F)$ equipped with its strong topology.

**Proof.** Let $B = \{f_n\}_{n \geq 1}$ and let $\varepsilon > 0$ be arbitrary. As in the proof of Lemma 3.1 we can find functions $\hat{f}_{n,k}$ with values in a compact $C_k \subset E$ ($C_k$ being a closed ball of a $k$-dimensional subspace of $E$) such that $|f_n - \hat{f}_{n,k}|_q \leq \varepsilon$ for every $n \geq 1$. Then assumption $(\Psi 2)$ implies

$$
||\Psi f_n - \Psi \hat{f}_{n,k}|| \leq L|f_n - \hat{f}_{n,k}|_q \leq \varepsilon L.
$$

(3.13)
On the other hand, the set \( \{ \hat{f}_{n,k} \}_{n \geq 1} \subset L^q(0,T;E) \) is weakly relatively compact in \( L^q(0,T;E) \). Next, assumption \((\Psi 3)\) guarantees that \( \{ \Psi \hat{f}_{n,k} \}_{n \geq 1} \) is relatively compact in \( L^p(0,T;F) \). Hence from (3.13) we see that \( \{ \Psi \hat{f}_{n,k} \}_{n \geq 1} \) is a relatively compact \( \varepsilon L \)-net of \( \Psi (B) \) with respect to the norm \( \| \cdot \| \). Since \( \varepsilon \) was arbitrary, we conclude that \( \Psi (B) \) is relatively compact in \( L^p(0,T;F) \).

Next we show that the graph

\[
\Lambda = \{ (f,w) : f \in B, w = \Psi f \}
\]

is weakly-strongly sequentially closed in \( L^q(0,T;E) \times L^p(0,T;F) \). To this end, assume \((f_k)\) and \((w_k)\) are sequences with \( f_k \in B \) and \( w_k = \Psi f_k, f_k \to f \) weakly and \( w_k \to w \) strongly for some \( f \in B \) and \( w \in L^p(0,T;F) \). We shall prove that \( w = \Psi f \). For an arbitrary number \( \varepsilon > 0 \), we have already seen that the proof of Lemma 3.1 provides a compact set \( P_\varepsilon \) and a sequence \( (f_k^\varepsilon) \) of \( P_\varepsilon \)-valued functions satisfying

\[
|f_k - f_k^\varepsilon|_q \leq \varepsilon
\]  
(3.14)

for every \( k \). The set \( \{ f_k^\varepsilon \}_{k \geq 1} \) being weakly relatively compact in \( L^q(0,T,E) \), a suitable subsequence \( (f_{k'}^\varepsilon) \) must be weakly convergent in \( L^q(0,T,E) \) towards some \( f^\varepsilon \). Consequently, \( \Psi f_{k'}^\varepsilon \to \Psi f^\varepsilon \) strongly in \( L^p(0,T;F) \). Also, Mazur's Lemma and (3.14) imply

\[
|f - f^\varepsilon|_q \leq \varepsilon.
\]  
(3.15)

Now assumption \((\Psi 2)\) and the triangle inequality yields

\[
\begin{align*}
\|w - \Psi f\| & \leq \|w - \Psi f_{k'}\| + \|\Psi f_{k'} - \Psi f_k^\varepsilon\| + \|\Psi f_k^\varepsilon - \Psi f^\varepsilon\| + \|\Psi f^\varepsilon - \Psi f\| \\
& \leq \|w - w_{k'}\| + L|f_{k'} - f_k^\varepsilon|_q + \|\Psi f_k^\varepsilon - \Psi f^\varepsilon\| + L|f^\varepsilon - f|_q.
\end{align*}
\]

Using (3.14), (3.15) and \( \|w - w_{k'}\| \to 0 \) and \( \|\Psi f_k^\varepsilon - \Psi f^\varepsilon\| \to 0 \) as \( k' \to \infty \) we deduce that

\[
\|w - \Psi f\| \leq 2\varepsilon L.
\]  
(3.16)

Since \( \varepsilon \) was arbitrary, (3.16) gives \( w = \Psi f \) and the proof of Lemma 3.2 is complete \( \blacksquare \).

**Proof of Theorem 3.1.** We apply Theorem 2.4 with \( x_0 := w_0, X \) the closed subspace of \( L^p(0,T;F) \) generated by \( K \), and \( Y := L^q(0,T;E) \). Notice that, since \( U \) is bounded in \( K \), there exists \( a > 0 \) such that \( \|w\| \leq a \) for all \( w \in U \). Then from assumption \((\Phi 2)\) one has \( |f(t)|_E \leq \nu_a(t) \) a.e. on \([0,T]\) for all \( f \in \Phi w \) and \( w \in U \). It follows that the same inequality is true for all \( f \in \text{ess} \Phi (U) \).
To guarantee that $\Psi$ is sequentially w-u.s.c. on $\overline{\mathcal{O}} \Phi(A)$ for any compact convex subset $A$ of $\overline{U}$ we have to show that

$$f_k \to f \text{ weakly, } f_k \in \overline{\mathcal{O}} \Phi(A) \implies \Psi f_k \to \Psi f \text{ strongly.}$$

Let $A_c \subset A$ be countable such that $\{f_k\}_{k \geq 1} \subset \overline{\mathcal{O}} \Phi(A_c)$. In virtue of Theorem 1.3, $A_c(t)$ is relatively compact in $F$ for a.e. $t \in [0,T]$. Then from (3.3) we deduce that $\beta_{E_0}(\Phi(A_c)(t) \cap E_0) = 0$ a.e. on $[0,T]$, for every separable closed subspace $E_0$ of $E$. As a result the set $\{f_k(t)\}_{k \geq 1}$ is relatively compact in $E$ for a.e. $t \in [0,T]$. Now Lemma 3.2 guarantees that $\Psi f_k \to \Psi f$ strongly.

It remains to check condition (2.5) for the couple $[\Phi, \Psi]$. Let $A \subset \overline{U}$ be a closed convex set with

$$A \subset \overline{\mathcal{O}}((\{w_0\} \cup \Psi(\overline{\mathcal{O}} \Phi(A))).$$

To prove that $A$ is compact it suffices that every sequence $(w_n^0)$ of $A$ has a convergent subsequence. Let $A_0 = \{w_n^1\}_{n \geq 1}$. Clearly, there exists a countable subset

$$A_1 = \{w_n^1\}_{n \geq 1}$$

of $A$, $f_n^1 \in \overline{\mathcal{O}} \Phi(A_1)$ and $v_n^1 = \Psi f_n^1$ with $A_0 \subset \overline{\mathcal{O}}(\{w_0\} \cup V^1)$, where $V^1 = \{v_n^1\}_{n \geq 1}$. Furthermore, there exists a countable subset

$$A_2 = \{w_n^2\}_{n \geq 1}$$

of $A$, $f_n^2 \in \overline{\mathcal{O}} \Phi(A_2)$ and $v_n^2 = \Psi f_n^2$ with $A_1 \subset \overline{\mathcal{O}}(\{w_0\} \cup V^2)$, where $V^2 = \{v_n^2\}_{n \geq 1}$, and so on. Hence for every $k \geq 1$ we find a countable subset

$$A_k = \{w_n^k\}_{n \geq 1}$$

of $A$ and correspondingly $f_n^k \in \overline{\mathcal{O}} \Phi(A_k)$ and $v_n^k = \Psi f_n^k$ such that $A_{k-1} \subset \overline{\mathcal{O}}(\{w_0\} \cup V^k)$ and $V^k = \{v_n^k\}_{n \geq 1}$. Let

$$A^* = \cup_{k \geq 0} A_k.$$ 

It is clear that $A^*$ is countable, $A_0 \subset A^* \subset A$ and $A^* \subset \overline{\mathcal{O}}(\{w_0\} \cup V^*)$, where $V^* = \cup_{k \geq 1} V^k$. Let $W^* := \{f_n^k\}_{n,k \geq 1}$. Since $A^*, V^*$ and $W^*$ are countable sets of strongly measurable functions, we may suppose that their values belong to a separable closed subspace $F_0$ of $F$ in the case of $A^*$ and $V^*$, respectively $E_0$ of $E$ in the case of $W^*$. Since $|f_n^k(t)| \leq \nu_a(t)$ a.e. on $[0,T]$, Lemma 3.1 guarantees

$$\beta_{F_0}(A^*(t)) \leq \beta_{F_0}(V^*(t)) \leq \eta(t, \beta_{E_0}(W^*(\cdot)))$$

(3.17)
while assumption \((\Phi3)\) gives
\[
\beta_{E_0}(W^*(s)) \leq \beta_{E_0}(\Phi(A^*)(s) \cap E_0) \leq \zeta(\beta_{F_0}(A^*(\cdot)))(s).
\] (3.18)

Since \(\eta\) is non-decreasing in its second variable, from (3.17) and (3.18) it follows that
\[
\beta_{F_0}(A^*(t)) \leq \eta(t, \zeta(\beta_{F_0}(A^*(\cdot)))).
\]

Moreover, the function \(\varphi\) given by \(\varphi(t) = \beta_{F_0}(A^*(t))\) belongs to \(L^p(0, T; \mathbb{R}_+).\) Consequently, \(\varphi \equiv 0\), and so \(\varphi(t) = \beta_{F_0}(A^*(t)) = 0\) a.e. on \([0, T].\) Then (3.18) and \(\zeta(0) = 0\) guarantee
\[
\beta_{E_0}(W^*(t)) = 0 \quad \text{a.e. on } [0, T].
\] (3.19)

Let \((v_i^*)\) be any sequence of \(V^*\) and let \((f_i^*)\) be the corresponding sequence of \(W^*\) with \(v_i^* = \Psi f_i^*\) for all \(i \geq 1.\) Using (3.19) we have that \((f_i^*)\) has a weakly convergent subsequence in \(L^q(0, T; E),\) say converging to \(f.\) Then the corresponding subsequence of \((v_i^*)\) converges to \(v = \Psi f\) in \(L^p(0, T; F)\). Hence \(V^*\) is relatively compact. Now Mazur’s Lemma guarantees that the set \(\mathcal{C}(\{w_0\} \cup V^*)\) is compact and so its subset \(A^*\) is relatively compact. Thus \(A_0\) possesses a convergent subsequence as we wished. Now the result follows from Theorem 2.4.

**Remark 3.1.**

(a) If the values of \(\Psi\) are in \(C(0, T; F),\) then any solution of inclusion problem (3.1) in \(K \subset L^p(0, T; F)\) \((1 \leq p \leq \infty)\) belongs to \(C(0, T; F).\)

(b) The existence theory in \(C(0, T; F)\) appears as a particular case, where \(p = \infty\) and \(K \subseteq C(0, T; F).\)

**Remark 3.2.**

(a) The typical example of a function \(\eta\) in assumption \((\Psi1)\) which occurs in applications is the one defined by \(\eta(t, \varphi) = \int_0^T k(t, s) \varphi(s)\) for \(k : [0, T]^2 \rightarrow \mathbb{R}_+,\) and \(k(t, \cdot) \in L^r(0, T)\) for a.e. \(t \in [0, T]\) (see Couchouron and Precup [5, 6], and O’Regan and Precup [21]). In this case condition \((\Psi2)\) is a consequence of condition \((\Psi1).\)

(b) For \(k(t, s) = \begin{cases} 0 & \text{if } t < s \\ m & \text{if } s \leq t \end{cases}\), where \(m > 0\) is a constant, the function \(\eta\) is defined as \(\eta(t, \varphi) = m \int_0^t \varphi(s)\) for a.e. \(t \in [0, T]\) (see Couchouron and Kamenskii [4], and Kamenskii, Obukhovskii and Zecca [16]). In this case, and if
\[
\zeta(\varphi)(t) = m_0 \varphi(t) + \int_0^t \delta(s) \varphi(s)\)\) (3.20)
where \( m_0 > 0 \) and \( \delta \in L^{r'}(0, T; \mathbb{R}_+) \) with \( r' > 2 \), the null function is the unique solution of inequality (3.4). Indeed, if \( \varphi(t) \leq \eta(t, \zeta(\varphi)) \), then

\[
\varphi(t) \leq m \int_0^t \left( m_0 \varphi(s) + \int_0^s \delta(\tau) \varphi(\tau) d\tau \right) ds
\]

\[
= m \int_0^t \left( m_0 e^{\theta s} \varphi(s) e^{-\theta s} + \int_0^s e^{\theta \tau} \delta(\tau) e^{-\theta \tau} d\tau \right) ds
\]

\[
\leq mm_0 e^{\theta s} |L^2(0, t)| \varphi(s) e^{-\theta s} |L^2(0, T)|
\]

\[
+ mT |e^{\theta s} |L^{r'}(0, t)| \delta|L^{r'}(0, T)| \varphi(s) e^{-\theta s} |L^2(0, T)|
\]

where \( \frac{1}{r} + \frac{1}{r'} + \frac{1}{2} = 1 \). It follows that

\[
\varphi(t) \leq me^{\theta t} \varphi(s) e^{-\theta s} |L^2(0, T)| \left( \frac{m_0}{\sqrt{2\theta}} + \frac{T |\delta|L^{r'}(0, T)}{(\theta r)^{1/r}} \right).
\]

Divide by \( e^{\theta t} \) and take the \( L^2 \)-norm to obtain

\[
|\varphi(s) e^{-\theta s} |L^2(0, T) \leq m \sqrt{T} |\varphi(s) e^{-\theta s} |L^2(0, T)| \left( \frac{m_0}{\sqrt{2\theta}} + \frac{T |\delta|L^{r'}(0, T)}{(\theta r)^{1/r}} \right).
\]

Clearly, if \( \theta \) is sufficiently large, this implies \( |\varphi(s) e^{-\theta s} |L^2(0, T) = 0 \). Thus \( \varphi = 0 \).

**Remark 3.3.** Let \( \Psi \) satisfy the following stronger compactness condition:

(\( \Psi 4 \)) If \( B \) is any bounded subset of \( L^q(0, T; E) \) for which there exists a function \( \nu \in L^q(0, T; \mathbb{R}_+) \) such that \( |f(t)|_E \leq \nu(t) \) a.e. on \([0, T] \), for all \( f \in B \), then \( \{ \Psi f \} \}_{f \in B} \) is relatively compact in \( L^p(0, T; F) \).

Then the conclusion of Theorem 4.1 is true without assumptions (\( \Psi 1 \)) and (\( \Phi 3 \)). Indeed, under assumption (\( \Psi 4 \)) the compactness of the set \( A \) satisfying (2.5) is immediate since \( \Psi(\overline{\Phi}(A)) \) is relatively compact in \( L^p(0, T; F) \).

Condition (\( \Psi 4 \)) has been required in Bader [3]. For a discussion on this condition, when \( \Psi \) is the mild solution operator for the initial value problem associated to an \( m \)-accretive map, see Vrabie [23].

**Example.** Let us consider the initial value problem for a functional-differential inclusion

\[
\begin{align*}
    u'(t) &\in (\Phi u)(t) \quad \text{a.e. on } [0, T] \\
    u(0) &= u_0
\end{align*}
\]

(3.21)

**Theorem 3.2.** Let \( E \) be a Banach space and let \( \Phi : C([0, T]; E) \rightarrow 2^{L^1(0, T; E)} \). Let assumptions (\( \Phi 1 \)) – (\( \Phi 3 \)) hold with \( p = \infty, q = 1, E = F, K = \).
$C([0,T]; E)$ and $\zeta$ given by (3.20). In addition, assume that there exists $a \in L^1(0,T; \mathbb{R}_+)$ and a non-decreasing function $b : \mathbb{R}_+ \to (0, \infty)$ such that

$$|f(t)|_E \leq a(t)b(|u(t)|_E)$$

a.e. on $[0,T]$, for all $u \in C([0,T]; E)$ and $f \in \Phi u$, and

$$\int_0^T a(s) \, ds < \int_{|u_0|_E}^{\infty} \frac{d\tau}{b(\tau)}.$$

Then problem (3.21) has a solution in $W^{1,1}(0,T; E)$.

**Proof.** Let $\Psi : L^1(0,T; E) \to C([0,T]; E)$ be defined by

$$(\Psi f)(t) = u_0 + \int_0^t f(s) \, ds.$$

We can easily see that $\Psi$ satisfies assumptions $(\Psi 1) - (\Psi 3)$ with $\eta(t, \varphi) = \int_0^t \varphi(s) \, ds$. Then recall Remark 3.2. On the other hand, a standard argument (see O'Regan and Precup [20: pp. 29] and Precup [22: pp. 74]) guarantees the existence of a number $R > 0$ with $|u(t)|_E < R$ for all $t \in [0,T]$ and any solution $u$ of $u \in \lambda \Psi \Phi u$, for $\lambda \in [0,1]$. Hence $\|u\| := \max_{t \in [0,T]} |u(t)|_E < R$ and so condition (L-S) holds with $U = \{u \in C([0,T]; E) : \|u\| < R\}$. Now the result follows from Theorem 3.1.

**References**


Received 13.02.2003