

Pseudodifferential Operators on \mathbb{R}^n with Variable Shifts

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Abstract. The aim of the paper is the study of pseudodifferential operators with shifts of the form

$$Au(x) = \sum_{j=1}^N a_j(x, D)V_{h_j} + \sum_{j=1}^N b_j(x, D)T_{g_j}$$

where $a_j(x, D) \in OPS_{1,0}^m$ and $b_j(x, D) \in OPS_{1,0}^{m-\varepsilon}$ ($\varepsilon > 0$) are pseudodifferential operators in the Hörmander classes, and V_{h_j} and T_{g_j} are shift operators of the form

$$\begin{aligned} V_{h_j}u(x) &= u(x - h_j) \\ T_{g_j}u(x) &= u(x - g_j(x)) \end{aligned} \quad (x \in \mathbb{R}^n)$$

where $h_j \in \mathbb{R}^n$ and the mappings $g_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$ have infinitely differentiable coordinate functions bounded with all their derivatives. We will investigate the Fredholm and semi-Fredholm properties of the operator A acting from $H^s(\mathbb{R}^n)$ into $H^{s-m}(\mathbb{R}^n)$ applying the limit operators method.

Keywords: Pseudodifferential operators, shifts, limit operators method

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0. Introduction

The aim of the paper is the study of the Fredholm and semi-Fredholm properties of the operator

$$Au(x) = \sum_{j=1}^N a_j(x, D)V_{h_j} + \sum_{j=1}^N b_j(x, D)T_{g_j} \quad (1)$$

acting from $H^s(\mathbb{R}^n)$ into $H^{s-m}(\mathbb{R}^n)$ where

$$a_j(x, D) \in OPS_{1,0}^m \quad \text{and} \quad b_j(x, D) \in OPS_{1,0}^{m-\varepsilon} \quad (\varepsilon > 0)$$

are pseudodifferential operators in the well-known Hörmander classes (see, for instance, [20]), and V_{h_j} and T_{g_j} are shift operators of the form

$$\begin{aligned} V_{h_j}u(x) &= u(x - h_j) \\ T_{g_j}u(x) &= u(x - g_j(x)) \end{aligned} \quad (x \in \mathbb{R}^n)$$

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where $h_j \in \mathbb{R}^n$ and the coordinate functions of the mappings $g_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are infinitely differentiable and bounded with all their derivatives. Moreover, we suppose that g_j are slowly oscillating at infinity.

Note that the study of integral operators with constant shifts on the half-line \mathbb{R}_+ goes back to the well-known book [5] (see also the recent monograph [4]). Differential-difference operators of form (1), where $g_j \in \mathbb{R}^n$ and the coefficients satisfy some additional conditions, are studied in the papers [13, 14], and pseudodifferential operators with shifts on compact manifolds are investigated in the books [1 - 3] (see also the book [9] dealing with ordinary differential operators with shifts). One can find in these books an extensive bibliography on the topic.

But this paper is the first where general pseudodifferential operators with variable shifts are studied. Our approach is essentially different from those of the cited papers and is based on the limit operators method. This method allows us to reduce the investigation of the Fredholm property of operators (1) to the problem of invertibility of limit operators with a simpler structure than that of the original operators (1). In our case, the limit operators are of two kinds:

1) pseudodifferential operators with constant shifts

2) operators of the form $\sum_{j=1}^M c_j V_{h_j}$, where c_j are infinitely differentiable functions bounded with all their derivatives.

The method of limit operators has been developed in the papers [10 - 12, 15, 17, 18] for the study of the Fredholm property of wide classes of pseudodifferential and convolution operators on \mathbb{R}^n and \mathbb{Z}^n . Note also that the method of limit operators recently was applied to the investigation of one-dimensional singular integral operators with slowly oscillating shifts [8].

Here we use an abstract scheme for the limit operators method presented in the paper [19].

The structure of the paper is as follows:

In Section 1 we present the abstract scheme of the limit operators method, and in Section 2 an auxiliary material on pseudodifferential operators needed in what follows. In Section 3 we consider a C^* -algebra generated by pseudodifferential operators of zero order with shifts and apply the abstract scheme of Section 1 for the investigation of operators in this algebra. At last, in Section 4 we use the results of Section 3 for the study of the Fredholm property of operators (1).

1. Axiomatic approach to the limit operators method

We start with recalling the axiomatic scheme for the application of the limit operators method developed in [19]. Let H be a Hilbert space and $L(H)$ the C^* -algebra of all bounded linear operators acting on H . Suppose that we are given

(A1) operators $P, \widehat{P} \in L(H)$ with $P\widehat{P} = \widehat{P}P = P$.

(A2) a countable set $\{U_\alpha\}_{\alpha \in \Lambda}$ of unitary operators on H such that, with $P_\alpha = U_\alpha P U_\alpha^*$

and $\widehat{P}_\alpha = U_\alpha \widehat{P} U_\alpha^*$,

$$\sum_{\alpha \in \Lambda} \|P_\alpha u\|^2 = \|u\|^2 \quad \text{and} \quad \sum_{\alpha \in \Lambda} \|\widehat{P}_\alpha u\|^2 \leq C \|u\|^2 \quad (2)$$

for all $u \in H$ with a constant $C > 0$ independent of u .

(A3) a sequence $(W_k)_{k \in \mathbb{N}}$ of unitary operators on H and an associated sequence $(D_k)_{k \in \mathbb{N}}$ of mappings from Λ into itself such that $W_k U_\alpha = U_{D_k(\alpha)} W_k$ for all $\alpha \in \Lambda$ and $k \in \mathbb{N}$, and such that the operators $\widehat{P}_k = W_k \widehat{P} W_k^*$ converge strongly to the identity operator on H . We also set $P_k = W_k P W_k^*$ and $P_{k,\alpha} = W_k P_\alpha W_k^*$ as well as $\widehat{P}_{k,\alpha} = W_k \widehat{P}_\alpha W_k^*$.

(A4) a bounded sequence $(Q_r)_{r \in \mathbb{N}}$ of operators in $L(H)$ such that:

- there is a distinguished set \mathcal{B} of sequences in Λ which contains all sequences (β_m) for which there exist a $k \in \mathbb{N}$ and a sequence $r_m \rightarrow \infty$ in \mathbb{N} such that

$$P_{k,\beta_m} Q_{r_m} \neq 0 \quad (m \in \mathbb{N}) \quad (3)$$

- every subsequence of a sequence in \mathcal{B} belongs to \mathcal{B}
- the set \mathcal{B} is invariant with respect to each of the mappings D_k , i.e. if $(\beta_m) \in \mathcal{B}$, then $(D_k \beta_m) \in \mathcal{B}$ for every k
- for each $r \in \mathbb{N}$ and each sequence $(\beta_m) \in \mathcal{B}$,

$$\lim_{m \rightarrow \infty} U_{\beta_m}^* Q_r U_{\beta_m} = I. \quad (4)$$

Since both U_α and W_k are unitary operators, one also has

$$\sum_{\alpha \in \Lambda} \|P_{k,\alpha} u\|^2 = \|u\|^2 \quad \text{and} \quad \sum_{\alpha \in \Lambda} \|\widehat{P}_{k,\alpha} u\|^2 \leq C \|u\|^2$$

for all $u \in H$ and $k \in \mathbb{N}$ and

$$P_{k,\alpha} \widehat{P}_{k,\alpha} = \widehat{P}_{k,\alpha} P_{k,\alpha} = P_{k,\alpha}$$

for all $\alpha \in \Lambda$ and $k \in \mathbb{N}$.

Definition 1. We say that the operator A_β is the *limit operator* of $A \in L(H)$ with respect to the sequence $\beta = (\beta_j) \in \mathcal{B}$ if there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$

$$\lim_{j \rightarrow \infty} \|(U_{\beta_j}^* A U_{\beta_j} - A_\beta) \widehat{P}_k\| = \lim_{j \rightarrow \infty} \|(\widehat{P}_k)^* (U_{\beta_j}^* A U_{\beta_j} - A_\beta)\| = 0.$$

The set of all limit operators of A with respect to sequences in \mathcal{B} will be denoted by $\lim_{\mathcal{B}}(A)$.

The limit operators have the following elementary properties:

Let $\beta \in B$ and let $A, B \in L(H)$ be operators for which the limit operators A_β and B_β exist. Then:

- (a) $\|A_\beta\| \leq \|A\|$.
- (b) $(A + B)_\beta$ exists and $(A + B)_\beta = A_\beta + B_\beta$.
- (c) $(A^*)_\beta$ exists and $(A^*)_\beta = (A_\beta)^*$.
- (d) If $C, C_n \in L(H)$ are operators with $\|C - C_n\| \rightarrow 0$ and if the limit operators $(C_n)_\beta$ exist for all sufficiently large n , then C_β exists and $\|C_\beta - (C_n)_\beta\| \rightarrow 0$.

Definition 2. Let $\mathcal{A}_0(H, \{P_{k,\alpha}\})$ denote the set of all operators $A \in L(H)$ with the following properties:

- (a) $\lim_{k \rightarrow \infty} \|[P_{k,\alpha}, A]\| = 0$ and $\lim_{k \rightarrow \infty} \|[P_{k,\alpha}, A^*]\| = 0$ uniformly with respect to $\alpha \in \Lambda$.
- (b) Every sequence in \mathcal{B} possesses a subsequence β for which the limit operator A_β exists.
- (c) There is a $k_0 \in \mathbb{N}$ such that $P_{k,\alpha}A = P_{k,\alpha}A\widehat{P}_{k,\alpha}$ for all $k \geq k_0$.

Further, let $\mathcal{A}(H, \{P_{k,\alpha}\})$ denote the closure of $\mathcal{A}_0(H, \{P_{k,\alpha}\})$ in $L(H)$.

Let $\nu(A) = \inf_{\|f\|=1} \|Af\|$ refer to the lower norm of the operator $A \in L(H)$. It is well-known that A is invertible from the left if and only if $\nu(A) > 0$ and invertible from the right if and only if $\nu(A^*) > 0$. Thus, A is invertible if and only if both $\nu(A) > 0$ and $\nu(A^*) > 0$.

For every non-zero (but not necessarily closed) subspace L of H we also consider the lower norm of the restriction $A|_L$ of A onto L . If, in particular, L is the range of a non-zero operator $P \in L(H)$, then we call

$$\nu(A|_{P(H)}) = \inf_{\|Pf\|=1} \|APf\|$$

the lower norm of A relative to P . The following result has been proved in [19].

Theorem 3. Let $A \in \mathcal{A}(H)$. Then

$$\liminf_{r \rightarrow \infty} \nu(A|_{Q_r(H)}) = \inf_{A_\beta \in \lim_{\mathcal{B}}(A)} \nu(A_\beta). \tag{5}$$

2. Auxiliary material on pseudodifferential operators

We say that $A = Op(a) = a(x, D)$ is a pseudodifferential operator in the class $OPS_{\rho,0}^0$ ($0 \leq \rho \leq 1$) with symbol $a(x, \xi)$ if

$$(Au)(x) = Op(a)u(x) = a(x, D)u(x) = \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} a(x, \xi) e^{i(x-y, \xi)} u(y) dy$$

$$u \in C_0^\infty(\mathbb{R}^n)$$

and $a(x, \xi)$ satisfies the estimates

$$|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-\rho|\alpha|} \quad (\langle \xi \rangle = (1 + |\xi|^2)^{1/2}, C_{\alpha\beta} > 0) \tag{6}$$

for all multiindices $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$, and A is a pseudodifferential operator with double symbol $a(x, y, \xi) \in S_{\rho,0,0}^0$ if

$$(Au)(x) = Op_d(a)u(x) = \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} a(x, y, \xi) e^{i(x-y, \xi)} u(y) dy$$

$$u \in C_0^\infty(\mathbb{R}^n)$$

where a satisfies the estimates

$$|\partial_x^\beta \partial_y^\gamma \partial_\xi^\alpha a(x, y, \xi)| \leq C_{\alpha\beta\gamma} \langle \xi \rangle^{m-\rho|\alpha|} \quad (C_{\alpha\beta\gamma} > 0)$$

for all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ and $\gamma = (\gamma_1, \dots, \gamma_n)$. We denote the class of such operators as $OPS_{\rho,0,0}^0$.

It is well-known that every pseudodifferential operator $A = Op_d(a)$ with double symbol $a \in S_{\rho,0,0}^0$ is a pseudodifferential operator in the class $OPS_{\rho,0}^0$ and its symbol $\sigma_A(x, \xi)$ is defined as

$$\sigma_A(x, \xi) = \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^{2n}} a(x, x+y, \xi+\eta) e^{-i(y,\eta)} dy d\eta$$

where the double integral is understood as oscillatory [20, 21]. It follows from the well-known Calderon-Vaillancourt theorem [21] that $A \in OPS_{0,0}^0$ is a bounded operator in $L^2(\mathbb{R}^n)$ and

$$\|Au\| \leq C \sum_{|\alpha|+|\beta| \leq m} \sup_{(x,\xi) \in \mathbb{R}^{3n}} |\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \tag{7}$$

where the constant $C > 0$ and $m \in \mathbb{N}$ are independent of A .

Definition 4 (see [6]). We say that the symbol $a \in S_{\rho,0}^m$ is *slowly oscillating at infinity* if for all multi-indices α, β

$$|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq C_{\alpha,\beta}(x) \langle \xi \rangle^{m-\rho|\alpha|}$$

where $\lim_{x \rightarrow \infty} C_{\alpha,\beta}(x) = 0$ for all α and $\beta \neq 0$. We denote the *class of slowly oscillating symbols* by $SL_{\rho,0}^m$ and the corresponding *class of pseudodifferential operators* by $OPSL_{\rho,0}^m$.

Proposition 5 (see [6]). Let $A_j = Op(a_j) \in OPSL_{\rho,0}^{m_j}$ ($j = 1, 2; 0 \leq \rho \leq 1$). Then $A_1 A_2 \in OPSL_{\rho,0}^{m_1+m_2}$ and $A_1 A_2 = Op(b)$ where $b(x, \xi) = a(x, \xi)b(x, \xi) + r(x, \xi)$ and r satisfies the estimates

$$|\partial_x^\beta \partial_\xi^\alpha r(x, \xi)| \leq C_{\alpha,\beta}(x) \langle \xi \rangle^{m_1+m_2-\rho-\rho|\alpha|}$$

where $\lim_{x \rightarrow \infty} C_{\alpha,\beta}(x) = 0$ for all α and β . If $\rho > 0$, then $Op(r) : H^s(\mathbb{R}^n) \rightarrow H^{s-m_1-m_2}(\mathbb{R}^n)$ is a compact operator.

We will denote by $C_b^\infty(\mathbb{R}^n)$ the class of C^∞ -functions on \mathbb{R}^n bounded with all their derivatives.

Proposition 6. Let $\varphi \in C_b^\infty(\mathbb{R}^n)$, let φ_α and $\varphi_{r,\alpha}$ are defined as $\varphi_\alpha(y) = \varphi(y - \alpha)$ and $\varphi_{r,\alpha}(x) = \varphi_\alpha(\frac{y}{r})$ for $y \in \mathbb{R}^n$ and let $A \in OPS_{0,0}^0$. Then

$$\lim_{r \rightarrow \infty} \|[A, \varphi_{r,\alpha} I]\| = \lim_{r \rightarrow \infty} \|[A, \varphi_{r,\alpha}(D)]\| = 0 \tag{8}$$

uniformly with respect to $\alpha \in \mathbb{Z}^n$.

Formula (8) follows easily from the formula for composition of pseudodifferential operators and estimate (7).

Let $f(x) \in C_0^\infty(\mathbb{R}^n)$ with $f(-x) = f(x)$, $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}^n$ and

$$f(x) = \begin{cases} 1 & \text{if } |x_i| \leq \frac{2}{3} \\ 0 & \text{if } |x_i| \geq \frac{3}{4} \end{cases} \quad (i = 1, \dots, n)$$

and let $f_k(x) = f(\frac{x}{k})$ ($k \in \mathbb{N}$).

Proposition 7. Let $a \in S_{0,0}^0$ and $a^k(x, y, \xi) = a(x, \xi) f_k(x - y)$. Then

$$\lim_{k \rightarrow \infty} \|Op(a) - Op_d(a^k)\| = 0.$$

Proof. It is easy to check that $a^k(x, y, \xi) \in S_{0,0,0}^0$. Then

$$Op(a) - Op_d(a^k) = Op_d(b^k)$$

where $b^k(x, y, \xi) = a(x, \xi) \psi_k(x - y)$ with $\psi_k = 1 - f_k$. The symbol of $B_k = Op_d(b^k)$ is given as

$$\begin{aligned} \sigma_{B_k}(x, \xi) &= \frac{1}{(2\pi)^n} \iint a(x, \xi + \eta) \psi_k(y) e^{-i(y,\eta)} dy d\eta \\ &= \frac{1}{(2\pi)^n} \iint \langle y \rangle^{-2l_1} \langle D_\eta \rangle^{2l_1} \{ \langle \eta \rangle^{-2l_2} a(x, \xi + \eta) \langle D_y \rangle^{2l_2} \psi_k(y) \} e^{-i(y,\eta)} dy d\eta. \end{aligned}$$

Let $2l_1 > n + 1$ and $2l_2 > n$. Then

$$|\sigma_{B_k}(x, \xi)| \leq \frac{C}{k} \sum_{|\gamma| \leq 2l_1} \sup_{x, \xi} |\partial_\xi^\gamma a(x, \xi)|$$

where the constant $C > 0$ does not depend on a and k . In the same way we obtain the estimates

$$|\partial_\xi^\alpha \partial_x^\beta \sigma_{B_k}(x, \xi)| \leq \frac{C}{k} \sum_{|\gamma| \leq 2l_1 + |\alpha|} \sup_{x, \xi} |\partial_\xi^\gamma \partial_x^\beta a(x, \xi)|.$$

Now the Calderon-Vaillancourt theorem provides that $\lim_{k \rightarrow \infty} \|B_k\| = 0$ ■

Note that the operators $\sum_{j=1}^N a_j(x, D) V_{h_j}$ where $a_j(x, D) \in OPS_{1,0}^m$ belong to $OPS_{0,0}^m$.

In what follows we need some class of symbols $\tilde{S}_{1,0}^m \subset S_{1,0}^m$.

Definition 8. We say that a symbol $a \in \widetilde{S}_{1,0}^m$ if $a \in S_{1,0}^m$ and there exists a function $a_0 \in C_b^\infty(\mathbb{R}_x^n \times S^{n-1})$ such that

$$\lim_{|\xi| \rightarrow +\infty} \sup_{(x, \xi) \in \mathbb{R}_x^n \times \mathbb{R}_\xi^n \setminus \{0\}} \left| |\xi|^{-m} a(x, \xi) - a_0\left(x, \frac{\xi}{|\xi|}\right) \right| = 0.$$

Proposition 9. Let $a \in \widetilde{S}_{1,0}^0$ and a sequence $h_m \rightarrow \infty$. Then there exists a subsequence h_{m_k} and a symbol $a_h \in S_{1,0}^0$ such that, for all α, β and for an arbitrary compact $K \subset \mathbb{R}_x^n$,

$$\lim_{k \rightarrow \infty} \sup_{K \times \mathbb{R}_\xi^n} \left| \partial_x^\beta \partial_\xi^\alpha a(x + h_{m_k}, \xi) - \partial_x^\beta \partial_\xi^\alpha a_h(x, \xi) \right| = 0. \tag{9}$$

Proof. Let $\widetilde{\mathbb{R}}_\xi^n$ be the compactification of \mathbb{R}_ξ^n obtained by association to each ray outgoing from the origin the infinitely distant point. All derivatives with respect to x of a symbol $a \in \widetilde{S}_{1,0}^0$ can be considered as continuous functions on $\mathbb{R}_x^n \times \widetilde{\mathbb{R}}_\xi^n$. By the Arzelà-Ascoli theorem the sequence $a(x + h_m, \xi)$ has a subsequence $a(x + h_{m_k}, \xi)$ such that

$$\lim_{k \rightarrow \infty} \sup_{K \times \mathbb{R}_\xi^n} \left| \partial_x^\beta a(x + h_{m_k}, \xi) - \partial_x^\beta a_h(x, \xi) \right| = 0$$

for each compact $K \subset \mathbb{R}_x^n$. Taking into account that for all $\alpha \neq 0$ and β

$$\lim_{\xi \rightarrow \infty} \sup_x \left| \partial_x^\beta \partial_\xi^\alpha a(x, \xi) \right| = 0$$

we obtain that there exists a subsequence h_{m_k} such that

$$\lim_{k \rightarrow \infty} \sup_{K \times \mathbb{R}_\xi^n} \left| \partial_x^\beta \partial_\xi^\alpha a(x + h_{m_k}, \xi) - \partial_x^\beta \partial_\xi^\alpha a_h(x, \xi) \right| = 0$$

for each compact $K \subset \mathbb{R}_x^n$ and for all multi-indices α and β . It follows from (9) that $a_h \in S_{1,0}^0$ ■

If we set

$$\varphi_\alpha^2(x) = \frac{f(x - \alpha)}{\sum_{\beta \in \mathbb{Z}^n} f(x - \beta)} \quad (\alpha \in \mathbb{Z}^n),$$

then

$$\sum_{\alpha \in \mathbb{Z}^n} \varphi_\alpha^2(x) = 1 \quad \text{and} \quad 0 \leq \varphi_\alpha(x) \leq 1 \quad (x \in \mathbb{R}^n).$$

To apply the abstract scheme of the limit operators method we set

$$U_\alpha = V_\alpha \quad (\alpha \in \mathbb{Z}^n), \quad (V_\alpha u)(x) = u(x - \alpha)$$

and

$$P = \varphi_0 I, \quad \widehat{P} = \phi I$$

for $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\text{supp } \phi = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $\phi(x) = 1$ on the ball $\{x \in \mathbb{R}^n : |x| \leq \frac{3}{4}\}$. The sequence $\{W_k\}_{k \in \mathbb{N}}$ is a sequence of dilations

$$(\gamma_k u)(x) = k^{-\frac{n}{2}} u\left(\frac{x}{k}\right).$$

Further,

$$\begin{aligned} P_\alpha &= (V_\alpha \varphi_0)I, & P_{k,\alpha} &= \gamma_k P_\alpha \gamma_k^{-1} \\ \widehat{P}_\alpha &= (V_\alpha \phi)I, & \widehat{P}_k &= \phi_k I, & \widehat{P}_{k,\alpha} &= \gamma_k (V_\alpha \phi) \gamma_k^{-1} \end{aligned}$$

where $\phi_k(x) = \phi_k(\frac{x}{k})$ and

$$Q_r = \chi_r I$$

where χ_r is the characteristic function of the set $\{x \in \mathbb{R}^n : |x| \geq r\}$ ($r \in \mathbb{N}$). It is evident that

$$\sum_{\alpha \in \mathbb{Z}^n} \|P_{k,\alpha} u\|^2 = \sum_{\alpha \in \mathbb{Z}^n} \|\varphi_{k,\alpha} u\|^2 = \|u\|^2$$

where $\varphi_{k,\alpha}(x) = \varphi(\frac{x}{k} - \alpha)$. It is easy to prove that

$$\sum_{\alpha \in \mathbb{Z}^n} \|\widehat{P}_{k,\alpha} u\|^2 \leq 2^n \|u\|^2.$$

Then conditions (A1) - (A4) are fulfilled with $\Lambda = \mathbb{Z}^n$ and \mathcal{B} being the set of all sequences in \mathbb{Z}^n tending to infinity.

Proposition 10. *Let $A = a(x, D) \in OPS_{1,0}^{\widetilde{0}}$. Then each sequence $\mathbb{Z}^n \ni h_m \rightarrow \infty$ has a subsequence h_{m_k} defining a limit operator $A_h \in OPS_{1,0}^0$.*

Proof. It follows from Proposition 9 that the sequence h_m has a subsequence h_{m_k} such that (9) holds. The Calderon-Vaillancourt theorem implies

$$\begin{aligned} & \|\widehat{P}_j (V_{h_{m_k}}^{-1} a(x, D) V_{h_{m_k}} - a_h(x, D))\| \\ &= \|\widehat{P}_j (a(x + h_{m_k}, D) - a_h(x, D))\| \\ &\leq \sup_{|x| \leq j, \xi \in \mathbb{R}^n} \sum_{|\alpha| + |\beta| \leq N} |\partial_x^\beta \partial_\xi^\alpha a(x + h_{m_k}, \xi) - \partial_x^\beta \partial_\xi^\alpha a_h(x, \xi)| \\ &\rightarrow 0 \end{aligned}$$

if $k \rightarrow \infty$, for every $j \in \mathbb{N}$. In the same way, applying the formulas of composition of pseudodifferential operators and the Calderon-Vaillancourt theorem, we obtain

$$\lim_{k \rightarrow \infty} \|(V_{h_{m_k}}^{-1} a(x, D) V_{h_{m_k}} - a_h(x, D)) \widehat{P}_j\| = 0$$

for every $j \in \mathbb{N}$ ■

Corollary 11. *The inclusion $OPS_{1,0}^{\widetilde{0}} \subset \mathcal{A}(L_2(\mathbb{R}^n))$ holds.*

Proposition 12. *Let $A = Op(a) \in OPSL_{1,0}^0 \cap OPS_{1,0}^{\widetilde{0}}$. Then all limit operators for A are invariant with respect to the operators V_h .*

Proof. By the Lagrange formula and by Definition 4,

$$\begin{aligned} & |a(x' + h_m, \xi) - a(x'' + h_m, \xi)| \\ &\leq \sup_{t \in (0,1)} \sum_{j=1}^n |\partial_{x_j} a((1-t)x' + tx'' + h_m, \xi)| |x' - x''| \\ &\rightarrow 0 \end{aligned}$$

if $h_m \rightarrow \infty$. Thus the limit $\lim_{m \rightarrow \infty} a(x + h_m, \xi) = a_h(\xi)$ does not depend on x . It means that the limit operator $A_h = a_h(D)$ is invariant with respect to shifts ■

3. Algebra of pseudodifferential operators of zero order with shifts

Let $g = (g_1, \dots, g_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, where

- (α) $g_j \in C_b^\infty(\mathbb{R}^n)$ for all $j = 1, \dots, n$
- (β) the mapping $F_g : \mathbb{R}^n \rightarrow \mathbb{R}^n, x \mapsto x - g(x)$ is invertible
- (γ) $\lim_{x \rightarrow \infty} \|dg(x)\| = 0$.

Proposition 13. *The set of the mappings $x \mapsto x - g(x)$ satisfying conditions (α) – (γ) is a group.*

Proof. Indeed:

1) Let $F_{g_1}(x) = x - g_1(x)$ and $F_{g_2}(x) = x - g_2(x)$. Then

$$(F_{g_2} \circ F_{g_1})(x) = F_{g_2 + g_1 \circ g_2}.$$

It is evident that $g_2 + g_1 \circ g_2$ satisfies conditions (α) – (γ) if g_1 and g_2 satisfy these conditions.

2) It follows from (β) that the mapping F_g is invertible. Hence

$$y = (F_g \circ F_g^{-1})(y) = F_g^{-1}(y) - (g \circ F_g^{-1})(y).$$

Thus

$$F_g^{-1}(y) = y + (g \circ F_g^{-1})(y)$$

where $g \circ F_g^{-1}$ satisfies conditions (α) – (γ) ■

We consider shift operators T_g of the form

$$(T_g u)(x) = u(x - g(x)).$$

The class of all shifts T_g where g satisfies conditions (α) – (γ) will be denoted by $\mathcal{R}(\mathbb{R}^n)$.

Proposition 14. *Let g satisfy conditions (α) – (γ). Then the operator T_g is bounded on $L^2(\mathbb{R}^n)$.*

Proof. We have

$$\|T_g u\|^2 = \int_{\mathbb{R}^n} |u(F_g(x))|^2 dx = \int_{\mathbb{R}^n} |u(y)|^2 |\det dF_g^{-1}(y)| dy \leq C \|u\|^2$$

where $C = \sup_{y \in \mathbb{R}^n} |\det dF_g^{-1}(y)| < \infty$ by conditions (β) and (γ) ■

Proposition 15.

- 1) Let $T_{g_1}, T_{g_2} \in \mathcal{R}(\mathbb{R}^n)$. Then $T_{g_1} T_{g_2} = T_{g_1 + g_2 \circ g_1} \in \mathcal{R}(\mathbb{R}^n)$.
- 2) Let $T_g \in \mathcal{R}(\mathbb{R}^n)$. Then T_g is invertible and $(T_g)^{-1} \in \mathcal{R}(\mathbb{R}^n)$ also. Moreover, $(T_g)^{-1} = T_{\tilde{g}}$ where $\tilde{g}(y) = -g(F_g^{-1}(y))$.
- 3) Let $T_g \in \mathcal{R}(\mathbb{R}^n)$. Then $(T_g)^* = |\det F_g(x)|(T_g)^{-1}$.

Proof. Assertions 1) and 2) follow from Proposition 13, assertion 3) can be proved by simple calculations ■

Proposition 16. *Let $T_g \in \mathcal{R}(\mathbb{R}^n)$. Then*

$$\lim_{\delta \rightarrow 0} \|[\varphi_{k,\alpha} I, T_g]\| = 0$$

uniformly with respect to $\alpha \in \mathbb{Z}^n$.

Proof. For every $u \in L^2(\mathbb{R}^n)$ one has

$$\begin{aligned} \|[\varphi_{k,\alpha} I, T_g]u\| &\leq \sup_{x \in \mathbb{R}^n} |\varphi_{k,\alpha}(x) - \varphi_{k,\alpha}(x + g(x))| \|T_g u\| \\ &\leq C \sup_{x \in \mathbb{R}^n} \left| \varphi\left(\frac{x}{k} - \alpha\right) - \varphi\left(\frac{x + g(x)}{k} - \alpha\right) \right| \|u\|. \end{aligned}$$

The function g is bounded due to assumption (α) and φ is uniformly continuous on \mathbb{R}^n . Then given $\varepsilon > 0$ there exist k_0 such that for $k > k_0$

$$\sup_{x \in \mathbb{R}^n, \alpha \in \mathbb{Z}^n} \left| \varphi\left(\frac{x}{k} - \alpha\right) - \varphi\left(\frac{x + g(x)}{k} - \alpha\right) \right| < \varepsilon.$$

This implies the assertion ■

Here are a few instances where requirements $(\alpha) - (\delta)$ are satisfied.

Example 17. If g is a constant function, then evidently $T_g = V_g \in \mathcal{R}(\mathbb{R}^n)$.

Example 18. Let

$$(T_g u)(x) = u(x - g(x))$$

and let conditions (α) and (γ) be fulfilled. If one of the conditions

$$\max_{1 \leq j \leq m} \sum_{k=1}^m \sup_x \left| \frac{\partial g_j(x)}{\partial x_k} \right| < 1 \quad \text{or} \quad \max_{1 \leq k \leq m} \sum_{j=1}^m \sup_x \left| \frac{\partial g_j(x)}{\partial x_k} \right| < 1 \quad (10)$$

is satisfied, then $T_g \in \mathcal{R}(\mathbb{R}^n)$. Indeed, conditions (10) imply that

$$F_g : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad x \mapsto x - g(x)$$

is a contraction. Thus, by the Banach fixed point theorem, F_g is invertible, that is condition (β) holds.

Proposition 19.

(a) *If $A = a(x, D)$ is in $OP S_{1,0}^m$ or in $OP \tilde{S}_{1,0}^m$ and $T_g \in \mathcal{R}(\mathbb{R}^n)$, then also $T_g A T_g^{-1}$ is in $OP S_{1,0}^m$ or in $OP \tilde{S}_{1,0}^m$, respectively. Moreover,*

$$T_g A T_g^{-1} = OP(a(F(x), (dF(x))^t \xi) |\det F'(x)| + R \quad (11)$$

where $R \in OP S_{1,0}^{m-1}$.

(b) *If $A = a(x, D) \in OP SL_{1,0}^m$, then $R = Op(r(x, \xi))$, where*

$$|\partial_x^\beta \partial_\xi^\alpha r(x, \xi)| \leq C_{\alpha,\beta}(x) \langle \xi \rangle^{m-1-|\alpha|}$$

with

$$\lim_{x \rightarrow \infty} C_{\alpha,\beta}(x) = 0 \quad (12)$$

for all α and β .

Proof. The proof follows from the well-known theorem on the change of variables in pseudodifferential operators (see, for instance, [20: Chapter 1/p. 31 - 38]) ■

Let us consider the operators of the form

$$A_{MN} = \sum_{j=1}^M a_j(x, D)V_{h_j} + \sum_{j=1}^N b_j(x, D)T_{g_j} \tag{13}$$

where $a_j(x, D) \in OPS_{1,0}^{\tilde{0}}$, $b_j(x, D) \in OPS_{1,0}^{-\varepsilon}$ ($\varepsilon > 0$) and $T_{g_j} \in \mathcal{R}(\mathbb{R}^n)$.

Proposition 20. *The sum, product and adjoint for operators of form (13) again is an operator of form (13).*

Proof. The proof for the sum is evident. Let us consider the product

$$\begin{aligned} & (a_1(x, D)V_{h_1} + b_1(x, D)T_{g_1})(a_2(x, D)V_{h_2} + b_2(x, D)T_{g_2}) \\ &= a_1(x, D)a_2(x - h_1, D)V_{h_1+h_2} + (b_1(x, D)T_{g_1}a_2(x, D)T_{g_1}^{-1})T_{g_1}V_{h_2} \\ & \quad + a_1(x, D)b_2(x - h_1, D)V_{h_1}T_{g_2} + (b_1(x, D)T_{g_1}b_2(x, D)T_{g_1}^{-1})T_{g_1}T_{g_2} \\ &= a(x, D)V_{h_1+h_2} + c_1(x, D)T_{g_1}V_{h_2} + c_2(x, D)V_{h_1}T_{g_2} + c_3(x, D)T_{g_1}T_{g_2} \end{aligned}$$

where

$$a(x, D) = a_1(x, D)a_2(x - h_1, D) \in OPS_{1,0}^{\tilde{0}}$$

and by Proposition 19 and according to the formula for composition of pseudodifferential operators [20] $c_j(x, D) \in OPS_{1,0}^{-\varepsilon}$. By Proposition 15,

$$T_{g_1}V_{h_2}, V_{h_1}T_{g_2}, T_{g_1}T_{g_2} \in \mathcal{R}(\mathbb{R}^n).$$

Thus the product of operators of form (13) is again an operator of form (13). At last, simple calculations, using the fact that the adjoint operator of $a(x, D) \in OPS_{1,0}^{-\varepsilon}$ ($\varepsilon \leq 0$) in $L_2(\mathbb{R}^n)$ is an operator in $OPS_{1,0}^{-\varepsilon}$ demonstrate that the adjoint operator to A_{MN} is again an operator of form (13) ■

Definition 21. We denote by \mathcal{P} the closure in $B(L^2(\mathbb{R}^n))$ of the set of all operators (13) and by \mathcal{J} the closure in $B(L^2(\mathbb{R}^n))$ of the set of operators of the form

$$\sum_{j=1}^N b_j(x, D)T_{g_j} \tag{14}$$

where $b_j(x, D) \in OPS_{1,0}^{-\varepsilon}$ ($\varepsilon > 0$) and $T_{g_j} \in \mathcal{R}(\mathbb{R}^n)$.

It follows from Proposition 20 that \mathcal{P} is a C^* -algebra and \mathcal{J} is a two-sided ideal in \mathcal{P} .

Proposition 22. *Let $T \in \mathcal{J}$ and $\chi \in C_0^\infty(\mathbb{R}^n)$. Then χT and $T\chi$ are compact operators.*

Proof. It is evident for operators of form (14). For the proof of the statement for operators in \mathcal{J} we use standard density arguments ■

Let us consider the question of existence of limit operators for operators in the algebra \mathcal{P} with respect to the set $\{V_\alpha\}_{\alpha \in \mathbb{Z}^n}$.

Proposition 23. *The C^* -algebra \mathcal{P} is contained in $\mathcal{A}(L_2(\mathbb{R}^n), \{P_{k,\alpha}\})$.*

Proof. 1) Let A_{MN} be an operator of form (14). Then Propositions 6, 7 and 16 provide the validity of assertions (a) and (c) of Definition 2. For general operators in \mathcal{P} the validity of these assertions follows from usual density arguments.

2) Let us consider the existence of limit operators for generators of the algebra \mathcal{P} . First we consider the operator $A = a(x, D)V_h$, where $a(x, D) \in OPS_{1,0}^{\tilde{0}}$. Then $V_{p_m}^{-1}AV_{p_m} = a(x + p_m, D)V_h$. It follows from Proposition 9 that there exists a subsequence p_{m_k} and an operator $a_p(x, D) \in OPS_{1,0}^0$ such that for every $j \in \mathbb{N}$

$$\begin{aligned} & \lim_{k \rightarrow \infty} \|\widehat{P}_j(a(x + p_{m_k}, D) - a_p(x, D))\| \\ &= \lim_{k \rightarrow \infty} \|(a(x + p_{m_k}, D) - a_p(x, D))\widehat{P}_j\| \\ &= 0. \end{aligned}$$

The last equalities imply

$$\lim_{k \rightarrow \infty} \|\widehat{P}_j(a(x + p_{m_k}, D)V_h - a_p(x, D)V_h)\| = 0$$

for each $j \in \mathbb{N}$ and

$$\lim_{k \rightarrow \infty} \|(a(x + p_{m_k}, D)V_h - a_p(x, D)V_h)\widehat{P}_j\| = 0$$

for large enough j . Thus $a_p(x, D)V_h$ is a limit operator of $a(x, D)V_h$.

Let us consider the operator $B = b(x, D)T_g$ where $b(x, D) \in OPS_{1,0}^{-\varepsilon}$ ($\varepsilon > 0$) and $T_g \in \mathcal{R}(\mathbb{R}^n)$. Then

$$V_{-p_m}BV_{p_m} = b(x + p_m, D)V_{p_m}^{-1}T_gV_{p_m}.$$

First we consider the operator $T_g \in \mathcal{R}(\mathbb{R}^n)$. Then one has

$$(V_{p_m}^{-1}T_gV_{p_m}u)(x) = u(x - g(x + p_m)).$$

Since the functions $x \mapsto g(x + p_m)$ are uniformly bounded with respect to $m \in \mathbb{N}$ and equicontinuous on compact subsets of \mathbb{R}^n , the Arzelà-Ascoli theorem implies the existence of a subsequence \tilde{p} of p such that the functions $x \mapsto g(x + \tilde{p}_m)$ converge uniformly on compacts in \mathbb{R}^n to a certain bounded function $g_{\tilde{p}}$. Note that the function $g_{\tilde{p}}$ is constant, that is $g_{\tilde{p}} \in \mathbb{R}^n$.

We proceed with showing that the strong limit of the operators $V_{\tilde{p}_m}^{-1}T_gV_{\tilde{p}_m}$ as $m \rightarrow \infty$ exists and that

$$s\text{-}\lim_{m \rightarrow \infty} V_{\tilde{p}_m}^{-1}T_gV_{\tilde{p}_m} = T_{g_{\tilde{p}}}. \tag{15}$$

Let $u \in C_0^\infty(\mathbb{R}^n)$. Thus u is uniformly continuous on \mathbb{R}^n , and there exists a compact subset Ω of \mathbb{R}^n such that $u(x + g(\tilde{p}_m + x)) - u(x + g_{\tilde{p}}) = 0$ whenever $x \notin \Omega$ (recall that g is bounded). Further, it is evident from the definition of $g_{\tilde{p}}$ that, for arbitrary $\delta > 0$,

there exists a $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$ and all $x \in \Omega$, $|g(\tilde{p}_m + x) - g_{\tilde{p}}| < \delta$. Since u is uniformly continuous, for each $\varepsilon > 0$ there exists a $m_0 \in \mathbb{N}$ such that

$$\sup_{x \in \Omega} |u(x - g(\tilde{p}_m + x)) - u(x + g_{\tilde{p}})| < \varepsilon \quad (m \geq m_0).$$

Thus $\lim_{k \rightarrow \infty} V_{\tilde{p}_m}^{-1} T_g V_{\tilde{p}_m} u = T_{g_{\tilde{p}}} u$ for every $u \in C_0^\infty(\mathbb{R}^n)$. Since these functions form a dense subset of $L^2(\mathbb{R}^n)$, this implies (15).

Let the sequence \tilde{p}_m be such that the strong limit (15) exists and for every $j \in \mathbb{N}$

$$\begin{aligned} & \lim_{m \rightarrow \infty} \|\widehat{P}_j(b(x + \tilde{p}_m, D) - b_{\tilde{p}}(x, D))\| \\ &= \lim_{m \rightarrow \infty} \|(b(x + \tilde{p}_m, D) - b_{\tilde{p}}(x, D))\widehat{P}_j\| \\ &= 0. \end{aligned} \tag{16}$$

The operators $\widehat{P}_j b(x + \tilde{p}_m, D)$ and $b(x + \tilde{p}_m, D)\widehat{P}_j$ are compact by Proposition 22, hence $\widehat{P}_j b_{\tilde{p}}(x, D)$ and $b_{\tilde{p}}(x, D)\widehat{P}_j$ are compact also. Thus applying (15) and (16) we obtain

$$\lim_{m \rightarrow \infty} \|\widehat{P}_j(b(x + \tilde{p}_m, D)V_{\tilde{p}_m}^{-1}T_g V_{\tilde{p}_m} - b_{\tilde{p}}(x, D)T_{g_{\tilde{p}}})\| = 0.$$

For the dual condition observe that, due to the boundedness of g , for every fixed j one can find an N such that

$$\begin{aligned} \widehat{P}_N V_{\tilde{p}_m}^{-1} T_g V_{\tilde{p}_m} \widehat{P}_j &= V_{\tilde{p}_m}^{-1} T_g V_{\tilde{p}_m} \widehat{P}_j \\ \widehat{P}_N T_{g_{\tilde{p}}} \widehat{P}_j &= T_{g_{\tilde{p}}} \widehat{P}_j \end{aligned}$$

for all m . Consequently,

$$\begin{aligned} & \|(b(x + \tilde{p}_m, D)V_{\tilde{p}_m}^{-1}T_g V_{\tilde{p}_m} - b_{\tilde{p}}(x, D)T_{g_{\tilde{p}}})\widehat{P}_j\| \\ & \leq \|(b(x + \tilde{p}_m, D) - b_{\tilde{p}}(x, D))\widehat{P}_N V_{\tilde{p}_m}^{-1}T_g V_{\tilde{p}_m} \widehat{P}_j\| \\ & \quad + \|b_{\tilde{p}}(x, D)\widehat{P}_N (V_{\tilde{p}_m}^{-1}T_g V_{\tilde{p}_m} - T_{g_{\tilde{p}}})\widehat{P}_j\|. \end{aligned} \tag{17}$$

The first term in the right side part of (17) tends to 0 by (16) and by uniform boundedness of the sequence $V_{\tilde{p}_m}^{-1}T_g V_{\tilde{p}_m}$, the second term tends to 0 by formula (15) and by compactness of $b_{\tilde{p}}(x, D)$.

Thus we proved that if A_{MN} has form (13) and $p_m \rightarrow \infty$, there exists a subsequence \tilde{p}_m which defines the limit operator $(A_{MN})_{\tilde{p}}$. The assertion of the proposition for arbitrary operators in \mathcal{P} follows from property (d) of limit operators. Thus assertion (d) of Definition 2 holds ■

We denote by $\text{Lim}_\infty A$ the set of all limit operators A_α of A defined by the sequences $\{P_{k, \alpha_m}\}$ with $\alpha_m \rightarrow \infty$.

Theorem 3 yields the following result.

Theorem 24. *Let $A \in \mathcal{P}$. Then*

$$\liminf_{r \rightarrow \infty} \nu(A|_{Q_r(L^2(\mathbb{R}^n))}) > 0 \tag{18}$$

if and only if

$$\inf \{ \nu(A_\alpha) : A_\alpha \in \text{Lim}_\infty(A) \} > 0. \tag{19}$$

Let, for $\psi \in C_b^\infty(\mathbb{R}^n)$, ψ_r be defined as $\psi_r(x) = \psi(\frac{x}{r})$. We denote by \mathcal{D}' the subset of bounded operators A in $L(L^2(\mathbb{R}^n))$ such that

$$\lim_{r \rightarrow \infty} \|[A, \psi_r I]\| = 0$$

for every function $\psi \in C_b^\infty(\mathbb{R}^n)$ where $\psi_r I$ is the operator of multiplication by ψ . It is easy to see that \mathcal{D}' is a C^* -subalgebra of $L(L^2(\mathbb{R}^n))$ and, by Proposition 6, $\mathcal{P} \subset \mathcal{D}'$.

Let $\rho \in C^\infty(\mathbb{R}^n)$ with

$$\rho(x) = \begin{cases} 1 & \text{if } |x| \geq 2 \\ 0 & \text{if } |x| \leq 1 \end{cases}$$

and let $\rho_r(x) = \rho(\frac{x}{r})$. We introduce the two-sided ideal J' in \mathcal{D}' as containing all operators $A \in \mathcal{D}'$ such that

$$\lim_{r \rightarrow \infty} \|\rho_r A\| = \lim_{r \rightarrow \infty} \|A \rho_r I\| = 0.$$

Proposition 25. *Condition (18) holds if and only if there exists an operator $L' \in \mathcal{D}'$ such that*

$$L' A = I + T' \tag{20}$$

where $T' \in J'$.

Proof. Let condition (18) hold. Then there exist $\delta > 0$ and $r_0 > 0$ such that

$$(\chi_{r_0} A^* A \chi_{r_0} u, \chi_{r_0} u) \geq \delta^2 \|\chi_{r_0} u\|^2$$

where χ_{r_0} is the operator of multiplication by the characteristic function of the set $\{x \in \mathbb{R}^n : |x| > r_0\}$. This inequality implies the existence of an operator $B \in \mathcal{D}'$ such that $BA\chi_{r_0} = \chi_{r_0}$. This implies

$$BA = I - BA(1 - \chi_{r_0})I + (1 - \chi_{r_0})I.$$

Since $(1 - \chi_{r_0})I \in J'$, we obtain

$$T = -BA(1 - \chi_{r_0})I + (I - \chi_{r_0})I \in J'.$$

Conversely, let (20) hold. Multiplying (20) from the right by the operator $\rho_r I$ we obtain $LA\rho_r I = (I + T)\rho_r I$. We take r such that $\|T\rho_r I\| < 1$ and let r_0 be such that $\chi_{r_0}\rho_r I = \chi_{r_0}I$. Then $LA\chi_{r_0}I = (I + T\rho_r I)\chi_{r_0}I$. Thus

$$(I + T\rho_r I)^{-1}LA\chi_{r_0}I = \chi_{r_0}I.$$

The last equality implies (18) ■

Let us consider the dual case of applications of the abstract scheme. In this case the unitary operators $U_\alpha = \widehat{V}_\alpha$ ($\alpha \in \mathbb{Z}^n$) where $(\widehat{V}_\alpha u)(x) = e^{i(\alpha, x)}u(x)$, $W'_k = \gamma_{1/k}$ ($k \in \mathbb{N}$), $P'_{k, \alpha} = \varphi_{k, \alpha}(D)$, $\phi_k(D) = \widehat{P}'_k$, $Q'_r = \chi_r(D)$ and \mathcal{B} is the set of all subsequences $\alpha_m \in \mathbb{Z}^n$ such that $\alpha_m \rightarrow \infty$.

Proposition 26. *The C^* -algebra \mathcal{P} is contained in $\mathcal{A}_0(L_2(\mathbb{R}^n), \{P'_{k, \alpha}\})$.*

Proof. It is enough to check conditions (a) - (c) of Definition 2 for the generators of \mathcal{P} . For operators of the form $A = a(x, D)V_h$ where $a(x, D) \in OPS_{1,0}^0$, conditions (a) and (c) follow from Propositions 6 and 7 because $A \in OPS_{0,0}^0$.

Let us consider the operator $B = b(x, D)T_g$ where $b(x, D) \in OPS_{1,0}^{-\varepsilon}$ ($\varepsilon > 0$) and $T_g \in \mathcal{R}(\mathbb{R}^n)$. We set $\psi_{k, \alpha}(D) = I - \varphi_{k, \alpha}(D)$. Then

$$\begin{aligned} \|\psi_{k, \alpha}(D)b(x, D)T_g\| &= \|\widehat{V}_{\alpha k}^{-1}\psi_{k,0}(D)\widehat{V}_{\alpha k}b(x, D)\widehat{V}_{\alpha k}^{-1}T_g\| \\ &= \|\psi_{k,0}(D)b(x, D + \alpha k)\widehat{V}_{\alpha k}^{-1}T_g\| \\ &\leq C\|\psi_{k,0}(D)b(x, D + \alpha k)\|. \end{aligned}$$

Since $b(x, D) \in OPS_{1,0}^{-\varepsilon}$ ($\varepsilon > 0$) and $\text{supp } \psi_{k,0} \subset \{\xi \in \mathbb{R}^n : |\xi| \geq k\}$, by the Calderon-Vaillancourt theorem,

$$\lim_{k \rightarrow \infty} \|\psi_{k,0}(D)b(x, D + \alpha k)\| = 0$$

uniformly with respect to α . Thus,

$$\lim_{k \rightarrow \infty} \|\psi_{k, \alpha}(D)b(x, D)T_g\| = 0 \tag{21}$$

uniformly with respect to α . In the same way,

$$\begin{aligned} \|b(x, D)T_g\psi_{k, \alpha}(D)\| &= \|T_gT_g^{-1}b(x, D)T_g\psi_{k, \alpha}(D)\| \\ &\leq C\|T_g^{-1}b(x, D)T_g\psi_{k, \alpha}(D)\| \\ &\rightarrow 0 \end{aligned} \tag{22}$$

if $k \rightarrow \infty$, uniformly with respect to α , since $T_g^{-1}b(x, D)T_g$ is a pseudodifferential operator of the class $OPS_{1,0}^{-\varepsilon}$ ($\varepsilon > 0$). Formulas (21) and (22) imply assertion (a) of Definition 2. Assertion (c) of Definition 2 follows from Proposition 7.

Let us consider the existences of the limit operators with respect to the unitary operators \widehat{V}_α . Let $\mathbb{Z}^n \ni \alpha_m \rightarrow \infty$. Then

$$\widehat{V}_{\alpha_m}^{-1}A\widehat{V}_{\alpha_m} = a(x, D + \alpha_m)e^{i(h, \alpha_m)}V_h.$$

There exist an infinitely distant point η_ω corresponding to the point $\omega \in S^{n-1}$ and a subsequence $\alpha_{m_j} \rightarrow \eta_\omega$. The numerical sequence $e^{i(h, \alpha_{m_j})}$ is bounded, thus there exists a convergent subsequence. We will suppose that the same sequence $e^{i(h, \alpha_{m_j})}$ converges to a complex number q_α with $|q_\alpha| = 1$. Let

$$a(x, \eta_\omega) = \lim_{t \rightarrow \infty} a(x, t\omega),$$

the last limit being uniformly with respect to $x \in \mathbb{R}^n$. Then by the Calderon-Vaillancourt theorem we obtain

$$\begin{aligned} & \lim_{k \rightarrow \infty} \left\| \widehat{P}'_k (\widehat{V}_{\alpha_{m_k}}^{-1} A \widehat{V}_{\alpha_{m_k}} - a(x, \eta_\omega) q_\alpha V_h) \right\| \\ &= \lim_{k \rightarrow \infty} \left\| (\widehat{V}_{\alpha_{m_k}}^{-1} A \widehat{V}_{\alpha_{m_k}} - a(x, \eta_\omega) q_\alpha V_h) \widehat{P}'_k \right\| \\ &= 0. \end{aligned}$$

Thus the limit operators for $a(x, D)V_h$ are operators of weighted shifts, that is the operators cV_h , where $c(x) = a(x, \eta_\omega)q_\alpha$.

Let us show that all limit operators for $B = b(x, D)T_g$, where $b(x, D) \in OPS_{1,0}^{-\varepsilon}$ ($\varepsilon > 0$) and $T_g \in \mathcal{R}(\mathbb{R}^n)$, are 0-operators. Indeed,

$$\widehat{V}_{\alpha_m}^{-1} B \widehat{V}_{\alpha_m} = b(x, D + \alpha_m) \widehat{V}_{\alpha_m}^{-1} T_g \widehat{V}_{\alpha_m}.$$

It easy to check that for each $k \in \mathbb{N}$

$$\lim_{m \rightarrow \infty} \left\| \widehat{P}'_k b(x, D + \alpha_m) \right\| = \lim_{m \rightarrow \infty} \left\| b(x, D + \alpha_m) \widehat{P}'_k \right\| = 0.$$

This implies

$$\left\| \widehat{P}'_k b(x, D + \alpha_m) \widehat{V}_{\alpha_m}^{-1} T_g \widehat{V}_{\alpha_m} \right\| \leq C \left\| \widehat{P}'_k b(x, D + \alpha_m) \right\| \rightarrow 0$$

and

$$\begin{aligned} \left\| b(x, D + \alpha_m) \widehat{V}_{\alpha_m}^{-1} T_g \widehat{V}_{\alpha_m} \widehat{P}'_k \right\| &= \left\| b(x, D + \alpha_m) \widehat{P}'_N \widehat{V}_{\alpha_m}^{-1} T_g \widehat{V}_{\alpha_m} \widehat{P}'_k \right\| \\ &\leq C \left\| b(x, D + \alpha_m) \widehat{P}'_N \right\| \\ &\rightarrow 0 \end{aligned}$$

if $\alpha_m \rightarrow \infty$. Thus the limit operators for the operator A_{MN} have the form

$$(A_{MN})_\alpha = \sum_{j=1}^M a_j(x, \eta_\omega) q_j^{(\alpha)} V_{h_j} \tag{23}$$

where $a_j(x, \eta_\omega) \in C_b^\infty(\mathbb{R}^n)$ and $|q_j^{(\alpha)}| = 1$. Property (d) of the limit operators provides the validity of assumption (b) of Definition 2. Moreover, all limit operators of $A \in \mathcal{P}$ with respect to the family $\{\widehat{V}_\alpha\}_{\alpha \in \mathbb{Z}^n}$ belong to the C^* -algebra \mathcal{Q} which is the closure in $B(L_2(\mathbb{R}^n))$ of operators of form (23) ■

We denote by $\text{Lim}'_\infty A$ the set of all limit operators of A defined by the sequences $\{\widehat{V}_{\alpha_m}\}$ with $\alpha_m \rightarrow \infty$. As a corollary of Theorem 3 we obtain

Theorem 27. *Let $A \in \mathcal{P}$. Then the condition*

$$\lim_{r \rightarrow \infty} \nu(A|_{Q'_r(L^2(\mathbb{R}^n))}) > 0 \tag{24}$$

holds if and only if

$$\inf_{A_\beta \in (\text{Lim}_\infty A)'} \nu(A_\beta) > 0. \tag{25}$$

We denote by \mathcal{D}'' the subset of bounded operators A in $L(L^2(\mathbb{R}^n))$ such that

$$\lim_{r \rightarrow \infty} \|[A, \psi_r(D)]\| = 0$$

for each function $\psi(\xi) \in C_b^\infty(\mathbb{R}^n)$. It is easy to see that \mathcal{D}'' is a C^* -subalgebra of $L(L^2(\mathbb{R}^n))$ and $\mathcal{P} \subset \mathcal{D}''$. We introduce the two-sided ideal J'' in \mathcal{D}'' which contains all operators $A \in \mathcal{D}''$ such that

$$\lim_{r \rightarrow \infty} \|\rho_r(D)A\| = \lim_{r \rightarrow \infty} \|A\rho_r(D)\| = 0.$$

Proposition 28. *Condition (24) holds if and only if there exists an operator $L'' \in \mathcal{D}''$ such that*

$$L''A = I + T'' \tag{26}$$

where $T'' \in J''$.

The proof is similar to that of Proposition 25 and it is thus omitted .

Theorems 24 and 27 have a very important corollary on semi-Fredholmnes and Fredholmness of operators in the C^* -algebra \mathcal{P} .

Theorem 29. *Let $A \in \mathcal{P}$. Then A is a Φ_+ -operator if and only if the conditions*

$$\inf_{A_\beta \in \text{Lim}_\infty A} \nu(A_\beta) > 0 \tag{27}$$

$$\inf_{A_\beta \in \text{Lim}'_\infty A} \nu(A_\beta) > 0 \tag{28}$$

hold.

Proof. Let conditions (27) - (28) be satisfied. Then there exist bounded operators L', L'' and operators $T' \in J'$ and $T'' \in J''$ such that $L'A = I + T'$ and $L''A = I + T''$. The operator $L = L'AL'' - L' - L''$ is such that $LA - I = T'T''$. The operator $T'T''$ is a compact one. To see this let ρ_r be defined as earlier. Then applying Proposition 6 we obtain

$$\lim_{r \rightarrow \infty} \|T'T''\rho_r I\| = \lim_{r \rightarrow \infty} \|T'T''\rho_r(D)\| = 0.$$

This implies that $T'T''$ can be approximated by the sequence of compact operators $T'T''(I - \rho_r I)(I - \rho_r(D))$ as $r \rightarrow \infty$. Hence $T'T''$ is a compact operator.

Inversely, let A be a Φ_+ -operator. Then the a priory estimate

$$\delta \|u\| \leq \|Au\| + \|Tu\| \delta > 0 \tag{29}$$

holds where T is a compact operator. Let U_γ ($\gamma \in \mathbb{Z}^n$) be one of the sequences of unitary operators (V_γ) and (\widehat{V}_γ) defining the limit operator A_γ , and let L_m be P_m or P'_m . Then it follows from (29) that for every m

$$\delta \|L_m u\| \leq \|U_\gamma^{-1} A U_\gamma L_m u\| + \|U_\gamma^* T U_\gamma L_m u\| \quad (\delta > 0). \tag{30}$$

The sequences U_γ and U_γ^* weakly converge to zero, consequently, $\lim_{\gamma \rightarrow \infty} \|U_\gamma^* T U_\gamma u\| = 0$. Passing to the limit if $\gamma \rightarrow \infty$ in estimate (30) we arrive the estimate

$$\delta \|L_m u\| \leq \|A_\gamma L_m u\|. \tag{31}$$

Taking into account that the sequence L_m strongly converges to the unit operator we can pass to the limit in (31) and obtain (27) - (28) ■

The following theorem is a corollary of Theorem 29.

Theorem 30. *Let $A \in \mathcal{P}$. Then:*

(a) *A is a Φ_- -operator if and only if*

$$\inf \{ \nu(A_\beta^*) : A_\beta \in \text{Lim}_\infty A \} > 0$$

$$\inf \{ \nu(A_\beta^*) : A_\beta \in \text{Lim}'_\infty A \} > 0.$$

(b) *A is a Fredholm operator if and only if all operators in $\text{Lim}_\infty A \cup \text{Lim}'_\infty A$ are uniformly invertible, i.e. if*

$$\sup \{ \|A_\beta^{-1}\| : A_\beta \in \text{Lim}_\infty A \cup \text{Lim}'_\infty A \} < \infty.$$

Example 31. Let us consider operators of the form

$$A = \sum_{j=1}^M a_j(x, D)V_{h_j} + \sum_{j=1}^N b_j(x, D)T_{g_j} \tag{32}$$

where

$$a_j(x, D) \in OPS_{1,0}^{\tilde{0}} \cap OPSL_{1,0}^0$$

$$b_j(x, D) \in OPS_{1,0}^{-\varepsilon} \cap OPSL_{1,0}^{-\varepsilon} \quad (\varepsilon > 0)$$

$$T_{g_j} \in \mathcal{R}(\mathbb{R}^n).$$

The limit operators for A with respect to the set of unitary operators $\{V_{\alpha_m}\}$ have the form

$$A_{(\alpha)} = \sum_{j=1}^M a_j^{(\alpha)}(D)V_{h_j} + \sum_{j=1}^N b_j^{(\alpha)}(D)V_{g_j^{(\alpha)}}$$

where

$$g_j^{(\alpha)} = \lim_{m \rightarrow \infty} g_j(\alpha_m)$$

$$a_j^{(\alpha)}(\xi) = \lim_{m \rightarrow \infty} a_j(\alpha_m, \xi)$$

$$b_j^{(\alpha)}(\xi) = \lim_{m \rightarrow \infty} b_j(\alpha_m, \xi)$$

since the symbols $a_j(x, \xi), b_j(x, \xi)$ and shifts T_{g_j} are slowly oscillating. The operators $A_{(\alpha)}$ are invariant with respect to the shifts V_h , hence $A_{(\alpha)}$ is invertible in $L_2(\mathbb{R}^n)$ if and only if

$$\inf_{\xi \in \mathbb{R}^n} |\hat{A}_{(\alpha)}(\xi)| > 0$$

where

$$\hat{A}_{(\alpha)}(\xi) = \sum_{j=1}^M a_j^{(\alpha)}(\xi)e^{i(h_j, \xi)} + \sum_{j=1}^N b_j^{(\alpha)}(\xi)e^{i(g_j^{(\alpha)}, \xi)}.$$

Thus we have an effective invertibility condition of limit operators corresponding to the sequences $\{V_{\alpha_m}\}$.

Moreover, one can show that all limit operators with respect to V_α are uniformly invertible if and only if

$$\lim_{R \rightarrow \infty} \inf_{|x| > R, \xi \in \mathbb{R}^n} \left| \sum_{j=1}^M a_j(x, \xi) e^{i(h_j, \xi)} + \sum_{j=1}^N b_j(x, D) e^{i(g_j(x), \xi)} \right| > 0.$$

The problem of invertibility of limit operators defined by the dual family of unitary operators $\{\widehat{V}_{\alpha_m}\}$ is more complicated. It follows from the proof of Proposition 26 that the limit operators have the form

$$A_{(\alpha)}(\omega) = \sum_{j=1}^M a_j(x, \eta_\omega) q_j^{(\alpha)} V_{h_j} \quad (\omega \in S^{n-1}) \tag{33}$$

where

$$\begin{aligned} a_j(x, \eta_\omega) &= \lim_{\xi \rightarrow \eta_\omega} a_j(x, \xi) \\ q_j^{(\alpha)} &= \lim_{m \rightarrow \infty} e^{i(h_j, \alpha_m)} \end{aligned}$$

and the sequence $\alpha_m \rightarrow \eta_\omega$ is such that the last limit exists.

The invertibility of operators of form (33) is a difficult problem which can be solved in some particular cases (see, for instance, monographs [1 - 3]).

4. Pseudodifferential operators of non-zero order with shifts

In this section we consider pseudodifferential operators of non-zero order with shifts of the form

$$A = \sum_{j=1}^M a_j(x, D) V_{h_j} + \sum_{j=1}^N b_j(x, D) T_{g_j} \tag{34}$$

where

$$\begin{aligned} a_j(x, D) &\in OPS_{1,0}^{\widetilde{0}} \cap OPSL_{1,0}^0 \\ b_j(x, D) &\in OPS_{1,0}^{-\varepsilon} \cap OPSL_{1,0}^{-\varepsilon} \quad (\varepsilon > 0) \\ T_{g_j} &\in \mathcal{R}(\mathbb{R}^n). \end{aligned}$$

We will consider the problem of Fredholmness for the operator A acting from $H^s(\mathbb{R}^n)$ into $H^{s-m}(\mathbb{R}^n)$. Applying the pseudodifferential operators $\langle D \rangle^s : H^s \rightarrow L_2(\mathbb{R}^n)$ of reduction order, which are isomorphisms, we can reduce the problem of Fredholmness of $A : H^s(\mathbb{R}^n) \rightarrow H^{s-m}(\mathbb{R}^n)$ to the corresponding problem for the operator

$$\widetilde{A} = \langle D \rangle^s A \langle D \rangle^{-(s-m)} : L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{R}^n).$$

Proposition 32.

- 1) The operator \widetilde{A} belongs to the C^* -algebra \mathcal{P} .

2) The limit operators of \tilde{A} defined by $\{V_{\alpha_k}\}$, where $\mathbb{Z}^n \ni \alpha_k \rightarrow \infty$, have the form

$$\tilde{A}^{(\alpha)} = \sum_{j=1}^M a_j^{(\alpha)}(D) \langle D \rangle^{-m} V_{h_j} + \sum_{j=1}^N b_j^{(\alpha)}(D) \langle D \rangle^{-m} V_{g_j^{(\alpha)}} \quad (35)$$

where

$$\begin{aligned} a_j^{(\alpha)}(\xi) &= \lim_{k \rightarrow \infty} a_j(\alpha_k, \xi) \\ b_j^{(\alpha)}(\xi) &= \lim_{k \rightarrow \infty} b_j(\alpha_k, \xi) \\ g_j^{(\alpha)} &= \lim_{k \rightarrow \infty} g(\alpha_k) \end{aligned} \quad (36)$$

with the sequence α_k such that the limits in (36) exist.

3) The limit operators of \tilde{A} defined by the sequence $\{\hat{V}_{\alpha_k}\}$, where $\alpha_k \rightarrow \eta_\omega$, have the form

$$\tilde{A}_{(\alpha)} = \sum_{j=1}^M \tilde{a}_j(x, \eta_\omega) d_j^{(\alpha)} V_{h_j} \quad (37)$$

with

$$\begin{aligned} \tilde{a}_j(x, \eta_\omega) &= \lim_{\xi \rightarrow \eta_\omega} a_j(x, \xi) \langle \xi \rangle^{-m} \\ d_j^{(\alpha)} &= \lim_{k \rightarrow \infty} e^{i(\alpha_k, h_j)} \end{aligned}$$

and the sequence $\alpha_k \rightarrow \eta_\omega$ is such that the last limit exists.

Proof. 1) Indeed,

$$\begin{aligned} \tilde{A} &= \sum_{j=1}^M \langle D \rangle^s a_j(x, D) \langle D \rangle^{-(s-m)} V_{h_j} \\ &\quad + \sum_{j=1}^N (\langle D \rangle^s b_j(x, D) T_{g_j} \langle D \rangle^{-(s-m)} T_{g_j}^{-1}) T_{g_j}. \end{aligned}$$

It is evident that

$$\langle D \rangle^s a_j(x, D) \langle D \rangle^{-(s-m)} \in OPS_{1,0}^{\tilde{S}^0} \cap OPSL_{1,0}^0.$$

By Proposition 19, $T_{g_j} \langle D \rangle^{-(s-m)} T_{g_j}^{-1}$ are pseudodifferential operators in the class $OPS_{1,0}^{\tilde{S}^{-\varepsilon_j}} \cap OPSL_{1,0}^{-\varepsilon_j}$. ■

2) Let us consider the limit operators defined by sequences V_{α_k} with $\alpha_k \rightarrow \infty$. It follows from Proposition 5 that $\tilde{a}_j(x, D) = \langle D \rangle^s a_j(x, D) \langle D \rangle^{-(s-m)}$ is a pseudodifferential operator with symbol

$$\tilde{a}_j(x, \xi) = a_j(x, \xi) \langle \xi \rangle^{-m} + r_j(x, \xi)$$

where $r_j(x, \xi)$ satisfies the estimates

$$|\partial_x^\beta \partial_\xi^\alpha r_j(x, \xi)| \leq C_{\alpha\beta}(x) \langle \xi \rangle^{-1} \quad (38)$$

with $\lim_{x \rightarrow \infty} C_{\alpha\beta}(x) = 0$ for all α and β . Thus the limit operators for $\tilde{a}_j(x, D)$ defined by the operators V_{α_k} with $\alpha_k \rightarrow \infty$ are $\tilde{a}_j^{(\alpha)}(D) = a_j^{(\alpha)}(D)\langle D \rangle^{-m}$ where

$$a_j^{(\alpha)}(\xi) = \lim_{k \rightarrow \infty} a_j(\alpha_k, \xi) \tag{39}$$

and the sequence $\alpha_k \rightarrow \infty$ is such that the limit in (39) exists. Let us consider the operators

$$\tilde{b}_j(x, D) = \langle D \rangle^s b_j(x, D) (T_{g_j} \langle D \rangle^{-(s-m)} T_{g_j}^{-1}).$$

By Propositions 5 and 19, $\tilde{b}_j(x, D)$ is a pseudodifferential operator with symbol

$$\langle \xi \rangle^s b_j(x, \xi) \langle (F'_{g_j}(x))^t \xi \rangle^{-(s-m)} |\det F'_{g_j}(x)| + r_j(x, \xi)$$

where $r_j(x, \xi)$ satisfies estimates (38). Since $\lim_{x \rightarrow \infty} dF_{g_j}(x) = E$ where E is the matrix identity, $\lim_{x \rightarrow \infty} |\det F'_{g_j}(x)| = 1$, and we obtain that the limit operators of $\tilde{b}_j(x, D)$ are

$$\tilde{b}_j^{(\alpha)}(D) = b_j^{(\alpha)}(D)\langle D \rangle^{-m}$$

where $b_j^{(\alpha)}(\xi) = \lim_{k \rightarrow \infty} b_j(\alpha_k, \xi)$ and the sequence $\alpha_k \rightarrow \infty$ is such that the last limit exists.

3) The calculation of limit operators defined by sequences \hat{V}_{α_k} with $\alpha_k \rightarrow \eta_\omega$ is similar to the calculation given in Proposition 26 ■

As a corollary of Theorem 30(b) and Proposition 32 we obtain the following

Theorem 33. *The operator $A : H^s(\mathbb{R}^n) \rightarrow H^{s-m}(\mathbb{R}^n)$ of form (34) is Fredholm if and only if all the limit operators defined in Proposition 32 are uniformly invertible, that is the limit operators are invertible and the norms of their inverses are uniformly bounded.*

Remark 34. The condition of uniform invertibility of operators (35) on $L_2(\mathbb{R}^n)$ is

$$\lim_{R \rightarrow \infty} \inf_{|x| > R, \xi \in \mathbb{R}^n} \left| \sum_{j=1}^M a_j(x, \xi) e^{i(h_j, \xi)} + \sum_{j=1}^N b_j(x, D) e^{i(g_j(x), \xi)} \right| \langle \xi \rangle^{-m} > 0.$$

Let us consider two examples, in which the conditions of Fredholmness have an effective form.

Example 35. We consider an operator with shifts of the form

$$A = a(x, D) + \sum_{j=1}^N b_j(x, D) T_{g_j}$$

where

$$\begin{aligned} a(x, D) &\in OPS_{1,0}^m \cap OPSL_{1,0}^m \\ b_j(x, D) &\in OPS_{1,0}^{m-\varepsilon} \cap OPSL_{1,0}^{m-\varepsilon} \quad (\varepsilon > 0) \\ T_{g_j} &\in \mathcal{R}(\mathbb{R}^n). \end{aligned}$$

In this case Theorem 33 implies that the operator $A : H^s(\mathbb{R}^n) \rightarrow H^{s-m}(\mathbb{R}^n)$ is a Fredholm one if and only if

- (a) $\lim_{R \rightarrow \infty} \inf_{|x| > R, \xi \in \mathbb{R}^n} |a(x, \xi) + \sum_{j=1}^N b_j(x, \xi) e^{i(g_j(x), \xi)}| \langle \xi \rangle^{-m} > 0$
- (b) $\inf_{(x, \omega) \in \mathbb{R}^n \times S^{n-1}} |\tilde{a}(x, \eta_\omega)| > 0$.

Note that the last condition is that of uniform ellipticity of the operator $a(x, D)$ on \mathbb{R}^n .

Example 36. Let us consider the operator

$$A = a_1(x, D)V_{h_1} + a_2(x, D)V_{h_2} + \sum_{j=1}^N b_j(x, D)T_{g_j}$$

where $h_1, h_2 \neq 0, h_1 \neq h_2$,

$$\begin{aligned} a_j(x, D) &\in OPS_{1,0}^m \cap OPSL_{1,0}^m \quad (j = 1, 2) \\ b_j(x, D) &\in OPS_{1,0}^{m-\varepsilon} \cap OPSL_{1,0}^{m-\varepsilon} \quad (\varepsilon > 0) \\ T_{g_j} &\in \mathcal{R}(\mathbb{R}^n). \end{aligned}$$

The limit operators defined by the sequences $\{\hat{V}_{\alpha_k}\}$ with $\alpha_k \rightarrow \eta_\omega$ have the form

$$\tilde{A}_{(\alpha)} = \tilde{a}_1(x, \eta_\omega) d_1^{(\alpha)} V_{h_1} + \tilde{a}_2(x, \eta_\omega) d_2^{(\alpha)} V_{h_2}$$

with

$$\begin{aligned} \tilde{a}_j(x, \eta_\omega) &= \lim_{\xi \rightarrow \eta_\omega} a_j(x, \xi) \langle \xi \rangle^{-m} \\ d_j^{(\alpha)} &= \lim_{k \rightarrow \infty} e^{i(\alpha_k, h_j)} \end{aligned} \quad (j = 1, 2)$$

and the sequence $\alpha_k \rightarrow \eta_\omega$ is such that the last limits exist. In this case one can give invertibility conditions of $\tilde{A}_{(\alpha)}$ on $L_2(\mathbb{R}^n)$, applying the analysis of the spectrum of the weighted shift operator $C = aV_h$, where a belongs to the C^* -algebra $SO(\mathbb{R}^n)$ defined as the algebra of bounded continuous functions a on \mathbb{R}^n such that

$$\lim_{x \rightarrow \infty} \sup_{y \in K} |a(x+y) - a(x)| = 0$$

for every compact $K \subset \mathbb{R}^n$. It follows from [3: Chapter 1] that if $\inf_{x \in \mathbb{R}^n} |a(x)| > 0$, then the spectrum $\sigma(C)$ of C is the ring

$$\sigma(C) = \left\{ z \in \mathbb{C} : \inf_{M_{SO} \setminus \mathbb{R}^n} |a(x)| \leq z \leq \sup_{M_{SO} \setminus \mathbb{R}^n} |a(x)| \right\} \quad (40)$$

where M_{SO} is the maximal ideal space of $SO(\mathbb{R}^n)$ (we consider functions in $SO(\mathbb{R}^n)$ as continuous functions on M_{SO}). It is well-known that

$$\begin{aligned} \inf_{M_{SO} \setminus \mathbb{R}^n} |a(x)| &= \lim_{R \rightarrow \infty} \inf_{|x| > R} |a(x)| \\ \sup_{M_{SO} \setminus \mathbb{R}^n} |a(x)| &= \lim_{R \rightarrow \infty} \sup_{|x| > R} |a(x)|. \end{aligned} \quad (41)$$

Formulas (40) - (41) imply that the operator $\tilde{A}_{(\alpha)} : L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{R}^n)$ is invertible if $\inf_x |\tilde{a}_1(x, \eta_\omega)| > 0$ and $\inf_x |\tilde{a}_2(x, \eta_\omega)| > 0$ and $1 \notin [m_1, M_1]$ where

$$m_1 = \lim_{R \rightarrow \infty} \inf_{|x| > R} \left| \frac{a_2(x, \eta_\omega)}{a_1(x, \eta_\omega)} \right| \quad \text{and} \quad M_1 = \lim_{R \rightarrow \infty} \sup_{|x| > R} \left| \frac{a_2(x, \eta_\omega)}{a_1(x, \eta_\omega)} \right|.$$

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