Solvability of
Two-Point Boundary Value Problems
for Fourth-Order
Nonlinear Differential Equations at Resonance

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Abstract. Under a resonance condition involving a two-point boundary value problem for a fourth-order nonlinear differential equation, we show its solvability.

Keywords: Fourth-order differential equation, two-point boundary value problem, solvability of boundary value problem, resonance

AMS subject classification: Primary 34K20, secondary 92D25

1. Introduction

Let $f : [0, 1] \times \mathbb{R}^4$ be a continuous function and $e \in L^1[0, 1]$. We consider the fourth-order differential equation

$$x^{(4)}(t) = f(t, x(t), x'(t), x''(t), x'''(t)) + e(t) \quad (0 < t < 1)$$

subject to the boundary value conditions

$$x'(0) = x'(1) = x'''(0) = x'''(1) = 0.$$ 

Boundary value problems of this form were used to understand the static equilibrium of an elastic beam supported by sliding clamps. We refer the
reader to [11, 12] and the references therein. For example, Gupta [12] studied the solvability of the boundary value problem
\[
\begin{align*}
- y^{(4)} + g(t, y(t)) &= e(t) \quad (0 < t < 1) \\
y'(0) &= y'(\pi) = y'''(0) = y'''(\pi) = 0
\end{align*}
\]
Since (2) implies that the linear operator \( Lx = x^{(4)} \) defined in a suitable Banach space is not invertible, we call (2) a resonance boundary value condition.

There are many other papers concerning the existence of solutions or positive solutions of fourth-order differential equations subjected to different kind of non-resonance boundary value conditions (see [1 - 6, 8, 10, 13, 14, 16] and the references therein).

To the best of our knowledge, the solvability of boundary value problem (1) - (2) has not been studied till now. The purpose of this paper is to establish an existence result for problem (1) - (2). Our method is based on the coincidence degree theory of Mawhin.

Now, we briefly recall some notations and an abstract existence result. Let \( X \) and \( Y \) be Banach spaces, \( L : \text{dom} \ L \subset X \to Y \) be a Fredholm operator of index zero, \( P : X \to X \) and \( Q : Y \to Y \) be projectors such that
\[
\begin{align*}
\text{Im} \ P &= \text{Ker} \ L \\
\text{Ker} \ Q &= \text{Im} \ L \\
X &= \text{Ker} \ L + \text{Ker} \ P \\
Y &= \text{Im} \ L + \text{Im} \ Q.
\end{align*}
\]
It follows that the reduced operator
\[ L|_{\text{dom} \ L \cap \text{Ker} \ P} : \text{dom} \ L \cap \text{Ker} \ P \to \text{Im} \ L \]
is invertible. We denote the inverse of that map by \( K_p \).

If \( \Omega \) is an open bounded subset of \( X \) and \( \text{dom} \ L \cap \Omega \neq \emptyset \), where \( \emptyset \) denotes the empty set, the map \( N : X \to Y \) will be called \( L \)-compact on \( \overline{\Omega} \) if \( QN(\overline{\Omega}) \) is bounded and the product map \( K_p(I - Q)N : \overline{\Omega} \to X \) is compact. The facts we use are [15: Theorem 2.4] and [7: Theorem IV.13].

**Theorem 1.** Let \( L \) be a Fredholm operator of index zero and let \( N \) be \( L \)-compact on \( \Omega \). Assume that the following conditions are satisfied:

(i) \( Lx \neq \lambda Nx \) for every \( (x, \lambda) \in [(\text{dom} \ L/\text{Ker} \ L) \cap \partial \Omega] \times (0, 1) \).

(ii) \( Nx \notin \text{Im} \ L \) for every \( x \in \text{Ker} \ L \cap \partial \Omega \).

(iii) \( \deg(\Lambda QN|_{\text{Ker} \ L}, \Omega \cap \text{Ker} \ L, 0) \neq 0 \), where \( \Lambda : \text{Im} \ L \to \text{Ker} \ L \) is some isomorphism.
Then the equation \( Lx = Nx \) has at least one solution in \( \text{dom } L \cap \Omega \).

We use the classical spaces \( C^3[0,1] \) and \( L^1[0,1] \). For \( x \in C^3[0,1] \), we use the norms \( \|x\|_\infty = \max_{t \in [0,1]} |x(t)| \) and

\[
\|x\| = \max \{ \|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty, \|x'''\|_\infty \}
\]

and denote the norm in \( L^1[0,1] \) by \( \|x\|_1 \). We also use the Sobolev space \( W^{4,1}(0,1) \) defined by

\[
W^{4,1} = \left\{ x : [0,1] \to \mathbb{R} \mid x, x', x'', x''' \text{ abs. cont.}, x^{(4)} \in L^1[0,1] \right\}
\]

with its usual norm.

2. Main results

In this section, we shall prove the existence result for problem (1) - (2). Let \( X = C^3[0,1] \) and \( Y = L^1[0,1] \). Define \( L \) to be the linear operator from \( \text{dom } L \subset X \) to \( Y \) with

\[
\text{dom } L = \left\{ x \in W^{4,1}(0,1) \mid x'(0) = x'(1) = x''(0) = x'''(1) = 0 \right\}
\]

and \( (Lx)(t) = x^{(4)}(t) \) for \( x \in \text{dom } L \cap X \), and we define \( N \) to be the nonlinear operator from \( X \) to \( Y \) with

\[
(Nx)(t) = f(t, x(t), x'(t), x''(t), x'''(t)) + e(t) \quad (0 < t < 1)
\]

for \( x \in X \). Thus problem (1) - (2) can be written as \( Lx = Nx \). We note that if \( x \in \text{dom } L \), then \( \|x\| = \max\{\|x\|_\infty, \|x''\|_\infty\} \), since \( \|x'\|_\infty \leq \|x''\|_\infty \leq \|x'''\|_\infty \).

**Lemma 1.** The following results hold:

(i) \( \ker L = \{ x \in X : x(t) = c \text{ for } 0 \leq t \leq 1 \text{ for some } c \in \mathbb{R} \} \).

(ii) \( \text{Im } L = \{ y \in Y : \int_0^1 y(s) \, ds = 0 \} \).

(iii) \( L \) is a Fredholm operator of index zero.

(iv) If \( \Omega \) is an open bounded subset such that \( \text{dom } L \cap \Omega \neq \emptyset \), then \( N \) is \( L \)-compact on \( \Omega \).

**Proof.** (i): For \( x \in \ker L \) we have \( x^{(4)}(t) = 0 \), thus \( x(t) = at^3 + bt^2 + ct + d \). On the other hand, \( x'(0) = x'(1) = x''(0) = x'''(1) = 0 \) implies that \( a = b = c = 0 \). So \( x(t) = d \) for \( t \in [0,1] \). Again, if \( x = d \), then \( x \in \ker L \). This completes the proof of assertion (i).
(ii): For \( y \in \operatorname{Im} L \) there is \( x \in \operatorname{dom} L \) such that \( x^{(4)} = y \). So
\[
x(t) = \int_0^t \frac{(t-s)^3}{6} y(s) \, ds + at^3 + bt^2 + ct + d.
\]
Since \( x'(0) = x'(1) = x''(1) = x'''(0) = 0 \), we get \( c = a = 0 \) and \( \int_0^1 y(s) \, ds = 0 \). Thus \( y \in \{ y \in Y : \int_0^1 y(s) \, ds = 0 \} \). On the other hand, if \( y \in Y \) and \( \int_0^1 y(s) \, ds = 0 \), let
\[
x(t) = \int_0^t \frac{(t-s)^3}{6} y(s) \, ds - \frac{t^2}{2} \int_0^1 \frac{(1-s)^2}{2} y(s) \, ds.
\]
Then \( x \in X \) and \( x'(0) = x'(1) = x''(0) = x'''(1) = 0 \). This implies \( y \in \operatorname{Im} L \), so assertion (ii) is valid.

(iii): Define the projector \( Q : Y \to Y \) by
\[
Qy(t) = \int_0^1 y(s) \, ds \quad (y \in Y).
\]
It is easy to check that, for \( y \in Y \), \( y - Qy \in \operatorname{Im} L \). So \( y = \operatorname{Im} L + R \), again \( \operatorname{Im} L \cap R = \{0\} \), hence \( Y = \operatorname{Im} L \oplus R \). Together with that \( \operatorname{Im} L \) is closed, thus \( L \) is a Fredholm operator of index zero.

(iv) Let \( \Omega \) be an open bounded subset in \( X \) such that \( \Omega \cap \operatorname{dom} L \neq \emptyset \). Define the projector \( P : X \to X \) by \( P(x) = x(0) \). Then the generalized inverse \( K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P \) of \( L \) can be written as
\[
(K_p y)(t) = \int_0^t \frac{(t-s)^3}{6} y(s) \, ds - \frac{t^2}{2} \int_0^1 \frac{(1-s)^2}{2} y(s) \, ds.
\]
In fact, for \( y \in \operatorname{Im} L \) we have
\[
(LK_p) y(t) = L \left( \int_0^t \frac{(t-s)^3}{6} y(s) \, ds - \frac{t^2}{2} \int_0^1 \frac{(1-s)^2}{2} y(s) \, ds \right) = y(t).
\]
Further, for \( x \in \operatorname{dom} L \cap \operatorname{Ker} P \) we have
\[
(K_p L x)(t) = K_p(x^{(4)}(t))
= \int_0^t \frac{(t-s)^3}{6} x^{(4)}(s) \, ds - \frac{t^2}{2} \int_0^1 \frac{(1-s)^2}{2} x^{(4)}(s) \, ds
= \frac{t^3}{6} x'''(0) + \frac{t^2}{2} x''(0) + tx'(0) + x(t) - x(0) - \frac{t^2}{2} x''(0)
= x(t).
\]
This shows $K_p = (L_{dom \cap Ker})^{-1}$. Furthermore, $X = Ker L \oplus Ker P$. In fact, for $x \in X$, $x(t) - x(0) \in Ker P$, so $X = Ker P + Ker L$, and again $Ker L \cap Ker P = \{0\}$. Then $X = Ker L \oplus Ker P$. From (4) we find

$$\|K_p y\|_\infty \leq \frac{1}{6} \|y\|_1 + \frac{1}{4} \|y\|_1 = \frac{5}{12} \|y\|_1$$

$$\|(K_p y)'\|_\infty \leq \frac{1}{2} \|y\|_1 + \frac{1}{2} \|y\|_1 = \|y\|_1$$

$$\|(K_p y)''\|_\infty = \left\| \int_0^t y(s) \, ds \right\|_\infty \leq \|y\|_1.$$ 

Since $(K_p y)'(0) = (K_p y)'(1) = 0$, there is $\xi \in (0, 1)$ such that $(K_p y)''(\xi) = 0$. Hence for $t \in (01)$ we have

$$|(K_p y)''(t)| = |(K_p y)''(t) - (K_p y)''(\xi)|$$

$$= |(K_p y)'''(\eta)(t - \xi)|$$

$$\leq |(K_p y)'''(\eta)|$$

for $\eta \in (t, \xi)$ or $\eta \in (\xi, t)$. So

$$\|(K_p y)''\|_\infty \leq \|(K_p y)'''\|_\infty \leq \|y\|_1.$$ 

It follows that $\|K_p y\| \leq \|y\|_1$ for $y \in Y$. It is easy to see that

$$(QN x)(t) = \int_0^1 \left( f(s, x(s), x'(s), x''(s), x'''(s)) + e(s) \right) \, ds$$

and

$$K_p(I - Q)Nx(t)$$

$$= \int_0^t \frac{(t - s)^3}{6} \left( f(s, x(s), x'(s), x''(s), x'''(s)) \bigg| s \right) \, ds$$

$$- \frac{t^2}{2} \int_0^t \frac{(1 - s)^2}{2} \left( f(s, x(s), x'(s), x''(s), x'''(s)) + e(s) \right) \, ds$$

$$- \left( \frac{t^4}{24} + \frac{t^2}{12} \right) \int_0^1 \left( f(s, x(s), x'(s), x''(s), x'''(s)) + e(s) \right) \, ds.$$ 

By using the Ascoli-Arzela theorem, we can prove that $QN(\bar{\Omega})$ is bounded and $K_p(I - Q)N : \bar{\Omega} \to X$ is compact. So $N$ is $L$-compact on $\Omega$. □
Theorem 2. Let \( f : [0, 1] \times \mathbb{R}^4 \to \mathbb{R} \) be a continuous function. Assume the following:

(A1) There exist functions \( a, b, c, d, g : [0, 1] \to \mathbb{R} \) and \( r \in L^1[0, 1] \) and a constant \( \theta \in [0, 1) \) such that

\[
|f(t, x, y, z, w)| \leq a(t)|x| + b(t)|y| + c(t)|z| + d(t)|w| + g(t)|w|^\theta + r(t);
\]

for all \( t \in [0, 1] \).

(A2) There exists a constant \( M > 0 \) such that if \( |w| > M \), then

\[
|f(t, x, y, z, w)| > -\alpha|x| + \beta|w| - L_1
\]

for all \( x, y, z \in \mathbb{R} \) and \( t \in [0, 1] \), where \( \beta > \alpha > 0 \) and \( L_1 > 0 \) are some constants.

(A3) There is a constant \( M_1 > 0 \) such that if \( |x(t)| > M_1 \) for all \( t \in [0, 1] \), then

\[
\int_0^1 \left( f(s, x(s), x'(s), x''(s), x'''(s)) + e(s) \right) ds \neq 0.
\]

(A4) \( \lim_{|c| \to \infty} \frac{|f(t, c, 0, 0, 0)|}{|c|} \in (0, +\infty) \).

(A5) There is a constant \( M_2 > 0 \) such that if \( |c| > M_2 \), then

\[
cf(t, c, 0, 0, 0) \begin{cases} 
\leq 0 & (0 \leq t \leq 1) \\
g\geq 0 & (0 \leq t \leq 1).
\end{cases}
\]

(A6) \( \|a\|_1 + \|b\|_1 + \|c\|_1 + \|d\|_1 < \frac{1}{2} (1 - \frac{\alpha}{\beta}) \).

Then for every \( e \in L^1[0, 1] \) problem (1) – (2) has at least one solution in \( C^3[0, 1] \).

Proof. Let

\[
\Omega_1 = \left\{ x \in \text{dom } L/\text{Ker } L : Lx = \lambda Nx \text{ for some } \lambda \in (0, 1) \right\}.
\]

If \( x \in \Omega_1 \), then \( x \notin \text{Ker } L, \lambda \neq 0 \) and \( Nx \in \text{Im } L \), thus \( QNx = 0 \), i.e.

\[
x^{(4)}(t) = \lambda f \left( t, x(t), x'(t), x''(t), x'''(t) \right) + e(t) \ (t \in [0, 1])
\]

\[
x'(0) = x'(1) = x'''(0) = x'''(1) = 0
\]

\[
\int_0^1 \left( f(s, x(s), x'(s), x''(s), x'''(s)) + e(s) \right) ds = 0.
\]
So there is $t_1 \in (0, 1)$ such that
\[ f(t_1, x(t_1), x'(t_1), x''(t_1), x'''(t_1)) = - \int_0^1 e(s) \, ds. \]
This yields
\[ |f(t_1, x(t_1), x'(t_1), x''(t_1), x'''(t_1))| \leq \|e\|_1. \]
Again, if $x \in \text{dom } L$, then $(I - P)x \in \text{dom } L \cap \text{Ker } P$ and $LPx = 0$. Thus, from Lemma 1,
\[ \|(I - P)x\| = \|K_P(I - P)x\| \leq \|L(I - P)x\|_1 = \|Lx\|_1 \leq \|Nx\|_1. \]
We consider two cases.

Case 1: $|x'''(t^*)| \leq M$ for some $t^* \in [0, 1]$. In this case we have
\[ |x'''(t)| = |x'''(t^*)| + \left| \int_t^{t^*} x^{(4)}(s) \, ds \right| \leq M + \|Lx\|_1 \leq M + \|Nx\|_1. \]
Since $x'(0) = x'(1) = x'''(0) = x'''(1) = 0$, there is $\xi \in (0, 1)$ such that $x''(\xi) = 0$, thus
\[ |x''(t)| = |x''(t) - x''(\xi)| = |x'''(\eta)(t - \xi)| \leq M + \|Nx\|_1. \]
Also, there is $\eta_1 \in [0, 1]$ such that
\[ |x'(t)| = |x'(t) - x'(0)| = |x''(\eta_1)t| \leq M + \|Nx\|_1. \]
We claim that there is a $t^{**} \in (0, 1)$ such that $|x(t^{**})| \leq M_1$. Otherwise, if $|x(t)| > M_1$ for all $t \in [0, 1]$, condition (A3) implies
\[ \int_0^1 \left( f(s, x(s), x'(s), x''(s), x'''(s)) + e(s) \right) ds \neq 0. \]
On the other hand, since $Lx \in \text{Im } L$, we have
\[ \int_0^1 \left( f(s, x(s), x'(s), x''(s), x'''(s)) + e(s) \right) ds = 0, \]
which is a contradiction. Thus
\[ |x(0)| = |x(t^{**})| + \left| \int_0^{t^{**}} x'(s) \, ds \right| \leq M_1 + M + \|Nx\|_1. \]
Hence
\[ \|Px\| = |x(0)| \leq \frac{\bar{a}}{\beta} \|x\|_{\infty} + \|Nx\|_{1} + c_1 \]

where
\[ c_1 = \max \left\{ M_1 + M, M_1 + \frac{1}{\beta} (L_1 + \|e\|_1) \right\}. \]

Thus we get
\[ \|x\| \leq \|Px\| + \|(I - P)x\| \leq \frac{\alpha}{\beta} \|x\|_{\infty} + 2 \|Nx\|_{1} + c_1. \]

From Property (A_1) we get
\[ \|x\| \leq \frac{\alpha}{\beta} \|x\|_{\infty} + 2 |a|_1 \|x\|_{\infty} + 2 |b|_1 \|x'\|_{\infty} + 2 |c|_1 \|x''\|_{\infty} \]
\[ + 2 |d|_1 \|x'''\|_{\infty} + 2 |g|_1 \|x''''\|_{\infty} + 2 |r|_1 + 2 |e|_1 + c_1 \]
\[ = \left( 2 |a|_1 + \frac{\alpha}{\beta} \right) \|x\|_{\infty} + 2 |b|_1 \|x'\|_{\infty} + 2 |c|_1 \|x''\|_{\infty} \]
\[ + 2 |d|_1 \|x'''\|_{\infty} + 2 |g|_1 \|x''''\|_{\infty} + 2 |r|_1 + 2 |e|_1 + c_1 \]
\[ \leq \left( 2 |a|_1 + \frac{\alpha}{\beta} \right) \|x\|_{\infty} + (2 |b|_1 + 2 |c|_1 + 2 |d|_1) \|x'''\|_{\infty} \]
\[ + 2 |g|_1 \|x''''\|_{\infty} + 2 |r|_1 + 2 |e|_1 + c_1. \]

i.e.
\[ \|x\| \leq \left( 2 |a|_1 + \frac{\alpha}{\beta} \right) \|x\|_{\infty} + (2 |b|_1 + 2 |c|_1 + 2 |d|_1) \|x'''\|_{\infty} \]
\[ + 2 |g|_1 \|x''''\|_{\infty} + 2 |r|_1 + 2 |e|_1 + c_1. \]

It is easy to check that \( \|x'\|_{\infty} \leq \|x''\|_{\infty} \leq \|x'''\|_{\infty} \). Together with \( \|x\|_{\infty} \leq \|x\| \), it follows from the above inequality that
\[ \|x\|_{\infty} \leq \frac{1}{1 - 2 |a|_1 - \frac{\alpha}{\beta}} \left[ 2 |b|_1 \|x'\|_{\infty} + 2 |c|_1 \|x''\|_{\infty} \right. \]
\[ + 2 |d|_1 \|x'''\|_{\infty} + 2 |g|_1 \|x''''\|_{\infty} + 2 |r|_1 + 2 |e|_1 + c_1 \left. \right] \]
\[ \leq \frac{1}{1 - 2 |a|_1 - \frac{\alpha}{\beta}} \left[ (2 |b|_1 + 2 |c|_1 + 2 |d|_1) \|x''\|_{\infty} \right. \]
\[ + 2 |g|_1 \|x''''\|_{\infty} + 2 |r|_1 + 2 |e|_1 + c_1 \left. \right]. \] (5)

**Case 2.** \( |x''(t)| > M \) for all \( t \in [0, 1] \). In this case from property (A_2) we obtain
\[ |x''(t_1)| \leq \frac{\bar{a}}{\beta} |x(t_1)| + \frac{L_1}{\beta} + \frac{1}{\beta} \left| f(t_1, x(t_1), x'(t_1), x''(t_1), x'''(t_1)) \right| \]
\[ \leq \frac{\bar{a}}{\beta} \|x\|_{\infty} + \frac{L_1}{\beta} (L_1 + \|e\|_1) \]
so that
\[ |x'''(t)| \leq |x'''(t_1)| + \left| \int_{t_1}^{t} x^{(4)}(s) \, ds \right| \]
\[ \leq \frac{\bar{\alpha}}{\beta} \|x\|_\infty + \frac{1}{\beta} (L_1 + \|e\|_1) + \|Nx\|_1. \]

Thus similarly to the above discussion, one has a \( \xi \in (0, 1) \) such that
\[ x''(\xi) = 0 \] and there is an \( \eta \in (0, 1) \) such that
\[ |x''(t)| = |x''(t) - x''(\xi)| \]
\[ = |x'''(\eta)(t - \eta)| \]
\[ \leq \frac{\bar{\alpha}}{\beta} \|x\|_\infty + \frac{1}{\beta} (L_1 + \|e\|_1) + \|Nx\|_1. \]

So we get
\[ |x'(t)| = |x'(t) - x'(0)| \]
\[ \leq |x''(\xi)| \]
\[ \leq \frac{\bar{\alpha}}{\beta} \|x\|_\infty + \frac{1}{\beta} (L_1 + \|e\|_1) + \|Nx\|_1. \]

From property \((A_3)\), there is a \( t^{**} \in (0, 1) \) such that \( |x(t^{**})| \leq M_1 \). Then, together with (5),
\[ \|Px\| = |x(0)| \]
\[ = \left| x(t^{**}) - \int_{0}^{t^{**}} x'(t) \, dt \right| \]
\[ \leq M_1 + \frac{\bar{\alpha}}{\beta} \|x\|_\infty + \frac{1}{\beta} (L_1 + \|e\|_1) + \|Nx\|_1 \]
\[ \leq \frac{\bar{\alpha}}{\beta} \|x\|_\infty + \|Nx\|_1 + c_1. \]

Thus
\[ \|x\| \leq \|Px\| + \|(I - P)x\| \leq \frac{\bar{\alpha}}{\beta} \|x\|_\infty + 2 \|Nx\|_1 + c_1. \]

So property \((A_1)\) implies
\[ \|x''\|_\infty \leq \|x\| \]
\[ \leq \frac{2\|a\|_1 + \frac{\bar{\alpha}}{\beta}}{1 - 2\|a\|_1 - \frac{\bar{\alpha}}{\beta}} \left[ (2\|b\|_1 + 2\|c\|_1 + 2\|d\|_1) \|x''\|_\infty \right. \]
\[ + 2\|g\|_1 \|x''\|_\infty^\theta + 2\|r\|_1 + 2\|e\|_1 + c_1 \right] \]
\[ + \left[ (2\|b\|_1 + 2\|c\|_1 + 2\|d\|_1) \|x''\|_\infty \right. \]
\[ + 2\|g\|_1 \|x''\|_\infty^\theta + 2\|r\|_1 + 2\|e\|_1 + c_1 \right] \]}
\[
\begin{align*}
= & \frac{1}{1 - 2\|a\|_1 - \frac{\alpha}{\beta}} \left[ (2\|b\|_1 + 2\|c\|_1 + 2\|d\|_1) \|x''\|_\infty \\
& + 2\|g\|_1 \|x''\|_\infty^\theta + 2\|r\|_1 + 2\|e\|_1 + c_1 \right].
\end{align*}
\]

We get (5). From (5) it follows that
\[
\|x''\|_\infty \leq \frac{2\|g\|_1 \|x''\|_\infty^\theta + c_1 + 2\|r\|_1 + 2\|e\|_1}{1 - 2\|a\|_1 - \frac{\alpha}{\beta}} (1 - 2\|a\|_1 - \frac{\alpha}{\beta}).
\]

Since \(\theta \in [0, 1]\), there is \(M_1^* > 0\) such that
\[
\|x''\|_\infty \leq M_1^*.
\]

Again, it is easy to prove that
\[
\|x''\|_\infty \leq \|x''\|_\infty \leq \|x''\|_\infty \leq \|x''\|_\infty \leq M_1^*.
\]

From property (A_3) we claim that there is \(t^{**} \in (0, 1)\) such that \(|x(t^{**})| \leq M_1\). Thus
\[
|x(t)| \leq \left| x(t^{**}) - \int_{t}^{t^{**}} x'(s) \, ds \right| \leq M_1 + \|x'||_\infty.
\]

Hence there is \(M_2^* > 0\) such that \(\|x\|_\infty \leq M_2^*\). Hence
\[
\|x\| \leq \max \{\|x\|_\infty, \|x'||_\infty, \|x''\|_\infty, \|x''''\|_\infty \} \leq \max \{M_1^*, M_2^*\}.
\]

Thus \(\Omega_1\) is bounded. Let
\[
\Omega_2 = \{x \in \text{Ker} \, L : Nx \in \text{Im} \, L\}.
\]

For \(x \in \Omega_2\), \(x \in \text{Ker} \, L\) and \(QNx = 0\), thus
\[
\int_{0}^{1} (f(s, c, 0, 0, 0) + e(s)) \, ds = 0, \quad \text{i.e.} \quad \int_{0}^{1} f(s, c, 0, 0, 0) \, ds = - \int_{0}^{1} e(s) \, ds.
\]

Thus there is \(t_0 \in (0, 1)\) such that
\[
f(t_0, c, 0, 0, 0) = - \int_{0}^{1} e(s) \, ds, \quad \text{so} \quad |f(t_0, c, 0, 0, 0)| \leq \|e\|_1.
\]

From property (A_4) we see that there is \(M^* > 0\) such that \(|c| \leq M^*\). Thus \(\Omega_2\) is bounded. Next, according condition (A_5), we have the following two cases.
**Case 1.** Suppose for any \( c \in R \), if \(|c| > M_2\), then \( cf(t, c, 0, 0, 0) \leq 0 \) for \( t \in [0, 1] \). Let

\[
\Omega_3 = \left\{ x \in \text{Ker} L : -\lambda x + (1 - \lambda)QN x = 0, \lambda \in [0, 1] \right\}.
\]

Now, similar to the proof of [6: Lemma 2.12], we prove that \( \Omega_3 \) is bounded. Suppose \( x_n(t) = c_n \in \Omega_3 \) and \(|c_n| \to \infty \) as \( n \to \infty \). Without loss of generality, suppose that \( c_n > M_2 \) for all \( n \). Then there is \( \lambda_n \in [0, 1] \) such that

\[
\lambda_n c_n = (1 - \lambda_n)QN(c_n), \quad \text{or} \quad \lambda_n = (1 - \lambda_n) \frac{QN(c_n)}{c_n}.
\]  

Without loss of generality, suppose \( \lambda_n \to \lambda_0 \) as \( n \to \infty \). Then

\[
\left| \frac{QN(c_n)}{c_n} \right| = \frac{1}{|c_n|} \left| \int_0^1 \left( f(s, c_n, 0, 0, 0) + e(s) \right) ds \right|
\leq \frac{1}{|c_n|} \left[ \|e\|_1 + \|a\|_1 |c_n| + \|r\|_1 \right]
= \|a\|_1 + \frac{\|e\|_1 + \|r\|_1}{|c_n|}.
\]

Thus \( \frac{QN(c_n)}{c_n} \) is bounded. So \( \lambda_n \to \lambda_0 \neq 1 \) by (6). Thus, for sufficiently large \( n \), \( \lambda_n \neq 1 \). Then

\[
\frac{\lambda_n}{1 - \lambda_n} = \frac{1}{c_n} \left( \int_0^1 \left( f(s, c_n, 0, 0, 0) + e(s) \right) ds \right).
\]

From property (A4), for sufficiently large \( n \), \(|f(t, c_n, 0, 0, 0)| \geq \alpha |c_n| \) for some \( \alpha > 0 \). Then property (A5) implies \( f(t, c_n, 0, 0, 0) < -\alpha c_n \). Thus, by Fatou’s Lemma,

\[
\limsup \left( \frac{1}{c_n} \int_0^1 f(s, c_n, 0, 0, 0) ds + \frac{1}{c_n} \int_0^1 e(s) ds \right)
\leq \limsup \frac{1}{c_n} \int_0^1 f(s, c_n, 0, 0, 0) ds
\leq \int_0^1 \limsup \frac{f(s, c_n, 0, 0, 0)}{c_n} ds
\leq -\alpha
< 0.
\]

This contradicts \( \frac{\lambda_n}{1 - \lambda_n} \geq 0 \). Then \( \Omega_3 \) is bounded.
Case 2. Suppose $|c| > M_2$. Then $cf(t, c, 0, 0, 0) \geq 0$ for $t \in [0, 1]$. Indeed, set
\[
\Omega_3 = \left\{ x \in \text{Ker} L : \lambda x + (1 - \lambda)QN x = 0 \text{ for all } \lambda \in (0, 1) \right\}.
\]
Like in the above argument, we can prove that $\Omega_3$ is bounded. In the following, we shall prove that all conditions of Theorem 1 are satisfied. Let $\Omega$ be a bounded open subset of $X$ such that
\[
\bigcup_{i=1}^{3} \overline{\Omega}_i \subset \Omega.
\]
By Lemma 1, $L$ is a Fredholm operator of index zero and $N$ is $L$-compact on $\overline{\Omega}$. By the above argument and the definition of $\Omega$, we have:
(i) $Lx \neq \lambda Nx$ for $(\lambda, x) \in [(\text{dom } L/\text{Ker } L) \cap \partial \Omega] \times (0, 1)$
(ii) $Nx \notin \text{Im } L$ for $x \in \text{Ker } L \cap \partial \Omega$.
At last, we prove that condition (iii) of Theorem M is satisfied. Let
\[
H(x, \lambda) = \pm \lambda x + (1 - \lambda)QN x.
\]
By the definition of $\Omega$, we see that $H(x, \lambda) \neq 0$ for $x \in \partial \Omega \cap \text{Ker } L$. Thus, by the homotopy property of degree, we have
\[
\deg(QN_{\text{Ker } L}, \Omega \cap \text{Ker } L, 0) = \deg(H(\cdot, 0), \Omega \cap \text{Ker } L, 0)
= \deg(H(\cdot, 1), \Omega \cap \text{Ker } L, 0)
= \deg(\pm \lambda I, \Omega \cap \text{Ker } L, 0)
\neq 0.
\]
Thus by Theorem 1, the equation $Lx = Nx$ has at least one solution in $\text{dom } L \cap \overline{\Omega}$. So problem (1) - (2) has at least one solution $\blacksquare$

References


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Received 14.05.2003