

Operator-Theoretic Positivstellensätze

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Abstract. We study the structure of positive polynomials with coefficients in an operator algebra as a non-commutative infinite-dimensional analogue of Hilbert's 17-th problem.

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0. Introduction

Hilbert has shown, as early as 1888, the existence of non-negative homogeneous polynomials in three variables, which are not sums of squares of homogeneous polynomials. This important remark led him to state, in 1900, the following problem:

Show that every polynomial in n variables, which is non-negative on the Euclidean space \mathbb{R}^n , can be represented as a sum of squares of rational functions.

An affirmative answer to this question, known as the 17-th problem of Hilbert, was given in 1927 by E. Artin. For subsequent contributions see [4, 19] (as well as their references).

Since the solution of Artin to the problem of Hilbert was given, preoccupations to represent positive polynomials as sums of squares of rational functions with universal denominators have been recorded. Generalizing results due to Pólya, Habicht, Delzel and others, Reznick proves, using algebraic methods (see [20] and its references), the following result:

If p is a homogeneous polynomial in n variables with $p(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, then there exist an integer $\nu \geq 0$ and homogeneous polynomials $(q_j)_{j \in J}$, J finite, such that

$$\|x\|^{2\nu} p(x) = \sum_{j \in J} q_j(x)^2 \quad (x \in \mathbb{R}^n)$$

where $\|x\|^2 = x_1^2 + \dots + x_n^2$ is the Euclidean norm of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

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This result of Reznick has been generalized in [17] (see also [16]), where the structure of positive polynomials on semi-algebraic sets defined by polynomial inequalities is studied, using functional-analytic methods.

The aim of the present paper is to investigate the structure of positive polynomials with coefficients in an arbitrary unital C^* -algebra as a non-commutative infinite-dimensional analogue of Hilbert's 17-th problem. Specifically, we consider polynomials p of several variables with values in an arbitrary unital C^* -algebra A . We describe a large class of polynomials p that are pointwise positive (in the operator-theoretic sense) on subsets of \mathbb{R}^n defined by inequalities of the form $p_0(t) \geq 0$, with p_0 a matrix-valued polynomial. In particular, if p and p_0 are homogeneous of even degree, we show that there exist an integer $\nu \geq 0$ and homogeneous polynomials q_j and q_{j0} ($j \in J$), J finite, such that

$$\|x\|^{2\nu} p(x) = \sum_{j \in J} \left(q_j(x)^* q_j(x) + q_{j0}(x)^* p_0(x) q_{j0}(x) \right) \quad (x \in \mathbb{R}^n)$$

which is an operator extension of [16: Theorem 1].

Other results, similar to those concerning ordinary positive polynomials (i.e., for $A = \mathbb{C}$), will be obtained. Our main tool is a positive measure on a suitable spectrum, which takes values in the dual of the algebra of coefficients. Its existence is derived from the positivity assumption by operator-theoretic techniques. This idea appeared in [17], where it was used for the scalar case $A = \mathbb{C}$. We generalize and unify here some of the results stated in the papers [16, 17]. In Section 2 we present a new, more direct approach to this type of problems. In Section 3 we combine our techniques with some ideas from [17] to obtain supplementary results. Similar problems have been studied in various contexts, either algebraic [4, 6, 12 - 14, 19, 20], analytic [2] or operator-theoretic [5 - 7, 15, 18].

1. Main tool

Throughout this text we denote by A a unital C^* -algebra whose unit will be designated by $\mathbf{1}$ (if not otherwise specified). The real subspace of the self-adjoint elements of A will be denoted by A_h .

Let $x = (x_1, \dots, x_n)$ be the current variable in \mathbb{R}^n , $n \geq 1$ fixed. We denote by $\mathbb{C}[x]$ and $\mathbb{R}[x]$ the spaces of polynomials in $x = (x_1, \dots, x_n)$ with coefficients in \mathbb{C} and \mathbb{R} , respectively. We often identify a polynomial p with its associated function $p(*)$ on \mathbb{R}^n . We shall also use the standard multi-index notation. In particular, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ is an arbitrary multi-index.

Remark 1. Let A^* be the (Banach space) dual of A . Let also $K \subset \mathbb{R}^n$ be a fixed compact set. We shall integrate A -valued bounded Borel-measurable functions against A^* -valued non-negative Borel measures on K . In this context, an A^* -valued measure m is a set map $B \rightarrow m(B)$, defined on the Borel measurable subsets of K with values in A^* , which is strongly countably additive, that is, the set map $B \rightarrow m(B)a$ is countably

additive (i.e., it is a scalar-measure) for each $a \in A$. The measure m is said to be *non-negative* if $m(B)a \geq 0$ for all $B \subset K$ measurable and all $a \geq 0$ in A . Various properties of the integral with respect to m hold in this context, too (see, for instance, [3]).

We shall briefly present some properties of such an integral, which will be later used.

Fix an A^* -valued non-negative measure m on K . If $f = \sum_j \chi_{B_j} a_j$ is an A -valued simple function, with $(B_j)_j$ a finite partition of K , we set

$$\int_K (dm, f) = \sum_j m(B_j) a_j. \tag{1}$$

If $f = \sum_j \chi_{B_j} a_j$ has self-adjoint values, i.e., $a_j \in A_h$ for all j , we have the estimate

$$\left| \int_K (dm, f) \right| \leq \sup_{t \in K} \|f(t)\| m(K) \mathbf{1}. \tag{2}$$

This follows from the estimates $-\|a_j\| \mathbf{1} \leq a_j \leq \|a_j\| \mathbf{1}$ for all j , implying

$$-\|a_j\| m(B_j) \mathbf{1} \leq m(B_j) a_j \leq \|a_j\| m(B_j) \mathbf{1}$$

due to the positivity of m . Summing up these inequalities, we infer easily (2). If a_j are not necessarily self-adjoint, then the right side of (2) should be multiplied by 2.

These estimates allow us to extend the integral $\int_K (dm, f)$ to functions f which are uniform limits of simple functions, in particular to A -valued continuous functions. Moreover, estimate (2) (or its more general version mentioned above) still holds for such functions.

Lemma 2. *Let m be an A^* -valued non-negative measure on K .*

(a) *If $f : K \rightarrow A$ is continuous and $f(t) \geq 0$ for all $t \in K$, then $\int_K (dm, f) \geq 0$.*

(b) *If $h : K \rightarrow \mathbb{C}$ is continuous and $a \in A$ is fixed, then $\int_K (dm, h \otimes a) = \int_K h dm_a$, where m_a is the scalar measure $m(*)a$.*

Proof.

(a) Fix $\varepsilon > 0$. Since f is continuous and K is compact, we can find a finite number of points $(t_j)_j$ in K , and a partition $(B_j)_j$ of K , such that if $g = \sum_j \chi_{B_j} a_j$ with $a_j = f(t_j) \geq 0$, then $\sup_{t \in K} \|f(t) - g(t)\| < \varepsilon$. Hence $\int_K (dm, g) = \sum_j m(B_j) a_j \geq 0$, since m is non-negative. Using (2) and letting $\varepsilon \rightarrow 0$ in

$$\int_K (dm, f) = \int_K (dm, g) + \int_K (dm, f - g) \geq \int_K (dm, f - g) \geq -\varepsilon m(K) \mathbf{1}$$

we infer easily the assertion.

(b) If $h = \sum_j \lambda_j \chi_{B_j}$ is a complex-valued simple function, with $(B_j)_j$ a finite partition of K , then

$$\int_K (dm, h \otimes a) = \sum_j \lambda_j m(B_j) a = \int h dm_a.$$

The general assertion follows by approximating h with simple functions ■

The following result is proved by a well-known method, combining an old idea of factorization, due to Gelfand and Naimark [11], with some operator-theoretic techniques, as in [9, 10] etc. It was also stated and used in various versions, for $A = \mathbb{C}$, in [17] (for a similar result see Proposition 14 below).

The algebra $\mathbb{C}[x] \otimes A$, that is, the algebra of polynomials in $x = (x_1, \dots, x_n)$, with coefficients in A (regarded as functions on \mathbb{R}^n) will be endowed with its natural involution, i.e., $p^*(x) = p(x)^*$ for all $x \in \mathbb{R}^n$.

Theorem 3. *Let $L : \mathbb{C}[x] \otimes A \rightarrow \mathbb{C}$ be linear such that $0 \leq L(x_j^2 p^* p) \leq L(p^* p)$ for all $j = 1, \dots, n$ and $p \in \mathbb{C}[x] \otimes A$. Then there exists an A^* -valued non-negative measure m on $C = [-1, 1]^n$ such that $L(p) = \int_C (dm, p)$ for any p .*

Proof. By the Cauchy-Schwarz inequality, the set $I = \{p : L(p^* p) = 0\}$ is a left ideal. Let \mathcal{H} be the completion of $(\mathbb{C}[x] \otimes A)/I$ with respect to the inner product $\langle \hat{p}, \hat{q} \rangle = L(q^* p)$. (We denote by \hat{p} the class of p modulo I .) The multiplications by x_j induce commuting selfadjoint operators $T_j \in B(\mathcal{H})$. Let E be the spectral measure of $T = (T_1, \dots, T_n)$. Since $\|T_j\| \leq 1$, the support of E is a subset of C . For $B \subset C$ measurable and $a \in A \subset \mathbb{C}[x] \otimes A$, set $m(B)a = \langle E(B)\hat{a}, \hat{\mathbf{1}} \rangle$. Obviously, $m(B)$ is linear and continuous on A and the map $B \rightarrow m(B)a$ is countably additive for each fixed a .

For a fixed $a \in A_h$ we denote by C_a the (commutative) unital C^* -algebra generated by a in A . Let also \mathcal{H}_a be the closure of $(\mathbb{C}[x] \otimes C_a + I)/I$ in \mathcal{H} . Apply L to the equality

$$(\|a^2\|\mathbf{1} - a^2)p^* p = q^* q \quad \text{where } q = (\|a^2\|\mathbf{1} - a^2)^{\frac{1}{2}} p \text{ and } p \in \mathbb{C}[x] \otimes C_a.$$

Then $L(a^2 p^* p) \leq \|a^2\|L(p^* p)$. Hence the multiplication by a induces an operator $T_a = T_a^* \in B(\mathcal{H}_a)$. Since $T_j \mathcal{H}_a \subset \mathcal{H}_a$ and $T_j T_a = T_a T_j$ on \mathcal{H}_a for all j , then T_a commutes with the joint spectral measure E_a of $T|_{\mathcal{H}_a}$. Now, $E_a(B) = E(B)|_{\mathcal{H}_a}$ for each measurable $B \subset C$. Hence $T_a E(B) = E(B) T_a$ on \mathcal{H}_a . If $a \geq 0$ and $b = a^{\frac{1}{2}}$, then

$$m(B)a = \langle E_a(B)\hat{a}, \hat{\mathbf{1}} \rangle = \langle T_b E_a(B)^2 T_b \hat{\mathbf{1}}, \hat{\mathbf{1}} \rangle = \langle E_a(B) T_b \hat{\mathbf{1}}, E_a(B) T_b \hat{\mathbf{1}} \rangle \geq 0$$

showing that m is positive.

Let $p(x) = x^\alpha \otimes a$ ($a \in A$). Using Lemma 2(b) for $h(t) = t^\alpha$, we obtain

$$L(p) = \langle \hat{p}, \hat{\mathbf{1}} \rangle = \langle T^\alpha \hat{a}, \hat{\mathbf{1}} \rangle = \int_C t^\alpha d\langle E(t)\hat{a}, \hat{\mathbf{1}} \rangle = \int_C t^\alpha dm_a(t) = \int_C (dm, p).$$

This equality holds for an arbitrary p , via a linearity argument ■

2. Positivstellensätze

Set $S = S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$, i.e., S is the unit sphere in \mathbb{R}^n . Fix $K \subset S$ – a non-empty compact set. We denote by \mathcal{A} the real space of functions of the form $p|_K$ with $p \in \mathbb{R}[x] \otimes A_h$. The space \mathcal{A} will be endowed with the finest (separated) locally convex topology. A basis of the topology of \mathcal{A} consists of all convex, absorbing and symmetric subsets. Let $\mathcal{A}_+ \subset \mathcal{A}$ be the set of all sums of elements of the form $p^*p|_K$ with $p \in \mathbb{C}[x] \otimes A$, which is a convex cone.

Lemma 4. *The constant function $\mathbf{1} \in \mathcal{A}$ belongs to the interior of \mathcal{A}_+ .*

Proof. Let $\mathcal{A}' \subset \mathcal{A}$ consist of all $p \in \mathcal{A}$ for which there is an $\varepsilon = \varepsilon_p > 0$ with $\mathbf{1} + \lambda p \in \mathcal{A}_+$ for any $\lambda \in (-\varepsilon, \varepsilon)$. The set \mathcal{A}' is a linear space. Indeed, $\mathbf{1} + \lambda(p + q) = \frac{1}{2}(\mathbf{1} + 2\lambda p + \mathbf{1} + 2\lambda q) \in \mathcal{A}_+$ if $|\lambda| < \min\{\frac{\varepsilon_p}{2}, \frac{\varepsilon_q}{2}\}$ and $p, q \in \mathcal{A}'$. Similarly, $\mathbf{1} + \lambda cp \in \mathcal{A}_+$ for all $c \in \mathbb{R}$ if $|\lambda|$ is sufficiently small.

Let ξ_j be the function $\mathbb{R}^n \ni x \rightarrow x_j \in \mathbb{R}$ restricted to K for all j . Then $\sum_j \xi_j^2 = 1$ on K . For any $\alpha \in \mathbb{Z}_+^n$ there are polynomial functions g_k such that $1 = (\sum \xi_j^2)^{|\alpha|}$ equals $\xi^{2\alpha} + \sum_k g_k^2$, where, as usual, $|\alpha| = \alpha_1 + \dots + \alpha_n$. Thus

$$\mathbf{1} \otimes \lambda^2 a^2 = \lambda^2 \xi^{2\alpha} \otimes a^2 + \sum_k \lambda^2 g_k^2 \otimes a^2$$

whence

$$\mathbf{1} + \lambda \xi^\alpha \otimes a = 2^{-1} \left[(\mathbf{1} + \lambda \xi^\alpha \otimes a)^2 + \sum_k (\lambda g_k \otimes a)^2 + [1 \otimes (\mathbf{1} - \lambda^2 a^2)^{1/2}]^2 \right]$$

for $a \in A_h$ and $|\lambda|$ sufficiently small. Hence any generator $\xi^\alpha \otimes a \in \mathcal{A}'$. Consequently, $\mathcal{A}' = \mathcal{A}$.

Set $U = (\mathcal{A}_+ - \mathbf{1}) \cap (\mathbf{1} - \mathcal{A}_+)$, which is a convex set containing zero. Let $f \in \mathcal{A}$. Then $\mathbf{1} + \lambda f \in \mathcal{A}_+$ for $|\lambda| < \varepsilon$. Therefore, $\lambda f \in \mathcal{A}_+ - \mathbf{1}$ and $-\lambda f \in \mathbf{1} - \mathcal{A}_+$, and so $\lambda f \in U$ for all $|\lambda| < \varepsilon$. In other words, U is absorbing. Since U is clearly symmetric, it follows that U is a neighbourhood of the origin. Hence $V = U + \mathbf{1} \subset \mathcal{A}_+$ is a neighbourhood of $\mathbf{1}$ in \mathcal{A} ■

For two integers $\nu_1, \nu_2 \geq 1$ and an arbitrary linear space \mathcal{L} , $M_{\nu_1 \times \nu_2}(\mathcal{L})$ stands for the space of all $\nu_1 \times \nu_2$ matrices with entries in \mathcal{L} . If $\nu = \nu_1 = \nu_2$, we denote $M_{\nu_1 \times \nu_2}(\mathcal{L})$ simply by $M_\nu(\mathcal{L})$.

Fix an integer $\nu \geq 1$. Let $\pi \in \mathbb{C}[x] \otimes M_\nu(\mathbb{C})$ and $q \in \mathbb{C}[x] \otimes M_{\nu \times 1}(A)$. If we identify $M_\nu(\mathbb{C})$ with $M_\nu(\mathbb{C}) \otimes \mathbf{1} \subset M_\nu(A)$, then $q^* \pi q \in \mathbb{C}[x] \otimes A$. Note also that $\mathbb{C}[x] \otimes M_{\nu \times 1}(A)$ and $\mathbb{C}[x] \otimes A^\nu$ are isomorphic, and we shall identify sometimes these two spaces.

Lemma 5. *Let L, m and C be as in Theorem 3. Let also $\pi \in \mathbb{C}[x] \otimes M_\nu(\mathbb{C})$. If $L(q^* \pi q) \geq 0$ for all $q \in \mathbb{C}[x] \otimes A^\nu$, then $\text{supp } m \subset \{t \in C : \pi(t) \geq 0\}$, where $\text{supp } m$ is the support of m .*

Proof. If $\pi = [\pi_{ik}]_{i,k}$ with π_{ik} ordinary polynomials, we set $\pi(T) = [\pi_{ik}(T)]_{i,k}$, where $T \in B(\mathcal{H})^n$ is defined in the proof of Theorem 3. Note that there exists a natural map $\mathbb{C}[x] \otimes A^\nu \rightarrow \mathcal{H}^\nu$ whose image is dense in \mathcal{H}^ν . Therefore,

$$\langle \pi(T) \hat{q}, \hat{q} \rangle = \langle \widehat{\pi q}, \hat{q} \rangle = L(q^* \pi q) \geq 0 \quad (q \in \mathbb{C}[x] \otimes A^\nu).$$

Hence $\pi(T) \geq 0$ in $B(\mathcal{H}^\nu)$. The normal commuting ν^2 -tuple of all entries of $\pi(T)$ generates, with the identity on \mathcal{H} , a commutative unital C^* -algebra B in $B(\mathcal{H})$. Let Γ be the spectrum of B and let $f : B \rightarrow C(\Gamma)$ be the Gelfand isomorphism. Since B is generated by $\{\pi_{ik}(T)\}_{1 \leq i,k \leq \nu}$, we may identify the spectrum Γ with a compact subset of \mathbb{C}^{ν^2} and the functions $f\pi_{ik}(T)$ with the coordinate functions z_{ik} of \mathbb{C}^{ν^2} restricted to Γ . Let also $I_\nu \otimes f : M_\nu(B) \rightarrow M_\nu(C(\Gamma))$ be the natural map induced by f , which is also an isomorphism, where I_ν is the identity on $M_\nu(\mathbb{C})$. As $M_\nu(B)$ is a C^* -algebra, then $\pi(T) \geq 0$ implies $\zeta := (I_\nu \otimes f)\pi(T) \geq 0$.

Factor $\zeta = \psi^* \psi$ in $M_\nu(C(\Gamma))$ with $\psi = [\psi_{ik}]_{i,k}$. Then $z_{jk} = \sum_i \overline{\psi_{ij}(z)} \psi_{ik}(z)$ for $z \in \Gamma$. Hence

$$\sum_{j,k} z_{jk} \xi_k \bar{\xi}_j = \sum_i \left| \sum_j \psi_{ij}(z) \xi_j \right|^2 \geq 0 \quad (\xi_j \in \mathbb{C}).$$

Therefore $\Gamma \subset \{z \in \mathbb{C}^{\nu^2} : [z_{jk}]_{j,k} \geq 0\}$. Now, regarding π as a map from \mathbb{C}^n into \mathbb{C}^{ν^2} and $\pi(T)$ as a ν^2 -tuple in B , since $\text{supp } m \subset \text{supp } E = \sigma(T)$, we derive $\pi(\sigma(T)) = \sigma(\pi(T)) = \Gamma$ by the spectral mapping theorem, where σ denotes the joint spectrum in the corresponding algebra, which completes the proof of the lemma ■

Remark 6. Suppose $n > 1$ and let $p, q \in \mathbb{R}[x] \otimes A_h$ be such that $p(x) = \|x\| q(x)$, with x in an open set $G \subset \mathbb{R}^n$. Then $p = q = 0$. Indeed, using a functional from the dual of A_h , we may reduce the problem to the case $p, q \in \mathbb{R}[x]$. The equality $p(x)^2 = \|x\|^2 q(x)^2$ in the open set G extends to the whole space. Since the polynomial $\|x\|^2$ is irreducible in $\mathbb{R}[x]$, this equality shows that $\|x\|^2$ is a divisor of p , and so $p(x)^2 = \|x\|^4 p_1(x)^2$. A successive application of this argument leads to the desired conclusion.

Throughout this section, if not otherwise specified, we shall assume that $n > 1$.

Remark 7. If $q \in \mathbb{C}[x] \otimes M_{\nu \times 1}(A)$, then $q^* q = \sum_{\alpha \in J} q_\alpha^* q_\alpha$ with $q_\alpha \in \mathbb{C}[x] \otimes A$ and J finite. Indeed, $q(x)$ is of the form $\sum_{\alpha \in J} x^\alpha \otimes a_\alpha$ with $a_\alpha = [a_{k\alpha}]_k \in M_{\nu \times 1}(A)$. If d is the cardinal of J , set $a = [a_{k\alpha}]_{k,\alpha} \in M_{\nu \times d}(A)$. Then $a^* a \in M_d(A)$, which is a C^* -algebra. Therefore, $b = (a^* a)^{\frac{1}{2}} \in M_d(A)$. Write $b = [b_{\alpha\beta}]_{\alpha,\beta}$ and define $q_\alpha = \sum_\beta x^\beta \otimes b_{\alpha\beta}$. Since $b^2 = a^* a$, we must have $\sum_\alpha b_{\alpha\beta}^* b_{\alpha\gamma} = \sum_k a_{k\beta}^* a_{k\gamma}$, implying $q^* q = \sum_{\alpha \in J} q_\alpha^* q_\alpha$.

If (x_0, x) is the variable in $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$, for each polynomial $p \in \mathbb{C}[x] \otimes \mathcal{L}$ (with \mathcal{L} an arbitrary linear space) we denote by $\tilde{p} \in \mathbb{C}[x_0, x] \otimes \mathcal{L}$ its *homogenization*, i.e., the polynomial $\tilde{p}(x_0, x) = x_0^{\text{deg } p} p(\frac{x}{x_0})$, where $\text{deg } p$ is the degree of p .

Let $\mathcal{R}[x]$ denote the space of all polynomials in $x, \|x\|, \|x\|^{-1}$, regarded as functions on $\mathbb{R}^n \setminus \{0\}$. For every function $f \in \mathcal{R}[x] \otimes A$ set

$$f^e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f^o(x) = \frac{f(x) - f(-x)}{2}.$$

We state and prove now the main result of this paper (see also [16: Theorems 1 and 3]).

Theorem 8. *Let $p \in \mathbb{R}[x] \otimes A_h$ be homogeneous, and let*

$$p_k \in \mathbb{R}[x] \otimes M_{\nu_k}(\mathbb{C})_h \quad (\nu_k \in \mathbb{Z}_+, k = 1, \dots, m)$$

be given. Assume

$$K_0 = \left\{ t \in S^{n-1} : p_k(t) \geq 0 \quad (k = 1, \dots, m) \right\} \neq \emptyset$$

and $p(t) > 0$ for all $t \in K_0$. Then there are homogeneous polynomials $q_j \in \mathbb{C}[x_0, x] \otimes A$ and $q_{jk} \in \mathbb{C}[x_0, x] \otimes M_{\nu_k \times 1}(A)$ ($j \in J, J$ finite, $k = 1, \dots, m$) and an integer $\theta \in \mathbb{Z}_+$ such that

$$\begin{aligned} \|x\|^\theta p(x) = & \sum_{j \in J} \left(q_j^*(\|x\|, x) q_j(\|x\|, x) \right. \\ & \left. + \sum_{k=1}^m \|x\|^{\kappa_k} q_{jk}^*(\|x\|, x) \tilde{p}_k(\|x\|, x) q_{jk}(\|x\|, x) \right) \end{aligned} \tag{3}$$

for all $x \in \mathbb{R}^n$, where κ_k equals 0 or 1 if the degree of p_k is even or odd, respectively.

If p_k are also homogeneous and all p, p_k ($k = 1, \dots, m$) have even degrees, then there are homogeneous polynomials $q_j \in \mathbb{C}[x] \otimes A$, $q_{jk} \in \mathbb{C}[x] \otimes M_{\nu_k \times 1}(A)$ ($j \in J, J$ finite) and an integer $\theta \in \mathbb{Z}_+$ such that

$$\|x\|^{2\theta} p(x) = \sum_{j \in J} \left(q_j^*(x) q_j(x) + \sum_{k=1}^m q_{jk}^*(x) p_k(x) q_{jk}(x) \right) \tag{4}$$

for all $x \in \mathbb{R}^n$.

Proof. We first discuss the general case, i.e., $\delta = \deg p$ and $\delta_k = \deg p_k$ arbitrary and p_k not necessarily homogeneous. Since the function $p(*)$ has self-adjoint values, and if $S := S^{n-1}$, we can find a neighbourhood $U \subset S$ of K_0 with $p > 0$ on U . It is easily seen that there exists an $\varepsilon > 0$ such that

$$K_0 \subset K_\varepsilon := \left\{ t \in S : p_k(t) + \varepsilon \mathbf{1}_{\nu_k} \geq 0 \quad (k = 1, \dots, m) \right\} \subset U$$

with $\mathbf{1}_{\nu_k}$ the identity on $M_{\nu_k}(\mathbb{C})$. Set $p_{1k} = p_k + \varepsilon \mathbf{1}_{\nu_k}$ and $K = K_\varepsilon$. Since p_k has self-adjoint values, each point $t_0 \in K_0$ has a neighbourhood V_0 in K . In particular, the interior K° of K in S is not empty. Fix a $\gamma > 0$ with $p \geq \gamma \mathbf{1}$ on K . Using the notation from Lemma 4, let \mathcal{C} be the positive cone in \mathcal{A} consisting of the restrictions to K of the functions as in the right side of (4), with p_k replaced by p_{1k} . By virtue of Remark 7, this change does not affect the general form of the functions from the right side of (4). Obviously, $\mathcal{C} \supset \mathcal{A}_+$.

Suppose $p|_K \notin \mathcal{C}$. By Lemma 4 and Mazur's theorem (see, for example, [1: Theorem 1.12]) we get a non-null functional f on \mathcal{A} with $\inf_{\mathcal{C}} f \geq f(p|_K)$. If there is a $c \in \mathcal{C}$ with $f(c) < 0$, then $\inf_{\mathcal{C}} f \leq \inf_{j \geq 1} f(jc) = -\infty$ that is false. Hence $f(\mathbf{1}) \geq \inf_{\mathcal{C}} f \geq 0$. Now $\inf_{\mathcal{C}} f \leq \inf_{j \geq 1} f(j^{-1}\mathbf{1}) = 0$. Hence $\inf_{\mathcal{C}} f = 0$.

Set $L_0 = fr$, where $r : \mathbb{R}[x] \otimes A_h \rightarrow \mathcal{A}$ is the restriction map, and let L be an extension of L_0 to the complex space $\mathbb{C}[x] \otimes A$. Since $\sum x_k^2 = 1$ on K , then

$$(1 - x_j^2)p^*p = \sum_{k \neq j} x_k^2 p^*p \in \mathcal{C} \quad \text{for every } p \in \mathbb{C}[x] \otimes A.$$

By Theorem 3, there exists an A^* -valued positive measure m on C such that $L(p) = \int (dm, p)$. Since $L \neq 0$, the Cauchy-Schwarz inequality $|L(q)|^2 \leq L(q^*q)L(\mathbf{1})$ gives $L(\mathbf{1}) > 0$. For $\pi = p|_K$ and $\pi = \pm(\|x\|^2 - 1) \otimes \mathbf{1}$, Lemma 5 implies that $\text{supp } m \subset K$. By Lemma 2 we get

$$0 \geq f(p|_K) = L(p) = \int_K (dm, p) \geq \int_K (dm, \gamma \mathbf{1}) = \gamma L(\mathbf{1}) > 0$$

which is impossible. Therefore $p|_K \in \mathcal{C}$. Hence p has a representation of the form

$$p(x) = \sum_j \left(q_j^*(x)q_j(x) + \sum_k q_{jk}^*(x)p_k(x)q_{jk}(x) \right)$$

on K for some polynomials q_j and q_{jk} .

We set

$$\begin{aligned} Q_j(x) &= \|x\|^{\frac{\delta}{2}} q_j\left(\frac{x}{\|x\|}\right) = \|x\|^{\frac{\delta}{2} - \text{deg} q_j} \tilde{q}_j(\|x\|, x) \\ Q_{jk}(x) &= \|x\|^{\frac{\delta}{2}} q_{jk}\left(\frac{x}{\|x\|}\right) = \|x\|^{\frac{\delta}{2} - \text{deg} q_{jk}} \tilde{q}_{jk}(\|x\|, x) \\ P_k(x) &= p_k\left(\frac{x}{\|x\|}\right) = \|x\|^{-\delta_k} \tilde{p}_k(\|x\|, x). \end{aligned}$$

Set also

$$Q = \sum_j (Q_j^* Q_j + \sum_k Q_{jk}^* P_k Q_{jk}).$$

Note that $Q \in \mathcal{R}[x] \otimes A$ is positive-homogeneous of degree δ , and that $p - Q$ vanishes on K . Therefore $p - Q$ vanishes in the open set $G = \{t \in \mathbb{R}^n : 0 \neq t \in \|t\|K^\circ\}$. Multiplying $p - Q$ by $\|x\|^N$ for a sufficiently large N , the resulting expression has a representation of the form $u + \|x\|v$ with u and v polynomials. Moreover, $u + \|x\|v = 0$ on G . Hence $u = v = 0$ by Remark 6, showing that $p = Q$ everywhere.

Now, choosing an integer τ such that

$$2\tau \geq \max \left\{ \delta, 2\text{deg } q_j, 2\text{deg } q_{jk} + \delta_k \quad (j \in J, k = 1, \dots, m) \right\}$$

we obtain easily a representation of form (3) of p , with $\theta = 2\tau - \delta$, and other polynomials q_j and q_{jk} (for instance, the new $q_j(\|x\|, x)$ will be $\|x\|^{\tau - \text{deg } q_j} \tilde{q}_j(\|x\|, x)$ etc.). For even δ and δ_k and homogeneous p_k , we note that

$$p = \sum_j \left(Q_j^{e*} Q_j^e + Q_j^{o*} Q_j^o + \sum_k Q_{jk}^{e*} P_k Q_{jk}^e + Q_{jk}^{o*} P_k Q_{jk}^o \right)$$

by identifying the function Q^e , which equals Q in this case. The terms of Q do not contain odd powers of $\|x\|^{\pm 1}$. Multiply by some $\|x\|^{2\theta}$ for a sufficiently large θ to get (4) ■

Remark 9.

1) In the statement of Theorem 8, with no loss of generality we may assume $\nu_1 = \dots = \nu_m = 1$ (since the positivity of a matrix whose entries are polynomials can be expressed in terms of polynomial inequalities). We prefer the actual statement which might be used in some applications.

2) If the polynomials p_1, \dots, p_m are homogeneous of even degrees, with no loss of generality we may assume that all degrees are equal by multiplying some polynomials, when necessary, with appropriate powers of $\|x\|^2$. Then the polynomial $\bigoplus_{k=1}^m p_k$ is non-negative if and only if all p_k are non-negative. Therefore, in the second part of Theorem 8 we may always assume $m = 1$.

3) The proof of Theorem 8 does not apply to the case $n = 1$ (see Remark 6). Nevertheless, we can get directly some representations similar to those given by (3) and (4) but only in terms of polynomials. When $n = 1$, an arbitrary homogeneous polynomial $p \in \mathbb{R}[x] \otimes A_h$ has necessarily the form $p(x) = x^\delta \otimes a$, that is, p is a monomial.

If δ is even and $p(x_0) > 0$ for one point $x_0 \in S^0 = \{-1, 1\}$, then $a > 0$ and we can write $x^\delta \otimes a = (x^{\frac{\delta}{2}} \otimes a^{\frac{1}{2}})^* x^{\frac{\delta}{2}} \otimes a^{\frac{1}{2}}$, which is a representation as in (4).

Assume that δ is odd and let $p_k(x) = x^{\delta_k} \otimes b_k$ ($k = 1, \dots, m$). For the sake of simplicity we assume $m = 2$, $b_1, b_2 \in A_h$ and $\delta_1 = \deg p_1$ odd and $\delta_2 = \deg p_2$ even. Since the degree of both p and p_1 is odd, we have either $K_0 = \{1\}$ or $K_0 = \{-1\}$ but $K_0 \neq \{-1, 1\}$. Assume $K_0 = \{1\}$. Then $b_1, b_2 \geq 0$ and $a > 0$. Fix a number $\varepsilon > 0$. Note that

$$x^{\delta_1} \otimes a = c_1^*(x^{\delta_1} \otimes b_1)c_1 + \varepsilon x(x^{\frac{\delta_1-1}{2}} \otimes c_1)^*(x^{\frac{\delta_1-1}{2}} \otimes c_1)$$

where $c_1 = (b_1 + \varepsilon \mathbf{1})^{-\frac{1}{2}} a^{\frac{1}{2}}$. Similarly,

$$x^{\delta_2} \otimes a = c_2^*(x^{\delta_2} \otimes b_2)c_2 + \varepsilon c_2^*(x^{\frac{\delta_2}{2}} \otimes b^{\frac{1}{2}})^*(x^{\frac{\delta_2}{2}} \otimes b^{\frac{1}{2}})c_2$$

where $c_2 = (b_2 + \varepsilon \mathbf{1})^{-\frac{1}{2}} a^{\frac{1}{2}}$. Let $\tau = \max\{\delta, \delta_1, \delta_2\}$. From the identity

$$x^{\tau-\delta}(x^\delta \otimes a) = \frac{1}{2}x^{\tau-\delta_1}(x^{\delta_1} \otimes a) + \frac{1}{2}x^{\tau-\delta_2}(x^{\delta_2} \otimes a)$$

we infer that p admits a representation of the form

$$\begin{aligned} x^\theta p(x) = & \sum_j \left(q_j(x)^* q_j(x) + x r_j(x)^* r_j(x) \right. \\ & \left. + \sum_k \left(q_{jk}(x)^* p_k(x) q_{jk}(x) + x r_{jk}(x)^* p_k(x) r_{jk}(x) \right) \right) \end{aligned}$$

with $\{q_j, r_j, q_{jk}, r_{jk}\}$ a finite number of monomials and $\theta \geq 0$ an integer. This formula still holds true for an arbitrary finite number of monomials p_1, \dots, p_m and for b_1, \dots, b_m matrices.

If $K_0 = \{-1\}$ and δ is odd, we obtain as above a representation of the form

$$(-x)^\theta p(x) = \sum_j \left(q_j(x)^* q_j(x) - x r_j(x)^* r_j(x) \right) + \sum_k \left(q_{jk}(x)^* p_k(x) q_{jk}(x) - x r_{jk}(x)^* p_k(x) r_{jk}(x) \right)$$

with $\{q_j, r_j, q_{jk}, r_{jk}\}$ a finite number of monomials and $\theta \geq 0$ an integer.

Remark 10. Let $p \in \mathbb{R}[x] \otimes A_h$ and $p_k \in \mathbb{R}[x] \otimes M_{\nu_k}(\mathbb{C})_h$ ($\nu_k \in \mathbb{N}, k = 1, \dots, m$) be as in Theorem 8. Let also $\omega \in \mathbb{R}[x]$ be a positive definite quadratic form. Then there are homogeneous polynomials $q_j \in \mathbb{C}[x_0, x] \otimes A$ and $q_{jk} \in \mathbb{C}[x_0, x] \otimes M_{\nu_k \times 1}(A)$ ($j \in J, J$ finite, $k = 1, \dots, m$) and an integer $\theta \in \mathbb{Z}_+$ such that

$$\omega^{\frac{\theta}{2}}(x)p(x) = \sum_{j \in J} \left(q_j^*(\omega(x)^{\frac{1}{2}}, x) q_j(\omega(x)^{\frac{1}{2}}, x) \right) + \sum_{k=1}^m \omega(x)^{\frac{\kappa_k}{2}} q_{jk}^*(\omega(x)^{\frac{1}{2}}, x) \tilde{p}_k(\omega(x)^{\frac{1}{2}}, x) q_{jk}(\omega(x)^{\frac{1}{2}}, x) \tag{5}$$

for all $x \in \mathbb{R}^n$, where κ_k equals 0 or 1 if the degree of p_k is even or odd, respectively. If p and p_k have even degrees, then there are homogeneous polynomials $q_j \in \mathbb{C}[x] \otimes A$ and $q_{jk} \in \mathbb{C}[x] \otimes M_{\nu_k \times 1}(A)$ ($j \in J, J$ finite, $k = 1, \dots, m$) and an integer $\theta \in \mathbb{Z}_+$ such that

$$\omega^\theta p = \sum_{j \in J} \left(q_j^* q_j + \sum_{k=1}^m q_{jk}^* p_k q_{jk} \right). \tag{6}$$

The assertion is obtained by applying Theorem 8 to $p \circ g^{-1}$ and $p_k \circ g^{-1}$, where $g \in M_n(\mathbb{C})$ is such that $\omega(x) = \|g(x)\|^2$ for all $x \in \mathbb{R}^n$.

Corollary 11. Let $p \in \mathbb{R}[x] \otimes A$ and $p_k \in \mathbb{R}[x] \otimes M_{\nu_k}(\mathbb{C})$ ($\nu_k \in \mathbb{N}; k = 1, \dots, m$). Suppose

$$\Sigma := \left\{ (t_0, t) \in \mathbb{R}^{n+1} \setminus \{0\} : \tilde{p}_k(t_0, t) \geq 0 \quad (k = 1, \dots, m) \right\} \neq \emptyset$$

and $\tilde{p}(t_0, t) > 0$ for any $(t_0, t) \in \Sigma$. Then there are $q_j \in \mathbb{C}[x_0, x] \otimes A$ and $q_{jk} \in \mathbb{C}[x_0, x] \otimes A^{\nu_k}$ ($j \in J, J$ finite, $k = 1, \dots, m$) and $\theta \in \mathbb{Z}_+$ such that

$$\varphi^\theta(x)p(x) = \sum_{j \in J} \left(q_j^*(\varphi(x), x) q_j(\varphi(x), x) \right) + \sum_{k=1}^m q_{jk}^*(\varphi(x), x) \tilde{p}_k(\varphi(x), x) q_{jk}(\varphi(x), x)$$

for all $x \in \mathbb{R}^n$, where $\varphi(x)^2 = 1 + \|x\|^2$.

If p and p_k have even degrees, then there are $q_j \in \mathbb{C}[x] \otimes A$ and $q_{jk} \in \mathbb{C}[x] \otimes A^{\nu_k}$ ($j \in J, J$ finite, $k = 1, \dots, m$) and $\theta \in \mathbb{Z}_+$ such that

$$\varphi^{2\theta} p = \sum_{j \in J} \left(q_j^* q_j + \sum_{k=1}^m q_{jk}^* p_k q_{jk} \right). \tag{7}$$

Proof. Apply Theorem 8 to $\tilde{p}(x_0, x)$ and $\tilde{p}_k(x_0, x)$ and then take $x_0 = 0$ ■

The next result is an extension of the essential part of the main result from [20].

Corollary 12. *If $p \in \mathbb{R}[x] \otimes A$ has even degree and $\tilde{p} > 0$ on $\mathbb{R}^{n+1} \setminus \{0\}$, then there are $q_j \in \mathbb{R}[x] \otimes A$ ($j \in J, J$ finite) and $\nu \in \mathbb{Z}_+$ such that $\varphi^{2\nu} p = \sum_j q_j^* q_j$.*

3. Related results

The integration procedure described in the first section can be also performed on not necessarily compact subsets of \mathbb{R}^n , provided one integrates bounded measurable functions. Such a situation will be encountered in Proposition 14 below.

Remark 13. Let $v : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^N$ ($N = (n + 1)^2$) be the map

$$v(x) = y \quad \begin{cases} x = (x_0, x_1, \dots, x_n) \\ y = (y_{jk})_{0 \leq j, k \leq n}, y_{jk} = x_j x_k. \end{cases}$$

As noticed in [17], the range of this map (which is related to the Veronese imbedding) is given by

$$v(\mathbb{R}^{n+1}) = \left\{ y = (y_{jk})_{0 \leq j, k \leq n} \mid \begin{cases} y_{jj} \geq 0 \\ y_{jk} = y_{kj} \\ y_{jk} y_{rs} = y_{jr} y_{ks} \end{cases} \quad (0 \leq j, k, r, s \leq n) \right\}.$$

We shall denote

$$V = \left\{ y = (y_{jk})_{0 \leq j, k \leq n} \in v(\mathbb{R}^{n+1}) : \|y\| = 1 \text{ and } y_{00} = 0 \right\}.$$

Set

$$\psi_{jk}(x) = \frac{x_j x_k}{1 + \|x\|^2} \quad (j, k = 0, \dots, n; x = (x_1, \dots, x_n) \in \mathbb{R}^n, x_0 = 1).$$

It is also noticed in [17] that if $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^N$ is given by $\Psi(t) = (\psi_{jk}(t))_{0 \leq j, k \leq n}$, then Ψ is injective and

$$\Psi(\mathbb{R}^n) = \left\{ y = (y_{jk})_{0 \leq j, k \leq n} \in \mathbb{R}^N \mid \begin{cases} y_{00} > 0 \\ y_{jk} = y_{kj} \\ y_{0j} y_{0k} = y_{00} y_{jk} \\ y_{00}^2 + \dots + y_{0n}^2 = y_{00} \end{cases} \right\} \subset v(\{1\} \times \mathbb{R}^n).$$

The next result is an operator version of [17: Theorem 3.2].

Proposition 14. *Let \mathcal{Q} be the complex algebra of bounded rational functions on \mathbb{R}^n generated by expressions of the form $\frac{x^\alpha}{(1 + \|x\|^2)^m}$ with $\alpha \in \mathbb{Z}_+^n, m \in \mathbb{Z}_+$ and $|\alpha| \leq 2m$. Let $\Lambda : \mathcal{Q} \otimes A \rightarrow \mathbb{C}$ be linear such that $\Lambda(q^* q) \geq 0$ for all q . Then there exist two A^* -valued non-negative measures μ on \mathbb{R}^n and ν on V such that*

$$\Lambda(\pi \circ \Psi) = \int_{\mathbb{R}^n} (d\mu, \pi \circ \Psi) + \int_V (d\nu, \pi) \quad (\pi \in \mathbb{C}[y] \otimes A) \tag{8}$$

where $y = (y_{jk})_{0 \leq j, k \leq n}$ is the current variable in \mathbb{R}^N .

Proof. The proof is similar to that of [17: Theorem 3.2], combined with some ideas from the proof of Theorem 3 above. For the convenience of the reader, we shall sketch this proof.

Let \mathcal{H} be the completion of the space $(\mathcal{Q} \otimes A)/I$, $I = \{q : \Lambda(q^*q) = 0\}$, with respect to the scalar product $\langle \hat{q}_1, \hat{q}_2 \rangle = \Lambda(q_2^*q_1)$, where $\hat{q} = q + I$. The multiplications by the functions ψ_{jk} induce commuting self-adjoint operators T_{jk} for all indices j, k . Indeed, the equality $\sum_{j,k} \psi_{jk}^2 = 1$ implies $\|T_{jk}\hat{q}\| \leq \|\hat{q}\|$ for all $\hat{q} \in (\mathcal{Q} \otimes A)/I$, showing that the operators T_{jk} are bounded (and symmetric), and hence self-adjoint. Moreover,

$$T_{00} \geq 0, \quad T_{jk} = T_{kj}, \quad T_{0j}T_{0k} = T_{00}T_{jk}, \quad T_{00}^2 + \dots + T_{0n}^2 = T_{00} \tag{9}$$

for all indices j, k .

Let E be the the joint spectral measure of the N -tuple $T = (T_{jk})_{0 \leq j, k \leq n}$, $N = (n + 1)^2$. We define an A^* -valued measure m via the relation $m(*)a = \langle E(*)\hat{a}, \hat{\mathbf{1}} \rangle$. As in the proof of Theorem 3, we can show that m is non-negative.

Using Gelfand’s theory, we deduce from (9) that the joint spectrum of T , and therefore the support of E , lies in the set $\Psi(\mathbb{R}^n) \cup V$. As $\Psi(\mathbb{R}^n) \cap V = \emptyset$, we may define the A^* -valued measures $\mu(*) = m(\Psi(*))$ and $\nu(*) = m(* \cap V)$, which leads us to (8). Indeed, if $g \in \mathbb{C}[y]$ ($y \in \mathbb{R}^N$) and $a \in A$, then

$$\begin{aligned} \Lambda((g \otimes a) \circ \Psi) &= \langle g(T)\hat{a}, \hat{\mathbf{1}} \rangle \\ &= \int (dm, g \otimes a) \\ &= \int_{\mathbb{R}^n} (d\mu, (g \otimes a) \circ \Psi) + \int_V (d\nu, g \otimes a). \end{aligned}$$

and the proof is complete ■

We need a version of Lemma 4 in the context of the space $\mathcal{Q} \otimes A$. For this we denote by \mathcal{B} the real space of the functions from $\mathcal{Q} \otimes A$ having self-adjoint values. The space \mathcal{B} will be endowed with the finest locally convex topology. Let $\mathcal{B}_+ \subset \mathcal{B}$ be the set of all sums of elements of the form p^*p with $p \in \mathcal{Q} \otimes A$, which is a convex cone.

Lemma 15. *The constant function $\mathbf{1} \in \mathcal{B}$ belongs to the interior of \mathcal{B}_+ .*

Proof. Let $\mathcal{B}' \subset \mathcal{B}$ consist of all $p \in \mathcal{B}$ for which there is an $\varepsilon > 0$ with $\mathbf{1} + \lambda p \in \mathcal{B}_+$ for any $\lambda \in (-\varepsilon, \varepsilon)$. The set \mathcal{B}' is a linear space, as in the proof of Lemma 4.

Let $\xi_{\alpha, m}$ be the function $\frac{x^\alpha}{(1+\|x\|^2)^m}$ for any $\alpha \in \mathbb{Z}_+^n$ and $m \geq 0$ an integer. Using the identity

$$1 = \frac{(1 + \|x\|^2)^{2m}}{(1 + \|x\|^2)^{2m}}$$

we infer the existence of some functions g_k in \mathcal{Q} such that $1 = \xi_{\alpha, m}^2 + \sum_k g_k^2$. Thus

$$1 \otimes \lambda^2 a^2 = \lambda^2 \xi_{\alpha, m}^2 \otimes a^2 + \sum_k \lambda^2 g_k^2 \otimes a^2$$

whence

$$\mathbf{1} + \lambda \xi_{\alpha,m} \otimes a = \frac{1}{2} \left[(\mathbf{1} + \lambda \xi_{\alpha,m} \otimes a)^2 + \sum_k (\lambda g_k \otimes a)^2 + [1 \otimes (\mathbf{1} - \lambda^2 a^2)^{\frac{1}{2}}]^2 \right]$$

for $a \in A_h$ and $|\lambda|$ sufficiently small. Hence any generator $\xi_{\alpha,m} \otimes a \in \mathcal{B}'$. Consequently, $\mathcal{B}' = \mathcal{B}$. The last part of the proof is similar to that of Lemma 4 and will be omitted ■

The next result is an extension of [17: Corollary 4.4].

Theorem 16. *Let $g_0, g_1 \in \mathbb{R}[x] \otimes A_h$ be such that $\deg g_0 < \deg g_1$. Let also $p_0 \in \mathbb{R}[x] \otimes M_\tau(\mathbb{C})_h$ be of even degree. Assume that $\tilde{g}_1(x') > 0$ for all $x' \in \mathbb{R}^{n+1} \setminus \{0\}$ and that $p(t) > 0$ whenever $p_0(t) \geq 0$, where $p = g_0 + g_1$. Then there are $q_j \in \mathbb{C}[x] \otimes A$, $q_{j0} \in \mathbb{C}[x] \otimes A^\tau$ and $\theta \in \mathbb{Z}_+$ such that*

$$\varphi^{2\theta} p = \sum_{j \in J} (q_j^* q_j + q_{j0}^* p_0 q_{j0}).$$

Proof. The hypothesis on p clearly implies that $\delta := \deg p = \deg g_1$ is even. Set $\psi(x) = \frac{1}{1+\|x\|^2}$ and $r(x) = \psi(x)^{\frac{\delta}{2}} p(x)$. Set also $r_0(x) = \psi(x)^{\frac{\delta_0}{2}} p_0(x)$, where $\delta_0 = \deg p_0$. We denote by \mathcal{C} the convex cone in \mathcal{B} (see the previous lemma) consisting of all finite sums of the form

$$\sum_{j \in J} (h_j^* h_j + h_{j0}^* r_0 h_{j0}) \quad (h_j \in \mathcal{Q} \otimes A, h_{j0} \in \mathcal{Q} \otimes A^\tau).$$

Obviously, $\mathcal{C} \supset \mathcal{B}_+$.

Assume that $r \notin \mathcal{C}$. As in the proof of Theorem 8, using Lemma 15 and Mazur's theorem we deduce the existence of a non-null linear functional Λ_0 on \mathcal{B} such that $\Lambda_0(\mathbf{1}) > \inf_{\mathcal{C}} \Lambda_0 = 0 \geq \Lambda_0(r)$. Let Λ be an extension of Λ_0 to the complex space $\mathcal{Q} \otimes A$.

With the notation from Proposition 14, since \tilde{p} is homogeneous of even degree δ , we can find a homogeneous polynomial $P \in \mathbb{R}[y] \otimes A_h$ of degree $\frac{\delta}{2}$ (not uniquely determined) such that

$$\tilde{p}(x') = P(y) \quad \begin{cases} x' = (x_0, \dots, x_n) \\ y = (y_{jk})_{j,k=0}^n, y_{jk} = x_j x_k. \end{cases}$$

Similarly, we can find a polynomial $P_0 \in \mathbb{R}[y] \otimes M_\tau(\mathbb{C})_h$ such that $\tilde{p}_0(x') = P_0(y)$. Therefore, with $x_0 = 1$, we have

$$r(x) = \psi(x)^{\frac{\delta}{2}} p(x) = \psi(x)^{\frac{\delta}{2}} \tilde{p}(1, x) = \psi(x)^{\frac{\delta}{2}} P((x_j x_k)_{j,k}) = P \circ \Psi(x)$$

for all $x \in \mathbb{R}^n$. Similarly, $r_0(x) = P_0 \circ \Psi(x)$.

According to Proposition 14, there are two A^* -valued non-negative measures μ on \mathbb{R}^n and ν on V such that

$$\Lambda(r) = \int_{\mathbb{R}^n} (d\mu, r) + \int_V (d\nu, P) \leq 0. \tag{10}$$

Note that $P(y) > 0$ if $y \in V$. Indeed, if $y \in V$, then $y_{00} = x_0^2 = 0$ implies $x_0 = 0$. We also have $\tilde{p}(x') = \tilde{g}_1(x') + x_0^d \tilde{g}_0(x')$ with $d \neq 0$, and so $\tilde{p}(x') = \tilde{g}_1(x') > 0$ whenever

$x' = (0, x)$ with $x \neq 0$ by the hypothesis. Consequently, $P(y) = \tilde{p}(x') > 0$ if $y \in V$, implying $\int_V(d\nu, P) > 0$ provided $\nu \neq 0$.

We shall prove that $\text{supp } \mu \subset \{t \in \mathbb{R}^n : p_0(t) \geq 0\}$. Note that $\Lambda(h^*r_0h) \geq 0$ for all $h \in \mathcal{Q} \otimes A^\tau$. We also have $r_0 = P_0 \circ \Psi$ for some polynomial P_0 . We proceed now as in the proof of Lemma 5. If $P_0 = [P_{jk}]_{j,k}$ with P_{jk} ordinary polynomials, we set $P_0(T) = [P_{jk}(T)]_{j,k}$, where $T \in B(\mathcal{H})^N$ is defined in the proof of Proposition 14. There exists a natural map $\mathcal{Q} \otimes A^\tau \rightarrow \mathcal{H}^\tau$ whose image is dense in \mathcal{H}^τ . Moreover, if $r_0 = [r_{jk}]_{j,k} = [P_{jk} \circ \Psi]_{j,k}$, the multiplication by r_{jk} induces in \mathcal{H} the operator $P_{jk}(T)$ for all j, k . Therefore, $\langle P_0(T)\hat{h}, \hat{h} \rangle = \langle \widehat{r_0 h}, \hat{h} \rangle = \Lambda(h^*r_0h) \geq 0$ for any $h \in \mathcal{Q} \otimes A^\tau$. Hence $P_0(T) \geq 0$ in $B(\mathcal{H}^\tau)$. The discussion from the proof of Lemma 5 can be applied to the normal commuting τ^2 -tuple of all entries of $P_0(T)$. We obtain $\text{supp } E \subset \{z \in \mathbb{R}^{\tau^2} : P_0(z) \geq 0\}$, whence

$$\text{supp } \mu \subset \Psi^{-1}(\text{supp } E) \subset \{x \in \mathbb{R}^n : r_0(x) \geq 0\} = \{x \in \mathbb{R}^n : p_0(x) \geq 0\}.$$

In particular, this shows that $\int_{\mathbb{R}^n}(d\mu, r) > 0$ since $r(x) = \psi(x)^{\frac{5}{2}}p(x) > 0$ whenever $p_0(x) \geq 0$, provided $\mu \neq 0$, by the hypothesis. But $m \neq 0$, and so either μ or ν is non-null, implying $\Lambda(r) > 0$, which contradicts (10). Consequently, $r = \sum_{j \in J}(h_j^*h_j + h_{j_0}^*r_0h_{j_0})$ with $h_j \in \mathcal{Q} \otimes A$ and $h_{j_0} \in \mathcal{Q} \otimes A^\tau$ for all $j \in J$, which clearly implies the assertion ■

Example. We justify in what follows the operator-valued generalization. With the notation from Theorem 8, let $A = M_2(\mathbb{C})$, $n = 2$, $m = 1$ and $\nu_1 = 2$. Also, for $\varepsilon > 0$, take

$$p_1(x) = \begin{pmatrix} x_1x_2 - x_1^2 & (1 - \varepsilon)x_1x_2 \\ (1 - \varepsilon)x_1x_2 & x_1x_2 \end{pmatrix}$$

and let $p(x) = p_\varepsilon(x) = [a_{ij}(x)]_{i,j=1}^2$ with

$$\begin{aligned} a_{11}(x) &= 2x_1^4 + 2x_2^4 + 5x_1x_2^3 - 4x_1^2x_2^2 \\ a_{12}(x) &= 2(2 - \varepsilon)x_1x_2^3 - (1 - \varepsilon)x_1^2x_2^2 \\ a_{21}(x) &= 2(2 - \varepsilon)x_1x_2^3 - (1 - \varepsilon)x_1^2x_2^2 \\ a_{22}(x) &= 4x_1^2x_2^2 + x_1x_2^3 + 2x_1^3x_2 - x_1^4 + 2(1 - \varepsilon)^2x_2^4. \end{aligned}$$

Then $K_0 = K \cup (-K)$, where

$$K = K_\varepsilon = \left\{ t \in \mathbb{R}^2 \mid t_1^2 + t_2^2 = 1 \text{ and } 0 \leq t_1 \leq (2\varepsilon - \varepsilon^2)t_2 \right\}.$$

Note that $\lim_{\varepsilon \rightarrow 0, t \in K_\varepsilon} p_\varepsilon(t) = 2I_2 > 0$. Then for ε sufficiently small we can apply Theorem 8 and p can be written in form (4). For instance, one can easily check that

$$p(x) = q_1^*(x)q_1(x) + q_{11}^*(x)p_1(x)q_{11}(x)$$

where

$$q_1(x) = \sqrt{2} \begin{pmatrix} x_2^2 & x_1x_2 \\ x_1^2 & -(1 - \varepsilon)x_2^2 \end{pmatrix} \quad \text{and} \quad q_{11} = \begin{pmatrix} a \\ b \end{pmatrix} \in M_{2 \times 1}(\mathbb{C}[x] \otimes A)$$

with

$$a(x) = \begin{pmatrix} 0 & x_1 \\ 2x_2 & 0 \end{pmatrix} \quad \text{and} \quad b(x) = \begin{pmatrix} -x_2 & 0 \\ 0 & x_1 + x_2 \end{pmatrix}.$$

More precisely,

$$\begin{aligned} q_{11}^*(x)p_1(x)q_{11}(x) &= (a^*(x) \ b^*(x)) \begin{pmatrix} x_1x_2 - x_1^2 & (1 - \varepsilon)x_1x_2 \\ (1 - \varepsilon)x_1x_2 & x_1x_2 \end{pmatrix} \begin{pmatrix} a(x) \\ b(x) \end{pmatrix} \\ &= (x_1x_2 - x_1^2)a^*(x)a(x) + (1 - \varepsilon)x_1x_2b^*(x)a(x) \\ &\quad + (1 - \varepsilon)x_1x_2a^*(x)b(x) + x_1x_2b^*(x)b(x). \end{aligned}$$

We omit the details, that are routine. Note that $p(1, 0) \not\geq 0$, and so the term $\sum_{j \in J} q_{j1}^* p_1 q_{j1}$ must be $\neq 0$ in any representation of form (4). Since $p_1(t) > 0$ if and only if $t_1 t_1 - t_1^2 > 0$ and $\det p_1(t) > 0$, one can also find $\theta \in \mathbf{Z}_+$ and $q_j, q_{jk} \in M_2(\mathbb{C}[x])$ for $j \in J$, J finite, and $k = 1, 2$ such that

$$\begin{aligned} \|x\|^{2\theta} p(x) &= \sum_{j \in J} \left(q_j^*(x)q_j(x) + (x_1x_2 - x_1^2)q_{j1}^*(x)q_{j1}(x) \right. \\ &\quad \left. + x_1^2x_2[(2\varepsilon - \varepsilon^2)x_2 - x_1]q_{j2}^*(x)q_{j2}(x) \right) \end{aligned}$$

($m = 2, \nu_1 = \nu_2 = 1$). We have $p(t) > 0$ if and only if $a_{11}(t), \det p(t) > 0$, but it does not seem to exist an obvious way of deriving (4) for $p(x)$ from corresponding representations of $a_{11}(x)$ and $\det p(x)$. This shows that the techniques from [16, 17] provide non-trivial results when applied to the case of A -valued polynomials.

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