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Fixed Point Theorems for a Class of Mixed Monotone Operators

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Abstract. In this paper we study a class of mixed monotone operators with convexity and concavity. In particular, we give conditions, both necessary and sufficient, for the existence and uniqueness of fixed points. Moreover, we sketch a simple application of our main theorem and generalize some previous results.

Keywords: *Banach space, cone, mixed monotone operator, fixed point*

AMS subject classification: 47H07, 47H10

1. Introduction

It is well known that mixed monotone operators are important for studying positive solutions of nonlinear differential and integral equations. In applications, in order to prove existence or uniqueness for solution of such equations, one usually considers the fixed points of some related operators. More information about mixed monotone operators may be found in [6]. There are many useful results about mixed monotone operators with convexity and concavity properties (see [1 - 5, 7 - 12]). In this paper, we study this class of operators and give sufficient and necessary conditions for the existence and uniqueness of fixed points without assuming the operators to be continuous or compact. In this way, we generalize and extend similar results from [2, 4, 6, 10 - 12].

Suppose that E is a real Banach space which is partially ordered by a cone $P \subset E$, i.e. $x \le y$ if and only if $y - x \in P$. By θ we denote the zero element of E. Recall that a non-empty closed convex set $P \subset E$ is a *cone* if it satisfies

$$\begin{aligned} x \in P, \lambda \ge 0 \;&\Rightarrow\; \lambda x \in P \\ x, -x \in P \;&\Rightarrow\; x = \theta. \end{aligned} \tag{1}$$

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Putting

$$P^\circ = \{x \in P : x \text{ is an interior point of } P\},$$

a cone P is said to be *solid* if its interior P° is non-empty. Moreover, P is called *normal* if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta \le x \le y$ implies $\|x\| \le N\|y\|$; in this case N is called the *normality constant* of P. In the case $y - x \in P^\circ$ we write $x \ll y$.

For instance, the usual cones of non-negative elements in $\mathbb{R}^N, l^p, l^\infty, C, L^p$, L^∞ and C are normal, the cone of non–negative functions in C^1 is not. On the other hand, the cone of non-negative functions in C and C^1 is solid, but in L^P it is not.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \le y \le \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e. $h \ge \theta$ and $h \ne \theta$), we denote by P_h the set

$$P_h = \left\{ x \in E \;\middle|\; \begin{array}{l} \text{there exist } \lambda(x), \mu(x) > 0 \text{ such} \\ \text{that } \lambda(x)h \le x \le \mu(x)h \end{array} \right\}.$$

It is easy to see that $P_h \subset P$.

Recall that $A : P_h \times P_h \to P_h$ is a *mixed monotone operator*, if $A(x, y)$ is non-decreasing in x and non-increasing in y, i.e. for all $x_1, x_2, y_1, y_2 \in P_h$,

$$x_1 \le x_2, y_2 \le y_1 \;\Rightarrow\; A(x_1, y_1) \le A(x_2, y_2).$$

A point $(x^*, y^*) \in P_h \times P_h$ is called a *coupled fixed point* of A if $A(x^*, y^*) = x^*$ and $A(y^*, x^*) = y^*$. Finally, an element $x^* \in P_h$ is called a *fixed point* of A if $A(x^*, x^*) = x^*$.

All the concepts discussed above can be found in [3]. Our paper is organized as follows. In the Section 2 we discuss mixed monotone operators with convexity and concavity properties in a simple special case. Afterwards, in Section 3, we pass to mixed monotone operators with convexity and concavity in the general case. A certain application of our results will be given in Section 4.

2. A special case

In this section, we give necessary and sufficient conditions for the existence and uniqueness of fixed points on mixed monotone operators with convexity and concavity in a special case. Let us begin with the following lemma.

Lemma 2.1. *Suppose that E is a real Banach space, P is a cone in E, $h > \theta$ and $A : P_h \times P_h \to P_h$. Then the following two statements are equivalent:*

(a) *For all $0 < t < 1$ there exists $0 < \alpha = \alpha(t) < 1$ such that*

$$A\Big(tu, \frac{1}{t}v\Big) \geq t^{\alpha(t)} A(u, v) \qquad (u, v \in P_h, u \leq v).$$

(b) *For all $0 < t < 1$ there exists $\eta = \eta(t) > 0$ such that*

$$A\Big(tu, \frac{v}{t}\Big) \geq t[1 + \eta(t)]A(u, v) \qquad (u, v \in P_h, u \leq v)$$

where $t[1 + \eta(t)] < 1$.

Proof. If assertion (a) holds, we take $\eta(t) = t^{\alpha(t)-1} - 1$ and get assertion (b). Conversely, if assertion (b) holds, we take $\alpha(t) = \frac{\ln[t(1+\eta(t))]}{\ln t}$; since $0 < t[1 + \eta(t)] < 1$, we easily get assertion (a) ∎

Theorem 2.1. *Suppose that E is a real Banach space, P is a normal cone in E, $h > \theta$, and $A : P_h \times P_h \to P_h$ is a mixed monotone operator. Assume property (a) of Lemma 2.1 is fulfilled. Then A has exactly one fixed point x^* in P_h if and only if, for some $u_0, v_0 \in P_h$ with $u_0 \leq v_0$, $u_0 \leq A(u_0, v_0)$ and $A(v_0, u_0) \leq v_0$. Moreover, constructing successively the sequences*

$$\begin{aligned} x_n &= A(x_{n-1}, y_{n-1}) \\ y_n &= A(y_{n-1}, x_{n-1}) \end{aligned} \qquad (n \geq 1), \tag{2}$$

for any initial value $(x_0, y_0) \in [u_0, v_0]$ we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*$.

Proof. Firstly we prove that, if A has a fixed point in P_h, then this fixed point is unique. In fact, if $x^*, y^* \in P_h$ are such that $A(x^*, x^*) = x^*$ and $A(y^*, y^*) = y^*$, then denote

$$a_0 = \sup\Big\{a > 0 \Big|\ ay^* \leq x^* \leq \frac{1}{a}y^*\Big\}.$$

Then $0 < a_0 \leq 1$; we claim that $a_0 = 1$. In fact, $0 < a_0 < 1$ would imply that

$$x^* = A(x^*, x^*) \geq A\Big(a_0 y^*, \frac{1}{a_0}y^*\Big) \geq a_0^{\alpha(a_0)} A(y^*, y^*) = a_0^{\alpha(a_0)} y^*.$$

But $0 < \alpha(a_0) < 1$ and $a_0^{\alpha(a_0)} > a_0$, contradicting the definition of a_0. We conclude that $x^* = y^*$.

`Proof of necessity:` Assume that x^* is a fixed point of A in P_h. Let $u_0 = v_0 = x^*$. Then $u_0 \leq A(u_0, v_0)$ and $A(v_0, u_0) \leq v_0$.

`Proof of sufficiency:` Let u_{n+1}, v_{n+1} $(n \geq 0)$ be as in (2). It is clear that

$$u_0 \leq u_1 \leq ... u_n \leq ... \leq v_n \leq ... \leq v_1 \leq v_0$$

and $\{u_n\}, \{v_n\} \subset P_h$. In what follows we prove that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. For any $n \in \mathbb{N}$, there exists $\mu > 0$ such that $\mu v_n \leq u_n \leq v_n$. Putting

$$t_n = \sup\{\mu > 0 : u_n \geq \mu v_n\}$$

we see that $0 < t_n \leq 1$ and t_n is non-decreasing. So $\lim_{n\to\infty} t_n = t$ for some $0 < t \leq 1$. We show that $t = 1$. In fact, otherwise it follows from $0 < t < 1$, $u_n \geq t_n v_n, v_n \leq \frac{1}{t_n} u_n$ and $t_n \leq t$ that

$$\begin{aligned}
u_{n+1} &= A(u_n, v_n) \\
&\geq A\Big(t_n v_n, \frac{1}{t_n} u_n\Big) \\
&= A\Big[\frac{t_n}{t}(t v_n), \frac{t}{t_n}(\frac{1}{t} u_n)\Big] \\
&\geq \frac{t_n}{t}\Big[1 + \eta(\frac{t_n}{t})\Big] A\Big(t v_n, \frac{1}{t} u_n\Big) \\
&\geq \frac{t_n}{t} A\Big(t v_n, \frac{1}{t} u_n\Big) \\
&\geq t_n[1 + \eta(t)] A(v_n, u_n) \\
&= t_n[1 + \eta(t)] v_{n+1}
\end{aligned}$$

where $\eta = \eta(\frac{t_n}{t}) > 0$ is that from Lemma 2.1/(b). By the definition of t_{n+1} we get $t_{n+1} \geq t_n[1 + \eta(t)]$. Letting $n \to \infty$ we obtain $t \geq t[1 + \eta(t)]$. But $\eta(t) > 0$, by Lemma 2.1, which is a contradiction. So we conclude that $t = 1$ as claimed.

For any natural number p we have

$$0 \leq u_{n+p} - u_n \leq v_n - u_n \leq v_n - t_n v_n \leq (1 - t_n) v_0.$$

From the normality of P it follows that $||u_{n+p} - u_n|| \leq N(1 - t_n)||v_0|| \to 0$ as $n \to \infty$. So $\{u_n\}$ is a Cauchy sequence; the same reasoning shows that $\{v_n\}$ is also a Cauchy sequence.

Now we prove that A has a fixed point $x^* \in P_h$. Because E is complete, there exist $u^*, v^* \in E$ such that $u_n \to u^*$ and $v_n \to v^*$. From the fact that $\{u_n\} \uparrow$, $\{v_n\} \downarrow$, $u_n \leq v_n$ and from the normality of P it follows that $u_n \leq u^* \leq v^* \leq v_n$. Consequently, $u^*, v^* \in P_h$ and $v^* - u^* \leq v_n - u_n \leq (1 - t_n) v_0$, and so $\|v^* - u^*\| \to 0$, i.e. $u^* = v^* =: x^*$. We know that

$$u_{n+1} = A(u_n, v_n) \leq A(u^*, v^*) = A(x^*, x^*) \leq A(v_n, u_n) = v_{n+1}.$$

So letting $n \to \infty$ yields $A(x^*, x^*) = x^*$.

Finally, if the sufficient condition holds, then for arbitrary $x_0, y_0 \in [u_0, v_0]$ and x_n, y_n as in (2) we get $u_n \le x_n \le v_n$ and $u_n \le y_n \le v_n$. Taking into account that P is normal we conclude that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*$ which completes the proof ∎

Theorem 2.2. *Suppose that E is a real Banach space, P is a normal cone in E, $h > \theta$ and $A : P_h \times P_h \to P_h$ is a mixed monotone operator. Assume property (a) of Lemma 2.1 holds for $u = v =: x$. Then A has exactly one fixed point x^* in P_h if and only if there exist $x_0 \in P_h$ and $t_0 \in (0,1)$ such that $t_0 x_0 \le A\big(t_0 x_0, \frac{x_0}{t_0}\big)$ and $A\big(\frac{x_0}{t_0}, t_0 x_0\big) \le \frac{x_0}{t}$.*

Proof. Set $u_0 = t_0 x_0$ and $v_0 = \frac{x_0}{t_0}$. Then $A(u_0, v_0) = A\big(t_0 x_0, \frac{x_0}{t_0}\big) \ge u_0$ and $A(v_0, u_0) = A\big(\frac{x_0}{t_0}, t_0 x_0\big) \le v_0$. Choosing u_{n+1}, v_{n+1} as in (2) we can complete the proof by the same reasoning as in the proof of Theorem 2.1 ∎

Remark 1. A comparison with the corresponding results in the literature ([2: Theorem 3.1], [4] and [11: Theorem 1]) shows that our hypotheses are simpler and weaker, while our assertions are stronger in the following sense:

- Only sufficient conditions were given in those papers, but we get conditions which are both sufficient and necessary.
- Tools used in those papers were the Hilbert metric, the Thompson metric, and the fixed point index; these methods cannot be used under the hypotheses of our theorems.
- We widen the range of α from $\alpha = const$ or $\alpha = \alpha(a, b)$ to $\alpha = \alpha(t)$ for $t \in (0,1)$.
- The corresponding theorems of [2, 4, 5] are corollaries of our Theorems 2.1 and 2.2.

Remark 2. The conclusion of Theorem 2.1 also holds when (1) is satisfied in $[u_0, v_0]$ only.

Corollary 2.1 (see [2]). *Suppose that E is a real Banach space, P is a normal cone in E, and $A : P^\circ \times P^\circ \to P^\circ$ is a mixed monotone operator such that for all $0 < t < 1$ there exists $0 < \beta < 1$ with $A\big(tx, \frac{1}{t}y\big) \ge t^\beta A(x,y)$ $(x, y \in P^\circ)$. Then A has exactly one fixed point x^* in P° and, for all $x_0, y_0 \in P^\circ$, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*$ where x_n, y_n $(n \ge 1)$ are as in (2).*

Proof. Choose a sufficiently small number t_0 such that $t_0 x_0 \le x_0 \le \frac{1}{t_0} x_0$, $t_0 x_0 \le y_0 \le \frac{1}{t_0} x_0$ and $t_0^{1-\beta} x_0 \le A(x_0, x_0) \le \big(\frac{1}{t_0}\big)^{1-\beta} x_0$. Putting $u_0 = t_0 x_0$ and $v_0 = \frac{1}{t_0} x_0$ we get $u_0 \le A(u_0, v_0)$ and $A(v_0, u_0) \le v_0$. So the hypotheses of Theorem 2.1 are satisfied and the conclusion follows ∎

Remember that an operator $A : x \to Ax$ is called

- *concave* if $A(tx_1 + (1-t)x_2) \geq tA(x_1) + (1-t)A(x_2)$
- $(-\alpha)$-*convex* if $A(tx) \leq t^{-\alpha}A(x)$

for all $0 < t < 1$ and all x.

Corollary 2.2 (see [4]). *Suppose that E is a real Banach space, P is a normal cone in E, and $A : P^\circ \times P^\circ \to P^\circ$ is a mixed monotone operator which satisfies the following assumptions:*

(a) *For fixed y, the operator $A(\cdot, y) : P^\circ \to P^\circ$ is concave while for fixed x the operator $A(x, \cdot) : P^\circ \to P^\circ$ is $(-\alpha)$-convex.*

(b) *There exist $u_0, v_0 \in P$ and $\varepsilon \geq \alpha > 0$ such that $\theta \ll u_0 \leq v_0$, $u_0 \leq A(u_0, v_0)$, $A(v_0, u_0) \leq v_0$ and $A(\theta, v_0) \geq \varepsilon A(u_0, v_0)$.*

Then A has exactly one fixed point x^* in $[u_0, v_0]$. Moreover, for all $x_0, y_0 \in [u_0, v_0]$ one has $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ $(n \to \infty)$, where x_n, y_n $(n \geq o)$ are as in (2).

Proof. It is easy to see that, if A has a fixed point in $P^\circ \times P^\circ$, then this fixed point is unique. So we only have to prove existence. In fact, for all $h \in P^\circ$ we have $P_h = P^\circ$. For $A : P_h \times P_h \to P_h$, condition (b) ensures that there exist $u_0, v_0 \in P_h$ such that $u_0 \leq A(u_0, v_0)$ and $A(v_0, u_0) \leq v_0$. We can prove that, for all $t \in (0, 1)$, one can find $0 < \alpha(t) < 1$ such that $A\big(tu, \frac{1}{t}v\big) \geq t^{\alpha(t)}A(u, v)$ for all $u, v \in [u_0, v_0]$. Indeed,

$$\begin{aligned}
A\Big(tu, \frac{1}{t}v\Big) &\geq t^\alpha A(tu, v) \\
&\geq t^\alpha \big[tA(u, v) + (1-t)A(\theta, v)\big] \\
&\geq t^{\alpha+1}A(u, v) + t^\alpha(1-t)\epsilon A(u, v) \\
&= \big[t^{\alpha+1} + t^\alpha(1-t)\epsilon\big]A(u, v) \\
&\geq \big[t^{\alpha+1} + t^\alpha(1-t)\alpha\big]A(u, v) \\
&= t^{\alpha(t)}A(u, v),
\end{aligned}$$

where $\alpha(t) = \frac{\ln(t^{\alpha+1} + t^\alpha(1-t)\alpha)}{\ln t}$, implies $0 < \alpha(t) < 1$. So all conditions of Theorem 2.1 are satisfied, and the assertion follows ∎

Corollary 2.3 (see [5]). Suppose that E is a real Banach space, P is a normal cone in E, and $A : P^\circ \times P^\circ \to P^\circ$ is a mixed monotone operator such that for all $[a, b] \subset (0, 1)$ one can find $\alpha = \alpha(a, b) \in (0, 1)$ such that $A\big(tx, \frac{1}{t}x\big) \geq t^\alpha A(x, x)$ $(x \in P^\circ)$. Then A has exactly one fixed point x^* in P° and $A^n(x_0, x_0) \to x^* = A(x^*, x^*)$ for all $x_0 \in P^\circ$ where $A^n(x_0, x_0) = A(x_{n-1}, x_{n-1})$ $(n \geq 1)$.

Proof. Firstly, for all $x_0 \in P^\circ$ there exists $0 < b < 1$ satisfying $bx_0 \leq A(x_0, x_0) \leq \frac{1}{b}x_0$. Secondly, by assumption we know that, for all $t_0 \in [a, b] \subset$

$(0,1)$, there is a $0<\beta=\beta(a,b)<1$ such that $A(t_0x_0,\frac{1}{t_0}x_0)\geq t_0^{\beta}A(x_0,x_0)$. Moreover,

$$A(t_0x_0,t_0x_0)\geq t_0^{\beta}A(x_0,x_0)$$
$$A\Big(\frac{1}{t_0}x_0,\frac{1}{t_0}x_0\Big)\leq t_0^{-\beta}A(x_0,x_0).$$

For all sequences $\{a_i\}_{i\geq 1}$ satisfying $0<a_i<1$ and $a_1>a_2>\ldots>a_n>\ldots>0$ we denote for $i\geq 1$

$$\beta_i=\inf\Big\{\beta\in(0,1):\ A(tx,tx)\geq t^{\beta}A(x,x)\ \forall\ t\in[a_i,b),x\in P^{\circ}\Big\}.$$

Then $\beta_1<\beta_2<\ldots\beta_n<\ldots<1$. It is clear that there exists $0<\beta\leq 1$ such that $\lim_{n\to\infty}\beta_n=\beta$.

Finally, if $b^{\frac{1}{1-\beta_1}}>a_1$, we choose $t_0\in(a_1,b^{\frac{1}{1-\beta_1}})\subset[a,b]$ and write $u_0=t_0x_0$ and $v_0=\frac{1}{t_0}x_0$. By assumption, we have then $u_0\leq A(u_0,v_0)$ and $A(v_0,u_0)\leq v_0$, which shows that all conditions of Theorem 2.2 hold. On the other hand, in the case $b^{\frac{1}{1-\beta_1}}\leq a_1$ we can choose a_n in the form $a_n=a_1b^{1/(1-\beta_{n-1})}$ $(n\geq 2)$. This implies that $a_n>0$ and the sequence $\{a_n\}$ is decreasing. It is easy to prove that there exists N_0 such that, for $N\geq N_0$,

$$a_N=a_1b^{\frac{1}{1-\beta_{N-1}}}<b^{\frac{1}{1-\beta_N}}<b.$$

So choosing $t_0\in(a_{N_0},b^{1/(1-\beta_{N_0})})\subset(a_{N_0},b)$ and denoting $u_0=t_0x_0$ and $v_0=\frac{1}{t_0}x_0$, we see that all conditions of Theorem 2.2 are satisfied as well. This completes the proof ∎

3. The general case

In order to study mixed monotone operators with convexity and concavity in the general case, we introduce the concept of an "adjoint sequence" and give necessary and sufficient conditions for the existence and uniqueness of fixed points.

Lemma 3.1. *Suppose that E is a real Banach space, P is a normal cone in E, $h>\theta$, and $A: P_h\times P_h\to P_h$. Then the following two statements are equivalent:*

(a) *For all $0<t<1$ and $u,v\in P_h$ there exists $0<t=\alpha(t,u,v)\leq 1$ such that $A\big(tu,\frac{1}{t}v\big)\geq t^{\alpha(t,u,v)}A(u,v)$.*

(b) *For all $0<t<1$ and $u,v\in P_h$ there exists $\eta=\eta(t,u,v)>0$ such that $A\big(tu,\frac{1}{t}v\big)\geq t[1+\eta(t,u,v)]A(u,v)$ where $t[1+\eta(t,u,v)]<1$.*

The method of proof is similar to that of Lemma 2.1 and therefore omitted.

For further reference we recall the concept of "adjoint sequence" in the following

Definition 3.1. Suppose that E is a real Banach space, P is a normal cone in E, $h > \theta$, $A : P_h \times P_h \to P_h$ is an operator, and $u_0, v_0 \in P_h$. If there exists $0 < \lambda_0 < 1$ such that $\lambda_0 v_0 \le u_0 \le v_0$, we define $u_{n+1}, v_{n+1} \ (n \ge 0)$ as in (2). Then, if $\{\eta_n\}$ is a sequence which satisfies $0 < \lambda_n = \lambda_0(1+\eta_n)^n < 1$ for $\lambda_n v_n \le u_n \le v_n \ (n \ge 0)$, we call $\{\eta_n\}$ an *adjoint sequence* of A with respect to λ_0, u_0, v_0.

Suppose A is a mixed monotone operator, and for all $0 < t < 1$ and $u, v \in P_h$ we can find $0 < t = \alpha(t,u,v) < 1$ such that $A\big(tu, \frac{1}{t}v\big) \ge t^{\alpha(t,u,v)} A(u,v)$. In this case we may choose $u_0, v_0 \in P_h$ with $u_0 \le v_0$ and $0 < \lambda_0 < 1$ such that $u_0 \ge \lambda_0 v_0$. Then A must have an adjoint sequence with respect to λ_0, u_0, v_0. In fact, by Lemma 3.2 there exists $\eta_1' = \eta_1'(\lambda_0, u_0, v_0)$ satisfying

$$A(u_0, v_0) \ge A\Big(\lambda_0 v_0, \frac{1}{\lambda_0} u_0\Big) \ge \lambda_0 \big[1 + \eta_1'(\lambda_0, u_0, v_0)\big] A(v_0, u_0)$$

and $\lambda_0(1+\eta_1) \in (0,1)$. Choosing $\eta_1 \le \eta_1'$ yields

$$A(u_0, v_0) \ge \lambda_0 \big[1 + \eta_1(\lambda_0, u_0, v_0)\big] A(v_0, u_0),$$

hence

$$\begin{gathered} \lambda_1 v_1 \le u_1 \le v_1 \\ 0 < \lambda_1 = \lambda_0(1+\eta_1) < 1. \end{gathered} \tag{3}$$

By (3) there exists $\eta_2' = \eta_2'(\lambda_0, u_0, v_0)$ such that

$$A(u_1, v_1) \ge A\Big(\lambda_1 v_1, \frac{1}{\lambda_1} u_1\Big) \ge \lambda_1 (1 + \eta_2') A(v_1, u_1)$$

and $0 < \lambda_1(1+\eta_2') < 1$. Choosing now $\eta_2 = \min(\eta_1, \eta_2')$ we get

$$\begin{gathered} \lambda_2 v_2 \le u_2 \le v_2 \\ 0 < \lambda_2 = \lambda_0(1+\eta_2)^2 < 1. \end{gathered}$$

By induction we get for all natural numbers n

$$\begin{gathered} \lambda_n v_n \le u_n \le v_n \\ 0 < \lambda_n = \lambda_0(1+\eta_n)^n < 1. \end{gathered}$$

We conclude that $\{\eta_n\}$ is an adjoint sequence of A with respect to λ_0, u_0, v_0 as claimed.

Theorem 3.1. *Suppose that E is a real Banach space, P is a normal cone in E, $h > \theta$, and $A : P_h \times P_h \to P_h$ is a mixed monotone operator. Assume that property (a) of Lemma 3.1 is fulfilled for some $\alpha(t, u, v)$. Then A has exactly one fixed point x^* in P_h if and only if there exist u_0, v_0 satisfying:*

(a) *$u_0 \leq A(u_0, v_0)$ and $A(v_0, u_0) \leq v_0$.*

(b) *If there exists $\lambda_0 > 0$ such that $u_0 \geq \lambda_0 v_0$, then one can find an adjoint sequence $\{\eta_n\}$ of A with respect to λ_0, u_0, v_0 such that $\lim_{n\to\infty} n\eta_n = \ln \frac{1}{\lambda_0}$.*

Moreover, if A has a fixed point x^* in P_h, then for all $x_0, y_0 \in P_h$ one has $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = x^*$ where $x_n, y_n \quad (n \geq 1)$ are as in (2).

Proof. `Sufficiency`: Let $u_{n+1}, v_{n+1} \quad (n \geq 0)$ as in Definition 3.1. Because A is a mixed monotone operator and assertion (a) holds, we have

$$u_0 \leq u_1 \leq \ldots \leq u_n \leq \ldots \leq v_n \leq \ldots \leq v_1 \leq v_0.$$

Furthermore, by assertion (b), the existence of $\lambda_0 > 0$ satisfying $u_0 \geq \lambda_0 v_0$ implies the existence of an adjoint sequence $\{\eta_n\}$ of A with respect to λ_0, u_0, v_0 such that $n\eta_n \to \ln \frac{1}{\lambda_0}$ as $n \to \infty$ and $u_n \geq \lambda_0(1+\eta_n)^n v_n$. Therefore

$$v_n - u_n \leq v_n - \lambda_0(1+\eta_n)^n v_n \leq \big[1 - \lambda_0(1+\eta_n)^n\big] v_0.$$

Consequently,

$$\|v_n - u_n\| \leq N\big[1 - \lambda_0(1+\eta_n)^n\big]\|v_0\|$$

where N is the normality constant of P. Taking into account that $\eta_n \to 0$ as $n \to \infty$ and

$$\lambda_0(1+\eta_n)^n = \lambda_0\big[(1+\eta_n)^{\frac{1}{\eta_n}}\big]^{n\eta_n} \to \lambda_0 \frac{1}{\lambda_0} = 1 \qquad (n \to \infty),$$

it follows that $\|v_n - u_n\| \to 0$ as $n \to \infty$. This shows that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences.

Since E is complete, P_h is closed, $u_n \uparrow$, $v_n \downarrow$, and $u_n \leq v_n$, we find $u^*, v^* \in P_h$ such that $u_n \to u^*, v_n \to v^* \quad (n \to \infty)$ and $u_n \leq u^* \leq v^* \leq v_n \quad (n \geq 0)$. This implies $u^* = v^* =: x^*$. We have

$$u_{n+1} = A(u_n, v_n) \leq A(u^*, v^*) \leq A(v_n, u_n) = v_{n+1}.$$

Moreover, $x^* \leq A(x^*, x^*) \leq x^*$, and so x^* is a fixed point of A. The proof of uniqueness is similar to that given in Theorem 2.1.

`Necessity`: Firstly, let x^* be a fixed point of A. For all $0 < t_0 < 1$ denote $u_0 = t_0 x^*$ and $v_0 = \frac{1}{t_0} x^*$. Then there exists $0 < \alpha(t_0, x^*) < 1$ such that

$$A(u_0, v_0) = A\Big(t_0 x^*, \frac{1}{t_0} x^*\Big) \geq t_0^{\alpha(t_0,x^*)} A(x^*, x^*) = t_0^{\alpha(t_0,x^*)} x^* \geq t_0 x^* = u_0$$

$$A(v_0, u_0) = A\Big(\frac{1}{t_0} x^*, t_0 x^*\Big) \leq \Big(\frac{1}{t_0}\Big)^{\alpha(t_0,x^*)} A(x^*, x^*) = \frac{1}{t_0} x^* = v_0.$$

This shows that assertion (a) holds.

Secondly, letting $u_{n+1}, v_{n+1} \quad (n \geq 0)$ as in Definition 3.1 we have

$$u_0 \leq u_1 \leq \ldots \leq u_n \leq \ldots \leq v_n \leq \ldots \leq v_1 \leq v_0.$$

It is not hard to prove that there exists $x^* \in P_h$ such that $u_n \to x^*$ and $v_n \to x^* \quad (n \to \infty)$. In fact, let us denote

$$\xi_n = \sup\Big\{t > 0 : \, tx^* \leq u_n, v_n \leq \frac{1}{t}x^*\Big\}.$$

Then $0 < \xi_n \leq 1$ and $\xi_n \uparrow$, and therefore we find ξ such that $\lim_{n\to\infty} \xi_n = \xi$. Obviously, $0 < \xi \leq 1$; we claim that $\xi = 1$. Indeed, assuming $0 < \xi < 1$ we get

$$\begin{aligned} u_{n+1} &= A(u_n, v_n) \\ &\geq A\Big(\xi_n x^*, \frac{1}{\xi_n} x^*\Big) \\ &= A\Big(\frac{\xi_n}{\xi}\xi_n x^*, \frac{\xi}{\xi_n}\frac{1}{\xi_n}x^*\Big) \\ &\geq \frac{\xi_n}{\xi} A\Big(\xi x^*, \frac{1}{\xi}x^*\Big) \\ &\geq \xi_n\big[1 + \eta(\xi, x^*)\big]x^*. \end{aligned}$$

Therefore, $\xi_{n+1} \geq \xi_n[1 + \eta(\xi, x^*)]$. This leads to $\xi \geq \xi[1 + \eta(\xi, x^*)]$, a contradiction.

Our result implies that

$$\|v_n - u_n\| \leq N\Big\|\Big(\frac{1}{\xi_n} - \xi_n\Big)x^*\Big\| \to 0 \qquad (n \to \infty).$$

So we get $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = x^*$. Moreover, from $u_0 \leq x^* \leq v_0$ we have $u_n \leq v_n \quad (n \geq 1)$. We may also easily prove that there exists c_n with $0 \leq c_n \leq 1$ and $c_n \to 1$ as $n \to \infty$ such that $u_n \geq c_n v_n$. In fact, from $u_n \geq \xi_n x^*$ and $v_n \leq \frac{1}{\xi_n}x^*$ we deduce that $u_n \geq \xi_n^2 v_n$. So letting $c_n = \xi_n^2$ proves the statement.

Let $c_n = \lambda_0(1 + \tilde{\eta}_n)^n$ where $\tilde{\eta}_n = \big(\frac{c_n}{\lambda_0}\big)^{\frac{1}{n}} - 1$. Then $u_n \geq \lambda_0(1 + \tilde{\eta}_n)^n v_n$ which shows that $\{\tilde{\eta}_n\}$ is an adjoint sequence of A with respect to λ_0, u_0, v_0. Furthermore, $n\tilde{\eta}_n \to \ln\frac{1}{\lambda_0}$ as $n \to \infty$. In fact,

$$c_n = \lambda_0(1 + \tilde{\eta}_n)^n = \lambda_0\big[(1 + \tilde{\eta}_n)^{\frac{1}{\tilde{\eta}_n}}\big]^{n\tilde{\eta}_n}$$

and

$$\ln c_n = \ln\lambda_0 + n\tilde{\eta}_n \ln\big[(1 + \tilde{\eta}_n)^{\frac{1}{\tilde{\eta}_n}}\big] \quad \Rightarrow \quad \lim_{n\to\infty} n\tilde{\eta}_n = \ln\frac{1}{\lambda_0}.$$

Finally, for all $x_0, y_0 \in P_h$ we choose a sufficiently small number t_0 and set $u_0 = t_0 x^*$ and $v_0 = \frac{1}{t_0}x^*$, where $x_0, y_0 \in [u_0, v_0]$. Then we get $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = x^*$. This completes the proof ∎

Remark 3. Theorem 2.1 is a special case of Theorem 3.1.

Theorem 3.2. *Suppose that E is a Banach space, P is a normal cone in E, $h > \theta$, and $A: P_h \times P_h \to P_h$ is a mixed monotone operator. Assume that property (a) of Lemma 2.1 with $\alpha(t) = 1$ is fulfilled for all $u, v \in P_h$ (in this case A is called sublinear). Then A has exactly one fixed point x^* in P_h if and only if there exist sequences $\{u_{k,0}\}_0^\infty, \{v_{k,0}\}_0^\infty \subset P_h$ such that*

(a) $u_{k,0} \leq A(u_{k,0}, v_{k,0})$ *and* $v_{k,0} \geq A(v_{k,0}, u_{k,0})$

(b) $u_{k,0} \leq u_{k+1,0}$ *and* $v_{k+1,0} \leq v_{k,0}$

(c) $\lambda_k v_{k,0} \leq u_{k,0} \leq v_{k,0}$ *and* $\lambda_k \to 1$ *as* $k \to \infty$.

Proof. `Sufficiency`: Note that

$$\begin{aligned} u_{k,l+1} &= A(u_{k,l}, v_{k,l}) \\ v_{k,l+1} &= A(v_{k,l}, u_{k,l}) \end{aligned} \qquad (k, l \geq 0).$$

By assertion (a), for k we have

$$\begin{aligned} u_{k,1} &= A(u_{k,0}, v_{k,0}) \geq u_{k,0} \\ v_{k,1} &= A(v_{k,0}, u_{k,0}) \leq v_{k,0} \\ u_{k,2} &= A(u_{k,1}, v_{k,1}) \geq A(u_{k,0}, v_{k,0}) = u_{k,1} \\ v_{k,2} &= A(v_{k,1}, u_{k,1}) \leq A(v_{k,0}, u_{k,0}) = v_{k,1} \\ &\vdots \end{aligned}$$

and so on, i.e. $u_{k,l} \leq u_{k,l+1}, v_{k,l} \geq v_{k,l+1}$ while assertion (b) for l implies $u_{k+1,l} \geq u_{k,l}; v_{k+1,l} \leq v_{k,l}$. We have the inequalities

$$\begin{aligned} &u_{0,0} \leq u_{0,1} \leq u_{0,2} \leq \ldots \leq u_{0,k} \leq \ldots v_{0,k} \leq \ldots v_{0,2} \leq v_{0,1} \leq v_{0,0} \\ &u_{1,0} \leq u_{1,1} \leq u_{1,2} \leq \ldots u_{1,k} \leq \ldots \leq v_{1,k} \leq \ldots v_{1,2} \leq v_{1,1} \leq v_{1,0} \\ &\quad\vdots \\ &u_{k,0} \leq u_{k,1} \leq u_{k,2} \leq \ldots u_{k,k} \leq \ldots \leq v_{k,k} \leq \ldots v_{k,2} \leq v_{k,1} \leq v_{k,0} \end{aligned}$$

and so on which in turn imply

$$u_{0,0} \leq u_{1,1} \leq \ldots \leq u_{k,k} \leq \ldots \leq v_{k,k} \leq \ldots v_{1,1} \leq v_{0,0}.$$

Moreover, by assertion (c), $u_{k,0} \geq \lambda_k v_{k,0}$ and

$$\begin{aligned} u_{k,1} &= A(u_{k,0}, v_{k,0}) \geq A(\lambda_k v_{k,0}, v_{k,0}) = A\big(\lambda_k v_{k,0}, \tfrac{1}{\lambda_k} \lambda_k v_{k,0}\big) \\ &\geq \lambda_k A(v_{k,0}, \lambda_k v_{k,0}) \geq \lambda_k A(v_{k,0}, u_{k,0}) = \lambda_k v_{k,1} \end{aligned}$$

and so on, i.e. $u_{k,k} \geq \lambda_k v_{k,k}$, we get

$$v_{k,k} - u_{k,k} \leq v_{k,k} - \lambda_k v_{k,k} = (1-\lambda_k) v_{k,k} \leq (1-\lambda_k) v_{0,0}.$$

Since $\lambda_k \to 1$ and P is normal, using a similar reasoning as in Theorem 2.1 we see that there exists $x^* \in P$ such that $A(x^*, x^*) = x^*$.

`Necessity`: Assume that x^* is a fixed point of A, i.e. $A(x^*, x^*) = x^*$. Denoting $u_{k,0} = \frac{k+1}{k+2} x^*$ and $v_{k,0} = \frac{k+2}{k+1} x^*$ we get

$$\begin{aligned}
A(u_{k,0}, v_{k,0}) &= A\Big(\frac{k+1}{k+2}x^*, \frac{k+2}{k+1}x^*\Big) \geq \frac{k+1}{k+2} A(x^*, x^*) = u_{k,0} \\
v_{k,0} &= \frac{k+2}{k+1}x^* = \frac{k+2}{k+1}A(x^*, x^*) = \frac{k+2}{k+1}A\Big(\frac{k+1}{k+2}v_{k,0}, \frac{k+2}{k+1}u_{k,0}\Big) \\
&\geq \frac{k+2}{k+1} \times \frac{k+1}{k+2} A(v_{k,0}, u_{k,0}) = A(v_{k,0}, u_{k,0}) \\
u_{k,0} &= \frac{k+1}{k+2}x^* \leq \frac{k+2}{k+3}x^* = u_{k+1,0} \\
v_{k,0} &= \frac{k+2}{k+1}x^* \geq \frac{k+3}{k+2}x^* = v_{k+1,0} \\
u_{k,0} &= \frac{k+1}{k+2}x^* = \frac{k+1}{k+2} \times \frac{k+1}{k+2} v_{k,0} \geq \frac{k+1}{k+2} \times \frac{k}{k+2} v_{k,0}.
\end{aligned}$$

Thus, if we choose $\lambda_k = \frac{k(k+1)}{(k+2)^2}$, then $u_{k,0} \geq \lambda_k v_{k,0}$ and $\lambda_k \to 1$ as $k \to \infty$. This completes the proof ∎

Remark 4. According to the best of our knowledge, for sublinear operators A, sufficient and necessary conditions for the existence and uniqueness of fixed points have not been given in the literature so far.

4. Example

Let $E = C_B(\mathbb{R}^N)$ denote the set of all bounded continuous functions on $\mathbb{R}^N$. Equipped with the natural norm $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}^N\}$, E is a real Banach space. The set $P = C_B^+(\mathbb{R}^N)$ of non-negative functions in $C_B(\mathbb{R}^N)$ is a normal and solid cone in $C_B(\mathbb{R}^N)$. We choose $h \equiv 1 \in E$ and consider the integral equation

$$x(t) = (Ax)(t) = \int_{\mathbb{R}^N} k(t,s)\Big[2 + x(s)^{\alpha(\|x\|)} + x(s)^{-\alpha(\|x\|)}\Big] ds \qquad (4)$$

where $\alpha : (0, +\infty) \to (0, 1)$ is non-decreasing.

Proposition 4.1. *Assume that* $k : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ *is continuous with* $k(t,s) \geq 0$ *and* $\frac{1}{31} \leq \int_{\mathbb{R}^N} k(t,s)\,ds \leq \frac{9}{130}$ $(t \in \mathbb{R}^N)$. *Then equation* (4) *has exactly one positive solution* x^* *satisfying* $0.1 \leq x^*(t) \leq 0.9$ $(t \in \mathbb{R}^N)$. *Moreover, starting from* $(x_0(t), y_0(t)) \in [0.1, 0.9] \times [0.1, 0.9]$ *and constructing successively the sequence* (2), *the corresponding results hold.*

Proof. Equation (4) can be written in the form $x = A(x,x)$, where $A(x,y) = A_1 x + A_2 y$ with

$$\left.\begin{aligned} (A_1 x)(t) &= \int_{\mathbb{R}^N} k(t,s)\big[2 + x(s)^{\alpha(\|x\|)}\big]ds \\ (A_2 y)(t) &= \int_{\mathbb{R}^N} k(t,s) y(s)^{-\alpha(\|y\|)} ds \end{aligned}\right. \qquad (t \in \mathbb{R}^N).$$

Let $u_0 = 0.1$ and $v_0 = 0.9$. Since $A : P_h \times P_h \to P_h$ is non-decreasing in x and non-increasing in y, for all $0 < t < 1$ and $u, v \in P_h$ we get

$$\begin{aligned} A\Big(tu, \frac{v}{t}\Big) &= \int_{\mathbb{R}^N} k(t,s)\Big[2 + t^{\alpha(t\|u\|)} u^{\alpha(t\|u\|)} + t^{\alpha(\frac{1}{t}\|v\|)} v^{-\alpha(\frac{1}{t}\|v\|)}\Big]ds \\ &\geq \int_{\mathbb{R}^N} k(t,s)\Big[2t^{\alpha(\frac{1}{t})} + t^{\alpha(t)} u^{\alpha(\|u\|)} + t^{\alpha(\frac{1}{t})} v^{-\alpha(\|v\|)}\Big]ds \\ &\geq t^{\alpha(\frac{1}{t})} \int_{\mathbb{R}^N} k(t,s)\Big[2 + u^{\alpha(\|u\|)} + v^{-\alpha(\|v\|)}\Big]ds \\ &= t^{\alpha(\frac{1}{t})} A(u,v) \\ &= t^{\beta(t)} A(u,v), \end{aligned}$$

and

$$\begin{aligned} A(u_0, v_0) &= \int_{\mathbb{R}^N} k(t,s)\big[2 + 0.1^{\alpha(0.1)} + 0.9^{-\alpha(0.1)}\big]ds \\ &\geq \int_{\mathbb{R}^N} k(t,s)\big[2 + 0.1 + 1\big]ds \\ &\geq 0.1 = u_0 \\ A(v_0, u_0) &= \int_{\mathbb{R}^N} k(t,s)\big[2 + 0.9^{\alpha(0.9)} + 0.1^{-\alpha(0.1)}\big]ds \\ &\leq \int_{\mathbb{R}^N} k(t,s)\big[2 + 10 + 1\big]ds \\ &\leq 0.9 = v_0. \end{aligned}$$

So all conditions of Theorem 2.1 are satisfied. Consequently, A has exactly one fixed point x^* in $[0.1, 0.9]$, and this fixed point is the unique positive solution of equation (4) in [0.1, 0.9].

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