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Solvability of Two-Point Boundary Value Problems for Fourth-Order Nonlinear Differential Equations at Resonance

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Abstract. Under a resonance condition involving a two-point boundary value problem for a fourth-order nonlinear differential equation, we show its solvability.

Keywords: *Fourth-order differential equation, two-point boundary value problem, solvability of boundary value problem, resonance*

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1. Introduction

Let $f : [0,1] \times \mathbb{R}^4$ be a continuous function and $e \in L^1[0,1]$. We consider the fourth-order differential equation

$$x^{(4)}(t) = f\big(t, x(t), x'(t), x''(t), x'''(t)\big) + e(t) \qquad (0 < t < 1) \tag{1}$$

subject to the boundary value conditions

$$x'(0) = x'(1) = x'''(0) = x'''(1) = 0. \tag{2}$$

Boundary value problems of this form were used to understand the static equilibrium of an elastic beam supported by sliding clamps. We refer the

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reader to [11, 12] and the references therein. For example, Gupta [12] studied the solvability of the boundary value problem

$$\left.\begin{aligned} -y^{(4)} + g(t, y(t)) = e(t) \quad (0 < t < 1) \\ y'(0) = y'(\pi) = y'''(0) = y'''(\pi) = 0 \end{aligned}\right\}.$$

Since (2) implies that the linear operator $Lx = x^{(4)}$ defined in a suitable Banach space is not invertible, we call (2) a *resonance boundary value condition.* There are many other papers concerning the existence of solutions or positive solutions of fourth-order differential equations subjected to different kind of non-resonance boundary value conditions (see [1 - 6, 8, 10, 13, 14, 16] and the references therein).

To the best of our knowledge, the solvability of boundary value problem (1) - (2) has not been studied till now. The purpose of this paper is to establish an existence result for problem (1) - (2). Our method is based on the coincidence degree theory of Mawhin.

Now, we briefly recall some notations and an abstract existence result. Let X and Y be Banach spaces, $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero, $P : X \to X$ and $Q : Y \to Y$ be projectors such that

$$\begin{aligned} \operatorname{Im} P &= \operatorname{Ker} L \\ \operatorname{Ker} Q &= \operatorname{Im} L \\ X &= \operatorname{Ker} L + \operatorname{Ker} P \\ Y &= \operatorname{Im} L + \operatorname{Im} Q. \end{aligned}$$

It follows that the reduced operator

$$L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is invertible. We denote the inverse of that map by K_p.

If Ω is an open bounded subset of X and $\operatorname{dom} L \cap \Omega \neq \emptyset$, where $\emptyset$ denotes the empty set, the map $N : X \to Y$ will be called *L-compact* on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the product map $K_p(I - Q)N : \overline{\Omega} \to X$ is compact. The facts we use are [15: Theorem 2.4] and [7: Theorem IV.13].

Theorem 1. *Let L be a Fredholm operator of index zero and let N be L-compact on Ω. Assume that the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in \big[(\operatorname{dom} L / \operatorname{Ker} L) \cap \partial\Omega\big] \times (0, 1)$.*

(ii) *$Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$.*

(iii) *$\deg\big(\Lambda QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0\big) \neq 0$, where $\Lambda : \operatorname{Im} L \to \operatorname{Ker} L$ is some isomorphism.*

Then the equation $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

We use the classical spaces $C^3[0,1]$ and $L^1[0,1]$. For $x \in C^3[0,1]$, we use the norms $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$ and

$$\|x\| = \max\left\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty, \|x'''\|_\infty\right\}$$

and denote the norm in $L^1[0,1]$ by $\|x\|_1$. We also use the Sobolev space $W^{4,1}(0,1)$ defined by

$$W^{4,1} = \left\{x : [0,1] \to \mathbb{R} \,\middle|\, x, x', x'', x''' \text{ abs. cont.}, x^{(4)} \in L^1[0,1]\right\}$$

with its usual norm.

2. Main results

In this section, we shall prove the existence result for problem (1) - (2). Let $X = C^3[0,1]$ and $Y = L^1[0,1]$. Define L to be the linear operator from $\operatorname{dom} L \subset X$ to Y with

$$\operatorname{dom} L = \left\{x \in W^{4,1}(0,1) \,\middle|\, x'(0) = x'(1) = x'''(0) = x'''(1) = 0\right\}$$

and $(Lx)(t) = x^{(4)}(t)$ for $x \in \operatorname{dom} L \cap X$, and we define N to be the nonlinear operator from X to Y with

$$(Nx)(t) = f\big(t, x(t), x'(t), x''(t), x'''(t)\big) + e(t) \qquad (0 < t < 1)$$

for $x \in X$. Thus problem (1) - (2) can be written as $Lx = Nx$. We note that if $x \in \operatorname{dom} L$, then $\|x\| = \max\{\|x\|_\infty, \|x'''\|_\infty\}$, since $\|x'\|_\infty \le \|x''\|_\infty \le \|x'''\|_\infty$.

Lemma 1. *The following results hold:*

(i) $\operatorname{Ker} L = \big\{x \in X : x(t) = c \;\; (0 \le t \le 1)$ *for some* $c \in \mathbb{R}\big\}$.

(ii) $\operatorname{Im} L = \big\{y \in Y : \int_0^1 y(s)\, ds = 0\big\}$.

(iii) L *is a Fredholm operator of index zero.*

(iv) *If* Ω *is an open bounded subset such that* $\operatorname{dom} L \cap \Omega \ne \emptyset$, *then* N *is* L*-compact on* $\overline{\Omega}$.

Proof. (i): For $x \in \operatorname{Ker} L$ we have $x^4(t) = 0$, thus $x(t) = at^3 + bt^2 + ct + d$. On the other hand, $x'(0) = x'(1) = x'''(0) = x'''(1) = 0$ implies that $a = b = c = 0$. So $x(t) = d$ for $t \in [0,1]$. Again, if $x = d$, then $x \in \operatorname{Ker} L$. This completes the proof of assertion (i).

(ii): For $y \in \operatorname{Im} L$ there is $x \in \operatorname{dom} L$ such that $x^{(4)} = y$. So

$$x(t) = \int_0^t \frac{(t-s)^3}{6} y(s)\, ds + at^3 + bt^2 + ct + d.$$

Since $x'(0) = x'(1) = x'''(1) = x'''(0) = 0$, we get $c = a = 0$ and $\int_0^1 y(s)\, ds = 0$. Thus $y \in \left\{y \in Y : \int_0^1 y(s)\, ds = 0\right\}$. On the other hand, if $y \in Y$ and $\int_0^1 y(s)\, ds = 0$, let

$$x(t) = \int_0^t \frac{(t-s)^3}{6} y(s)\, ds - \frac{t^2}{2} \int_0^1 \frac{(1-s)^2}{2} y(s)\, ds.$$

Then $x \in X$ and $x'(0) = x'(1) = x'''(0) = x'''(1) = 0$. This implies $y \in \operatorname{Im} L$, so assertion (ii) is valid.

(iii): Define the projector $Q : Y \to Y$ by

$$Qy(t) = \int_0^1 y(s)\, ds \qquad (y \in Y).$$

It is easy to check that, for $y \in Y$, $y - Qy \in \operatorname{Im} L$. So $y = \operatorname{Im} L + R$, again $\operatorname{Im} L \cap R = \{0\}$, hence $Y = \operatorname{Im} L \oplus R$. Together with that $\operatorname{Im} L$ is closed, thus L is a Fredholm operator of index zero.

(iv) Let Ω be an open bounded subset in X such that $\Omega \cap \operatorname{dom} L \neq \Phi$. Define the projector $P : X \to X$ by $P(x) = x(0)$. Then the generalized inverse $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ of L can be written as

$$(K_p y)(t) = \int_0^t \frac{(t-s)^3}{6} y(s)\, ds - \frac{t^2}{2} \int_0^1 \frac{(1-s)^2}{2} y(s)\, ds. \tag{4}$$

In fact, for $y \in \operatorname{Im} L$ we have

$$(LK_p)y(t) = L\left(\int_0^t \frac{(t-s)^3}{6} y(s)\, ds - \frac{t^2}{2} \int_0^1 \frac{(1-s)^2}{2} y(s)\, ds\right) = y(t).$$

Further, for $x \in \operatorname{dom} L \cap \operatorname{Ker} P$ we have

$$\begin{aligned}
(K_p L x)(t) &= K_p(x^{(4)}(t)) \\
&= \int_0^t \frac{(t-s)^3}{6} x^{(4)}(s)\, ds - \frac{t^2}{2} \int_0^1 \frac{(1-s)^2) x^{(4)}(s)}{2}\, ds \\
&= \frac{t^3}{6} x'''(0) + \frac{t^2}{2} x''(0) + t x'(0) + x(t) - x(0) - \frac{t^2}{2} x''(0) \\
&= x(t).
\end{aligned}$$

This shows $K_p = (L|_{\mathrm{dom}\, L \cap \mathrm{Ker}\, P})^{-1}$. Furthermore, $X = \mathrm{Ker}\, L \bigoplus \mathrm{Ker}\, P$. In fact, for $x \in X$, $x(t) - x(0) \in \mathrm{Ker}\ P$, so $X = \mathrm{Ker}\ P + \mathrm{Ker}\, L$, and again $\mathrm{Ker}\, L \cap \mathrm{Ker}\, P = \{0\}$. Then $X = \mathrm{Ker}\, L \oplus \mathrm{Ker}\, P$. From (4) we find

$$\begin{aligned}
\|K_p y\|_\infty &\le \frac{1}{6}\|y\|_1 + \frac{1}{4}\|y\|_1 = \frac{5}{12}\|y\|_1 \\
\|(K_p y)'\|_\infty &\le \frac{1}{2}\|y\|_1 + \frac{1}{2}\|y\|_1 = \|y\|_1 \\
\|(K_p y)'''\|_\infty &= \left\| \int_0^t y(s)\, ds \right\|_\infty \le \|y\|_1.
\end{aligned}$$

Since $(K_p y)'(0) = (K_p y)'(1) = 0$, there is $\xi \in (0,1)$ such that $(K_p y)''(\xi) = 0$. Hence for $t \in (01)$ we have

$$\begin{aligned}
|(K_p y)''(t)| &= |(K_p y)''(t) - (K_p y)''(\xi)| \\
&= |(K_p y)'''(\eta)(t - \xi)| \\
&\le |(K_p y)'''(\eta)|
\end{aligned}$$

for $\eta \in (t, \xi)$ or $\eta \in (\xi, t)$. So

$$\|(K_p y)''\|_\infty \le \|(K_p y)'''\|_\infty \le \|y\|_1.$$

It follows that $\|K_p y\| \le \|y\|_1$ for $y \in Y$. It is easy to see that

$$(QNx)(t) = \int_0^1 \Big(f\big(s, x(s), x'(s), x''(s), x'''(s)\big) + e(s) \Big) ds$$

and

$$\begin{aligned}
&K_p(I - Q)Nx(t) \\
&\quad = \int_0^t \frac{(t-s)^3}{6} \Big(f\big(s, x(s), x'(s), x''(s), x'''(s)\ big) + e(s) \Big) ds \\
&\qquad - \frac{t^2}{2} \int_0^1 \frac{(1-s)^2}{2} \Big(f\big(s, x(s), x'(s), x''(s), x'''(s)\big) + e(s) \Big) ds \\
&\qquad - \Big(\frac{t^4}{24} + \frac{t^2}{12} \Big) \int_0^1 \Big(f\big(s, x(s), x'(s), x''(s), x'''(s)\big) + e(s) \Big) ds.
\end{aligned}$$

By using the Ascoli-Arzela theorem, we can prove that $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact. So N is L-compact on $\overline{\Omega}$ ∎

Theorem 2. *Let $f : [0,1] \times R^4 \to R$ be a continuous function. Assume the following:*

($\mathbf{A}_1$) *There exist functions $a, b, c, d, g : [0,1] \to \mathbb{R}$ and $r \in L^1[0,1]$ and a constant $\theta \in [0,1)$ such that*

$$|f(t,x,y,z,w)| \le a(t)|x| + b(t)|y| + c(t)|z| + d(t)|w| + g(t)|w|^\theta + r(t);$$

for all $t \in [0,1]$.

($\mathbf{A}_2$) *There exists a constant $M > 0$ such that if $|w| > M$, then*

$$|f(t,x,y,z,w)| > -\overline{\alpha}|x| + \overline{\beta}|w| - L_1$$

for all $x, y, z \in R$ and $t \in [0,1]$, where $\overline{\beta} > \overline{\alpha} > 0$ and $L_1 > 0$ are some constants.

($\mathbf{A}_3$) *There is a constant $M_1 > 0$ such that if $|x(t)| > M_1$ for all $t \in [0,1]$, then*

$$\int_0^1 \Big(f\big(s, x(s), x'(s), x''(s), x'''(s)\big) + e(s)\Big) ds \ne 0.$$

($\mathbf{A}_4$) $\lim_{|c|\to\infty} \frac{|f(t,c,0,0,0)|}{|c|} \in (0, +\infty)$.

($\mathbf{A}_5$) *There is a constant $M_2 > 0$ such that if $|c| > M_2$, then*

$$cf(t,c,0,0,0) \begin{cases} \le 0 \\ or \\ \ge 0 \end{cases} \qquad (0 \le t \le 1).$$

($\mathbf{A}_6$) $\|a\|_1 + \|b\|_1 + \|c\|_1 + \|d\|_1 < \frac{1}{2}\big(1 - \frac{\overline{\alpha}}{\overline{\beta}}\big)$.

Then for every $e \in L^1[0,1]$ problem (1) – (2) has at least one solution in $C^3[0,1]$.

Proof. Let

$$\Omega_1 = \Big\{x \in \operatorname{dom} L/\operatorname{Ker} L : \ Lx = \lambda Nx \ \text{ for some } \lambda \in (0,1)\Big\}.$$

If $x \in \Omega_1$, then $x \notin \operatorname{Ker} L, \lambda \ne 0$ and $Nx \in \operatorname{Im} L$, thus $QNx = 0$, i.e.

$$x^{(4)}(t) = \lambda f\big(t, x(t), x'(t), x''(t), x'''(t)\big) + e(t) \quad (t \in [0,1])$$

$$x'(0) = x'(1) = x'''(0) = x'''(1) = 0$$

$$\int_0^1 \Big(f\big(s, x(s), x'(s), x''(s), x'''(s)\big) + e(s)\Big) ds = 0.$$

So there is $t_1 \in (0,1)$ such that

$$f\big(t_1, x(t_1), x'(t_1), x''(t_1), x'''(t_1)\big) = -\int_0^1 e(s)\,ds.$$

This yields

$$\big|f\big(t_1, x(t_1), x'(t_1), x''(t_1), x'''(t_1)\big)\big| \le \|e\|_1.$$

Again, if $x \in \operatorname{dom} L$, then $(I-P)x \in \operatorname{dom} L \cap \operatorname{Ker} P$ and $LPx = 0$. Thus, from Lemma 1,

$$\|(I-P)x\| = \|K_p L(I-P)x\| \le \|L(I-P)x\|_1 = \|Lx\|_1 \le \|Nx\|_1.$$

We consider two cases.

Case 1: $|x'''(t^*)| \le M$ for some $t^* \in [0,1]$. In this case we have

$$|x'''(t)| = |x'''(t^*)| + \left|\int_t^{t^*} x^{(4)}(s)\,ds\right| \le M + \|Lx\|_1 \le M + \|Nx\|_1.$$

Since $x'(0) = x'(1) = x'''(0) = x'''(1) = 0$, there is $\xi \in (0,1)$ such that $x''(\xi) = 0$, thus

$$|x''(t)| = |x''(t) - x''(\xi)| = |x'''(\eta)(t-\xi)| \le M + \|Nx\|_1.$$

Also, there is $\eta_1 \in [0,1]$ such that

$$|x'(t)| = |x'(t) - x'(0)| = |x''(\eta_1)t| \le M + \|Nx\|_1.$$

We claim that there is a $t^{**} \in (0,1)$ such that $|x(t^{**})| \le M_1$. Otherwise, if $|x(t)| > M_1$ for all $t \in [0,1]$, condition (A_3) implies

$$\int_0^1 \Big(f\big(s, x(s), x'(s), x''(s), x'''(s)\big) + e(s)\Big)ds \ne 0.$$

On the other hand, since $Lx \in \operatorname{Im} L$, we have

$$\int_0^1 \Big(f\big(s, x(s), x'(s), x''(s), x'''(s)\big) + e(s)\Big)ds = 0,$$

which is a contradiction. Thus

$$|x(0)| = |x(t^{**})| + \left|\int_0^{t^{**}} x'(s)\,ds\right| \le M_1 + M + \|Nx\|_1.$$

Hence

$$\|Px\| = |x(0)| \le \frac{\bar{\alpha}}{\bar{\beta}}\|x\|_\infty + \|Nx\|_1 + c_1$$

where

$$c_1 = \max\Big\{M_1 + M, M_1 + \frac{1}{\bar{\beta}}(L_1 + \|e\|_1)\Big\}.$$

Thus we get

$$\|x\| \le \|Px\| + \|(I-P)x\| \le \frac{\bar{\alpha}}{\bar{\beta}}\|x\|_\infty + 2\|Nx\|_1 + c_1.$$

From Property (A_1) we get

$$\begin{aligned}\|x\| &\le \frac{\bar{\alpha}}{\bar{\beta}}\|x\|_\infty + 2\|a\|_1\|x\|_\infty + 2\|b\|_1\|x'\|_\infty + 2\|c\|_1\|x''\|_\infty \\ &\quad + 2\|d\|_1\|x'''\|_\infty + 2\|g\|_1\|x'''\|_\infty^\theta + 2\|r\|_1 + 2\|e\|_1 + c_1 \\ &= \Big(2\|a\|_1 + \frac{\bar{\alpha}}{\bar{\beta}}\Big)\|x\|_\infty + 2\|b\|_1\|x'\|_\infty + 2\|c\|_1\|x''\|_\infty \\ &\quad + 2\|d\|_1\|x'''\|_\infty + 2\|g\|_1\|x'''\|_\infty^\theta + 2\|r\|_1 + 2\|e\|_1 + c_1 \\ &\le \Big(2\|a\|_1 + \frac{\bar{\alpha}}{\bar{\beta}}\Big)\|x\|_\infty + \big(2\|b\|_1 + 2\|c\|_1 + 2\|d\|_1\big)\|x'''\|_\infty \\ &\quad + 2\|g\|_1\|x'''\|_\infty^\theta + 2\|r\|_1 + 2\|e\|_1 + c_1,\end{aligned}$$

i.e.

$$\begin{aligned}\|x\| &\le \Big(2\|a\|_1 + \frac{\bar{\alpha}}{\bar{\beta}}\Big)\|x\|_\infty + \big(2\|b\|_1 + 2\|c\|_1 + 2\|d\|_1|\big)|x'''\|_\infty \\ &\quad + 2\|g\|_1\|x'''\|_\infty^\theta + 2\|r\|_1 + 2\|e\|_1 + c_1.\end{aligned}$$

It is easy to check that $\|x'\|_\infty \le \|x''\|_\infty \le \|x'''\|_\infty$. Together with $\|x\|_\infty \le \|x\|$, it follows from the above inequality that

$$\begin{aligned}\|x\|_\infty &\le \frac{1}{1 - 2\|a\|_1 - \frac{\bar{\alpha}}{\bar{\beta}}}\Big[2\|b\|_1\|x'\|_\infty + 2\|c\|_1\|x''\|_\infty \\ &\quad + 2\|d\|_1\|x'''\|_\infty + 2\|g\|_1\|x'''\|_\infty^\theta + 2\|r\|_1 + 2\|e\|_1 + c_1\Big] \\ &\le \frac{1}{1 - 2\|a\|_1 - \frac{\bar{\alpha}}{\bar{\beta}}}\Big[\big(2\|b\|_1 + 2\|c\|_1 + 2\|d\|_1\big)\|x'''\|_\infty \\ &\quad + 2\|g\|_1\|x'''\|_\infty^\theta + 2\|r\|_1 + 2\|e\|_1 + c_1\Big].\end{aligned} \tag{5}$$

Case 2. $|x'''(t)| > M$ for all $t \in [0,1]$. In this case from property (A_2) we obtain

$$\begin{aligned}|x'''(t_1)| &\le \frac{\bar{\alpha}}{\bar{\beta}}|x(t_1)| + \frac{L_1}{\bar{\beta}} + \frac{1}{\bar{\beta}}\big|f\big(t_1, x(t_1), x'(t_1), x''(t_1), x'''(t_1)\big)\big| \\ &\le \frac{\bar{\alpha}}{\bar{\beta}}\|x\|_\infty + \frac{1}{\bar{\beta}}(L_1 + \|e\|_1)\end{aligned}$$

so that

$$\begin{aligned}|x'''(t)| &\le |x'''(t_1)| + \left|\int_{t_1}^{t} x^{(4)}(s)\,ds\right| \\ &\le \frac{\bar{\alpha}}{\bar{\beta}}\|x\|_\infty + \frac{1}{\bar{\beta}}(L_1 + \|e\|_1) + \|Nx\|_1.\end{aligned}$$

Thus similarly to the above discussion, one has a $\xi \in (0,1)$ such that $x''(\xi) = 0$ and there is an $\eta \in (0,1)$ such that

$$\begin{aligned}|x''(t)| &= |x''(t) - x''(\xi)| \\ &= |x'''(\eta)(t-\eta)| \\ &\le \frac{\bar{\alpha}}{\bar{\beta}}\|x\|_\infty + \frac{1}{\bar{\beta}}(L_1 + \|e\|_1) + \|Nx\|_1.\end{aligned}$$

So we get

$$\begin{aligned}|x'(t)| &= |x'(t) - x'(0)| \\ &\le |x''(\xi)| \\ &\le \frac{\bar{\alpha}}{\bar{\beta}}\|x\|_\infty + \frac{1}{\bar{\beta}}(L_1 + \|e\|_1) + \|Nx\|_1.\end{aligned}$$

From property (A_3), there is a $t^{**} \in (0,1)$ such that $|x(t^{**})| \le M_1$. Then, together with (5),

$$\begin{aligned}\|Px\| &= |x(0)| \\ &= \left|x(t^{**}) - \int_0^{t^{**}} x'(t)\,dt\right| \\ &\le M_1 + \frac{\bar{\alpha}}{\bar{\beta}}\|x\|_\infty + \frac{1}{\bar{\beta}}(L_1 + \|e\|_1) + \|Nx\|_1 \\ &\le \frac{\bar{\alpha}}{\bar{\beta}}\|x\|_\infty + \|Nx\|_1 + c_1.\end{aligned}$$

Thus

$$\|x\| \le \|Px\| + \|(I-P)x\| \le \frac{\bar{\alpha}}{\bar{\beta}}\|x\|_\infty + 2\|Nx\|_1 + c_1.$$

So property (A_1) implies

$$\begin{aligned}\|x'''\|_\infty &\le \|x\| \\ &\le \frac{2\|a\|_1 + \frac{\bar{\alpha}}{\bar{\beta}}}{1 - 2\|a\|_1 - \frac{\bar{\alpha}}{\bar{\beta}}}\Big[\big(2\|b\|_1 + 2\|c\|_1 + 2\|d\|_1\big)\|x'''\|_\infty \\ &\quad + 2\|g\|_1\|x'''\|_\infty^\theta + 2\|r\|_1 + 2\|e\|_1 + c_1\Big] \\ &\quad + \Big[\big(2\|b\|_1 + 2\|c\|_1 + 2\|d\|_1\big)\|x'''\|_\infty \\ &\quad + 2\|g\|_1\|x'''\|_\infty^\theta + 2\|r\|_1 + 2\|e\|_1 + c_1\Big]\end{aligned}$$

$$= \frac{1}{1-2\|a\|_1-\frac{\bar{\alpha}}{\bar{\beta}}}\Big[\big(2\|b\|_1+2\|c\|_1+2\|d\|_1\big)\|x'''\|_\infty + 2\|g\|_1\|x'''\|_\infty^\theta+2\|r\|_1+2\|e\|_1+c_1\Big].$$

We get (5). From (5) it follows that

$$\|x'''\|_\infty \le \frac{2\|g\|_1\|x'''\|_\infty^\theta + c_1 + 2\|r\|_1 + 2\|e\|_1}{1-2\|a\|_1+2\|b\|_1+2\|c\|_1+2\|d\|_1-\frac{\bar{\alpha}}{\bar{\beta}}}\Big(1-2\|a\|_1-\frac{\bar{\alpha}}{\bar{\beta}}\Big).$$

Since $\theta \in [0,1]$, there is $M_1^* > 0$ such that

$$\|x'''\|_\infty \le M_1^*.$$

Again, it is easy to prove that

$$\left.\begin{array}{r}\|x''\|_\infty \le \|x'''\|_\infty \\ \|x'\|_\infty \le \|x''\|_\infty \le \|x'''\|_\infty\end{array}\right\} \le M_1^*.$$

From property (A_3) we claim that there is $t^{**} \in (0,1)$ such that $|x(t^{**})| \le M_1$. Thus

$$|x(t)| \le \left|x(t^{**}) - \int_t^{t^{**}} x'(s)\,ds\right| \le M_1 + \|x'\|_\infty.$$

Hence there is $M_2^* > 0$ such that $\|x\|_\infty \le M_2^*$. Hence

$$\|x\| \le \max\big\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty, \|x'''\|_\infty\big\} \le \max\{M_1^*, M_2^*\}.$$

Thus Ω_1 is bounded. Let

$$\Omega_2 = \big\{x \in \operatorname{Ker} L:\ Nx \in \operatorname{Im} L\big\}.$$

For $x \in \Omega_2$, $x \in \operatorname{Ker} L$ and $QNx = 0$, thus

$$\int_0^1 \big(f(s,c,0,0,0)+e(s)\big)ds = 0, \quad \text{i.e. } \int_0^1 f(s,c,0,0,0)\,ds = -\int_0^1 e(s)\,ds.$$

Thus there is $t_0 \in (0,1)$ such that

$$f(t_0,c,0,0,0) = -\int_0^1 e(s)\,ds, \quad \text{so} \ \ |f(t_0,c,0,0,0)| \le \|e\|_1.$$

From property (A_4) we see that there is $M^* > 0$ such that $|c| \le M^*$. Thus Ω_2 is bounded. Next, according condition (A_5), we have the following two cases.

Case 1. Suppose for any $c \in R$, if $|c| > M_2$, then $cf(t,c,0,0,0) \le 0$ for $t \in [0,1]$. Let

$$\Omega_3 = \Big\{x \in \operatorname{Ker} L :\, -\lambda x + (1-\lambda)QNx = 0, \lambda \in [0,1]\Big\}.$$

Now, similar to the proof of [6: Lemma 2.12], we prove that Ω_3 is bounded. Suppose $x_n(t) = c_n \in \Omega_3$ and $|c_n| \to \infty$ as $n \to \infty$. Without loss of generality, suppose that $c_n > M_2$ for all n. Then there is $\lambda_n \in [0,1]$ such that

$$\lambda_n c_n = (1-\lambda_n)QN(c_n), \qquad \text{or} \;\; \lambda_n = (1-\lambda_n)\frac{QN(c_n)}{c_n}. \tag{6}$$

Without loss of generality, suppose $\lambda_n \to \lambda_0$ as $n \to \infty$. Then

$$\begin{aligned}\left|\frac{QN(c_n)}{c_n}\right| &= \frac{1}{|c_n|}\left|\int_0^1 \big(f(s,c_n,0,0,0)+e(s)\big)ds\right| \\ &\le \frac{1}{|c_n|}\big[\|e\|_1 + \|a\|_1|c_n| + \|r\|_1\big] \\ &= \|a\|_1 + \frac{\|e\|_1+\|r\|_1}{|c_n|}.\end{aligned}$$

Thus $\frac{|QN(c_n)}{c_n|}$ is bounded. So $\lambda_n \to \lambda_0 \ne 1$ by (6). Thus, for sufficiently large n, $\lambda_n \ne 1$. Then

$$\frac{\lambda_n}{1-\lambda_n} = \frac{1}{c_n}\left(\int_0^1 \big(f(s,c_n,0,0,0)+e(s)\big)ds\right).$$

From property (A_4), for sufficiently large n, $|f(t,c_n,0,0,0)| \ge \alpha|c_n|$ for some $\alpha > 0$. Then property (A_5) implies $f(t,c_n,0,0,0) < -\alpha c_n$. Thus, by Fatou's Lemma,

$$\begin{aligned}&\limsup\left(\frac{1}{c_n}\int_0^1 f(s,c_n,0,0,0)\,ds + \frac{1}{c_n}\int_0^1 e(s)\,ds\right) \\ &\quad\le \limsup\frac{1}{c_n}\int_0^1 f(s,c_n,0,0,0)\,ds \\ &\quad\le \int_0^1 \limsup\frac{f(s,c_n,0,0,0)}{c_n}\,ds \\ &\quad\le -\alpha \\ &\quad< 0.\end{aligned}$$

This contradicts $\frac{\lambda_n}{1-\lambda_n} \ge 0$. Then Ω_3 is bounded.

Case 2. Suppose $|c| > M_2$. Then $cf(t, c, 0, 0, 0) \geq 0$ for $t \in [0, 1]$. Indeed, set

$$\Omega_3 = \Big\{x \in \operatorname{Ker} L : \lambda x + (1-\lambda)QNx = 0 \text{ for all } \lambda \in (0,1)\Big\}.$$

Like in the above argument, we can prove that Ω_3 is bounded. In the following, we shall prove that all conditions of Theorem 1 are satisfied. Let Ω be a bounded open subset of X such that

$$\sqcup_{i=1}^{3} \overline{\Omega}_i \subset \Omega.$$

By Lemma 1, L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By the above argument and the definition of Ω, we have:

(i) $Lx \neq \lambda Nx$ for $(\lambda, x) \in [(\operatorname{dom} L/\operatorname{Ker} L) \cap \partial\Omega] \times (0, 1)$

(ii) $Nx \notin \operatorname{Im} L$ for $x \in \operatorname{Ker} L \cap \partial\Omega$.

At last, we prove that condition (iii) of Theorem M is satisfied. Let

$$H(x, \lambda) = \pm\lambda x + (1-\lambda)QNx.$$

By the definition of Ω, we see that $H(x, \lambda) \neq 0$ for $x \in \partial\Omega \cap \operatorname{Ker} L$. Thus, by the homotopy property of degree, we have

$$\begin{aligned}
\deg\big(QN_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0\big) &= \deg\big(H(\cdot, 0), \Omega \cap \operatorname{Ker} L, 0\big) \\
&= \deg\big(H(\cdot, 1), \Omega \cap \operatorname{Ker} L, 0\big) \\
&= \deg\big(\pm \lambda I, \Omega \cap \operatorname{Ker} L, 0\big) \\
&\neq 0.
\end{aligned}$$

Thus by Theorem 1, the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$. So problem (1) - (2) has at least one solution ∎

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