A Certain Series
Associated with Catalan’s Constant

V. S. Adamchik

Abstract. A parametric class of series generated by integration of complete elliptic integrals
\[ \sum_{r\neq k=0}^{\infty} \frac{(2k)^2}{(k+r)16^k} \] is evaluated in closed form. Alternative proofs to results of Ramanujan and others are given. Also, a particular case of the Saalschützian hypergeometric series \( _4F_3(1) \) is derived.

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1. Introduction

The subject of our interest is the hypergeometric series generated by elliptic integrals

\[ S(r) = \sum_{k=0}^{\infty} \frac{(2k)^2}{(k+r)16^k} = \frac{1}{r} \,_4F_3\left(\frac{1}{2}, \frac{1}{2}, r; 1, r + 1; 1\right). \tag{1} \]

This series has a long and interesting story. About a century ago Ramanujan (see [8: p. 351] and [3: p. 39]) in his first letter to Hardy stated without proof a particular case of (1), when the parameter \( r = n \) is a positive integer, namely

\[ S(r) = \frac{16^n}{\pi n^2 \left(\frac{2n}{n}\right)^2} \sum_{k=0}^{n-1} \frac{(2k)^2}{16^k}. \tag{2} \]

In 1927, when Ramanujan’s collected papers were published and result (2) became publicly known, it attracted a great deal of attention. Different proofs were given by Watson [13] and Darling [4], later Bailey [2] and Hodgkinson [9] generalized (2) to

\[ _3F_2(a, b, c + n - 1; c, a + b + n; 1) = \frac{\Gamma(n)\Gamma(a + b + n)}{\Gamma(a + n)\Gamma(b + n)} \sum_{k=0}^{n-1} \frac{(a)_k(b)_k}{(c)_kk!} \]

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which gives Ramanujan’s result when \( a = b = \frac{1}{2} \) and \( c = 1 \). Ramanujan (see [11: pp. 237 - 239] and [3: p. 45]) also stated a complementary formula to (2), when the parameter \( r = n + \frac{1}{2} \) is a half integer, namely

\[
S(n + \frac{1}{2}) = \frac{4}{\pi} \left( \frac{2n}{n} \right)^2 \left( 2G + \sum_{k=0}^{n-1} \frac{16^k}{\binom{2k}{k}^2 (2k+1)^2} \right).
\] (3)

Here \( G \) is Catalan’s constant defined by

\[
G = \frac{1}{2} \int_0^1 K(k) \, dk = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}
\]

and \( K \) is the complete elliptic integral of the first kind, given by

\[
K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}.
\]

As mentioned in [3: p. 47], Ramanujan’s proofs of formulas (2) and (3) most likely were based on the recurrence equation

\[
(r + \frac{1}{2})^2 S(r + 1) - r^2 S(r) = \frac{1}{\pi}
\] (4)

subject to initial conditions. This equation is derived from the fact that \( S(r) \) is generated by integration of complete elliptic integrals as

\[
S(r) = \frac{2}{\pi} \int_0^1 z^{r-1} K(z) \, dz \quad (\Re(r) > 0).
\] (5)

In 1981, unawared of Ramanujan’s equation (4), Dutka [5] employed (5) to rediscover formulas (2) and (3). In Section 2 we outline the derivation of equation (4), as well as its solution. In view of (4), it is pretty straightforward to see that for any rational \( r = n + p \), where \( n \) is a positive integer and \( 0 < p \leq 1 \), series (1) has a closed form representation

\[
S(n + p) = \frac{(p)_n^2}{(p + \frac{1}{2})_n^2} \left( S(p) + \frac{1}{\pi p^2} \sum_{k=0}^{n-1} \frac{(p + \frac{1}{2})_k^2}{(p + \frac{1}{2})_k^2} \right).
\]

Here \((p)_n = p(p+1) \cdots (p+n-1)\) is the Pochhammer symbol. There are only three known cases when the function \( S(p) \) is expressible in terms other than hypergeometric functions, namely \( p \in \{1, \frac{1}{2}, \frac{1}{4}\} \) with

\[
S(1) = 3F_2(\frac{1}{2}, \frac{1}{2}, 1; 1, 2; 1) = 2F_1(\frac{1}{2}, \frac{1}{2}; 2; 1) = \frac{4}{\pi}
\]

\[
S(\frac{1}{2}) = 2F_2(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{1}{2}; 1) = \frac{8G}{\pi}
\]

\[
S(\frac{1}{4}) = 4F_2(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}; 1, \frac{5}{4}; 1) = \frac{\Gamma(\frac{1}{4})^4}{4\pi^2}
\]
where $\Gamma(z)$ is the Euler gamma function. All these cases are due to Ramanujan (see [3]). Glasser [6] made a conjecture that it is possible to express $S(\frac{1}{2^k})$ for $k \geq 3$ in finite terms, however this is remained to be seen.

It does not appear to have been previously studied the case when the parameter $r$ in (1) is a negative integer (assuming that the term $r = -k$ is dropped from summation):

$$S(r) = \sum_{-r \neq k=0}^{\infty} \frac{(2k)^2}{(k+r)16^k}.$$  \hfill (6)

A few particular cases of (6) appeared in the handbooks by Adams and Hippisley [1] and by Hansen [7]:

\begin{align*}
S(-1) &= - \frac{2G+1}{\pi} + \log 2 - \frac{1}{2} \\
S(-2) &= - \frac{18G+13}{16\pi} + \frac{9}{16} \log 2 - \frac{21}{64}.
\end{align*}

In the present paper, using contour integration technique, we will show that for negative integer $r$ sum (6) is solvable in closed form by

$$S(r) = -S(\frac{1}{2} - r) + \frac{4}{16^{-r}} \left( -\frac{2r}{-r} \right)^2 \left( H_r - H_{-2r} + \log 2 \right)$$

where $H_n$ are the harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

As a consequence of this result, in Section 3 we derive the new representation for Saalschützian $\binom{4}{3}F_3(1)$ series with a special set of the parameters

\begin{align*}
(n - \frac{1}{2})4F_3(1, 1, n + \frac{1}{2}, n + \frac{1}{2}; 2, n + 1, n + 1; 1) \\
= \frac{4n^2}{2n-1} (H_{n-1} + \log 4) - \frac{16^n}{(\frac{n}{2^n})^3} F_2(\frac{1}{2}, \frac{1}{2}, n - \frac{1}{2}; 1, n + \frac{1}{2}; 1).
\end{align*}

2. Evaluation

We consider two cases, namely when $r$ is positive and negative. We denote

$$S^+(r) = S(r) \quad (\Re(r) > 0)$$
$$S^-(r) = S(r) \quad (\Re(r) \leq 0).$$

Let $r$ be a positive integer. We transform series (1) to a definite integral involving complete elliptic integrals. Multiplying the summand by $x^{k+r}$ and differentiating it with respect to $x$, we get

$$g(r, x) = x^{r-1} \sum_{k=0}^{\infty} \left( \frac{2k}{k} \right)^2 \frac{x^k}{16^k} = \frac{2}{\pi} x^{r-1} K(x)$$

for $|x| < 1$ where $K(x)$ is the elliptic integral. Integrating both sides of (7), we arrive at

$$S^+(r) = \int_{0}^{1} g(r, x) \, dx = \frac{2}{\pi} \int_{0}^{1} x^{r-1} K(x) \, dx \quad (\Re(r) > 0).$$

\hfill (8)
In the next subsections we evaluate $S^+(r)$ by first developing a recurrent equation for $S^+(r)$ and then solving it by iteration. The result depends on the disparity of $r$.

Now let us consider the second case when $r$ is a negative integer. We split the series $S(r)$ into two sums as

$$S^-(r) = \sum_{k=0}^{\infty} \frac{(2k)^2}{(k + r)16^k} = \left( \sum_{k=0}^{\infty} \frac{r-1}{16^k} + \sum_{k=-r+1}^{\infty} \frac{(2k)^2}{(k + r)16^k} \right).$$

Leaving the first sum unchanged, and converting the second sum into an elliptic integral (by applying the same reasoning as above), we obtain

$$S^-(r) = \sum_{k=0}^{r-1} \frac{(2k)^2}{(k + r)16^k} + \int_0^1 x^{r-1} \left( \frac{2}{\pi} K(x) - \sum_{k=0}^{r-1} \frac{(2k)^2}{16^k} \right) dx \quad (9)$$

for $\Re(r) \leq 0$. In Subsection 2.3, using contour integration technique, we establish a functional relation transforming $S^-(r)$ into $S^+(r)$.

2.1 $S^+(r)$ for $r$ a non-negative integer. Consider the system of indefinite integrals

$$\begin{align*}
k_p(x) &= \int x^p K(x) \, dx \\
e_p(x) &= \int x^p E(x) \, dx
\end{align*}$$

(10)

where the parameter $p$ is a positive integer or zero, and $E(x)$ and $K(x)$ are complete elliptic integrals. Using integration by parts, the above integral system can be reduced to the system of coupled recurrent equations

$$\begin{align*}
k_p(x) &= x^p k_0(x) - 2p(k_p(x) - k_{p-1}(x) + e_{p-1}(x)) \\
e_p(x) &= x^p e_0(x) - \frac{2}{3} p(e_{p-1}(x) + e_p(x) + k_p(x) - k_{p-1}(x))
\end{align*}$$

with initial conditions

$$\begin{align*}
2k_0(x) &= E(x) + (x - 1)K(x) \\
\frac{3}{2} e_0(x) &= (x + 1)E(x) + (x - 1)K(x).
\end{align*}$$

Eliminating $e_{p-1}(x)$ from the first equation, and $k_{p-1}(x)$ and $k_p(x)$ from the second, the system is simplified to

$$\begin{align*}
k_p(x) &= \frac{4p^2}{(2p + 1)^2} k_{p-1}(x) + \frac{2x^p E(x) + 2(2p + 1)(x - 1)x^p K(x)}{(2p + 1)^2} \\
e_p(x) &= \frac{4p^2}{(2p + 1)(2p + 3)} e_{p-1}(x) + \frac{2(1 - 2p + (2p + 1)x)x^p E(x) + 2(x - 1)x^p K(x)}{(2p + 1)(2p + 3)}
\end{align*}$$

.$$
Now we compute the values of \( k_p(x) \) and \( e_p(x) \) at the limiting points \( x = 0 \) and \( x = 1 \). We get two recurrent equations

\[
\begin{align*}
k_p(0) &= 0 \quad (p \geq 0) \\
k_0(1) &= 2 \\
k_p(1) &= \frac{4p^2}{(2p + 1)^2} k_{p-1}(1) + \frac{2}{(2p + 1)^2} \quad (p \geq 1)
\end{align*}
\]

and

\[
\begin{align*}
e_p(0) &= 0 \quad (p \geq 0) \\
e_p(1) &= \frac{4p^2}{(2p + 1)(2p + 3)} e_{p-1}(1) + \frac{4}{(2p + 1)(2p + 3)} \quad (p \geq 1).
\end{align*}
\]

In view of formulas (8) and (11) we conclude that

\[
S^+(r) = \frac{2}{\pi} (k_{r-1}(1) - k_{r-1}(0)) = \frac{2}{\pi} k_{r-1}(1)
\]

where \( S^+(r) \) satisfies the recurrence relation

\[
\begin{align*}
S^+(1) &= \frac{4}{\pi} \\
(r + \frac{1}{2})^2 S^+(r + 1) - r^2 S^+(r) &= \frac{1}{\pi} \quad (r \geq 1)
\end{align*}
\]

This recurrence equation can be solved by iteration (see Section 4 for details).

We have proven

**Proposition 2.1.** Let \( n \) be a positive even. Then \( S(n) \) defined by (1) evaluates to

\[
S(n) = \frac{16^n}{\pi n^2 (2^n)^2} \sum_{k=0}^{n-1} \left( \frac{2k}{n} \right)^2 \frac{1}{16^k}.
\]

**2.2 \( S^+(r) \) for \( r \) a positive half-integer.** Consider slightly different (than (10)) system of indefinite integrals

\[
\begin{align*}
\hat{k}_p(x) &= \int x^{p-\frac{1}{2}} K(x) \, dx \\
\hat{e}_p(x) &= \int x^{p-\frac{1}{2}} E(x) \, dx
\end{align*}
\]

where the parameter \( p \) is a positive integer or zero, and \( E(x) \) and \( K(x) \) are complete elliptic integrals. Using integration by parts, we transform (13) to the system of recurrent equations

\[
\begin{align*}
p^2 \hat{k}_r(x) &= (p - \frac{1}{2})^2 \hat{k}_{p-1}(x) + \frac{1}{2} x^{p-\frac{1}{2}} \left( E(x) + 2p(x-1)K(x) \right) \\
p(p + 1) \hat{e}_r(x) &= (p - \frac{1}{2})^2 \hat{e}_{p-1}(x) + x^{p-\frac{1}{2}} \left( (p(x-1) + 1)E(x) + \frac{x-1}{2} K(x) \right)
\end{align*}
\]
where
\[ \hat{k}_0(x) = \pi \sqrt{x} 3F_2\left(\frac{1}{2}, \frac{1}{2}, 1; 1, \frac{3}{2}; x\right) \]
\[ \hat{e}_0(x) = \pi \sqrt{x} 3F_2\left(-\frac{1}{2}, \frac{1}{2}, 1; 1, \frac{3}{2}; x\right) \]
and \(3F_2(x)\) is the hypergeometric function. By computing the limits at \(x = 0\) and \(x = 1\), system (14) yields
\[ \hat{k}_p(0) = 0 \quad (p \geq 0) \]
\[ \hat{k}_0(1) = 4G \]
\[ \hat{k}_p(1) = \frac{(p - \frac{1}{2})^2}{p^2} \hat{k}_{p-1}(1) + \frac{1}{2p^2} \quad (p \geq 1) \]
where \(G\) is Catalan’s constant. Therefore,
\[ S^+\left(\frac{1}{2}\right) = \frac{8G}{\pi} \]
\[ (r + \frac{1}{2})^2 S^+(r + 1) - r^2 S^+(r) = \frac{1}{\pi}, \tag{15} \]
Solving this recurrence by iteration (see Section 4 for details), we have proven

**Proposition 2.2.** Let \(n\) be a positive integer. Then \(S(n + \frac{1}{2})\) defined by (1) evaluates to
\[ S(n + \frac{1}{2}) = \frac{4}{\pi} \left(\frac{2^n}{n}\right)^2 \left(2G + \sum_{k=0}^{n-1} \frac{16^k}{(2k)2(2k + 1)^2}\right). \tag{16} \]

### 2.3 \(S^-(r)\) for \(r\) a negative integer.
Recall formula (9). Observing that the finite sum inside of the integrand \(\sum_{k=0}^{-r} \left(\frac{2k}{2}\right)^k x^k\) is the Taylor expansion of \(\frac{2}{\pi} K(x)\) at \(x = 0\), we pull that sum out of integration, by understanding integration in the Hadamard sense (finite part). Computing limits at the end points and obliterating logarithmic and polynomial order singularities, we get
\[ S^{-}(r) = \text{f.p.} \frac{2}{\pi} \int_0^1 x^{-r-1} K(x) \, dx. \]
Comparing this integral with formula (8) immediately implies that
\[ S^{-}(r) = S^{+}(r) + F(r) \]
where \(F(r)\) is an unknown function. The necessity of \(F\) becomes obvious once we recall that in the original series we skip the term \(k = -r\), when \(r\) is a negative integer. In order to find \(F\), we derive a contour integral representation for the sum \(S(r)\) as
\[ S(r) = \frac{1}{2\pi i} \int_{C} \frac{\Gamma(s)\Gamma\left(\frac{1}{2} - s\right)}{\Gamma\left(\frac{1}{2} + s\right)\Gamma\left(1 - s\right) r - s} \, ds, \tag{17} \]
The contour \((\gamma - i\infty, \gamma + i\infty)\) is a straight line lying in the strip \(0 < \gamma = \Re(s) < \frac{1}{2}\). In fact, evaluating integral (17) by residues at single poles \(s = 0, -1, -2, \ldots\), lying to the left of the contour, we arrive at series (1). However, if \(r\) is a negative integer, the integrand in (17) has a double pole at \(s = r\). According to the definition of \(S^- (r)\) we must skip this pole. Thus, we have

\[
S^- (r) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma (s) \Gamma (\frac{1}{2} - s)}{\Gamma (1 - s) \Gamma (1 + s)} \frac{ds}{r - s} - \lim_{s \to r} \left( \frac{\Gamma (s) \Gamma (\frac{1}{2} - s)}{\Gamma (1 - s) \Gamma (1 + s)} \right).
\]

As a matter of fact, the contour integral herein can also be computed via residues at the poles \(s = \frac{1}{2}, \frac{3}{2}, \ldots\), lying to the right of the contour. Evaluating the integral via those poles allows us to avoid the double pole at \(s = r\). This yields

\[
\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma (s) \Gamma (\frac{1}{2} - s)}{\Gamma (1 - s) \Gamma (1 + s)} \frac{ds}{r - s} = -\sum_{k=0}^{\infty} \frac{(2k)!^2}{k! (k - r + 1/2)^{16k}} = -S^+ (\frac{1}{2} - r).
\]

Finally, computing the residue

\[
\lim_{s \to r} \left( \frac{\Gamma (s) \Gamma (\frac{1}{2} - s)}{\Gamma (1 - s) \Gamma (1 + s)} \right) = \frac{4}{16^{-r} (-2r)^2} (H_{-2r} - H_{-r} - \log 2)
\]

we establish

**Proposition 2.3.** Let \(r\) be a negative integer or zero. Then

\[
S^- (r) = -S^+ (\frac{1}{2} - r) - \frac{4}{16^{-r} (-2r)^2} (H_{-2r} - H_{-r} + \log 2)
\]

where \(S^+ (\frac{1}{2} - r)\) is defined in Proposition 2.2.

### 2.4 \(S^- (r)\) for \(r\) a negative half integer.

This case immediately follows from the previous subsection, taking into consideration that the integrand in (17) has only a single pole at \(s = r\).

**Proposition 2.4.** Let \(n\) be a positive integer. Then \(S^- (-n + \frac{1}{2}) = -S^+ (n)\).
3. Special cases of hypergeometric functions

In this section we derive a particular case of the Saalschützian hypergeometric series \( _4F_3(1) \). We begin by recalling that the hypergeometric series

\[
p_{p+1}F_p(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; 1)
\]

is called Saalschützian if the parameters \( a_i \) and \( b_i \) satisfy the relation

\[
1 + a_1 + \ldots + a_{p+1} = b_1 + \ldots + b_p.
\]

**Proposition 3.1.** Let \( n \) be a positive integer. Then

\[
\frac{(2n - 1)^2}{8n^2} _4F_3(1, 1, n + \frac{1}{2}, n + \frac{1}{2}; 2, n + 1, n + 1; 1)
\]

\[
= -\frac{4G}{\pi} + H_{n-1} + \log 4 - \frac{2}{\pi} \sum_{k=0}^{n-2} \frac{16^k}{(2k + 1)^2 (\binom{2k}{k})^2} (19)
\]

where \( G \) is Catalan’s constant and \( H_n \) are harmonic numbers.

**Proof.** In view of formula (18) with \( r = -n \) \( (n \in \mathbb{N}_0) \) we have

\[
S^-(n) = -S^+(n + \frac{1}{2}) - \frac{4}{16_n} \left(\frac{2n}{n}\right)^2 (H_n - H_{2n} + \log 2) \tag{20}
\]

where \( S^+(n + \frac{1}{2}) \) is defined in (16). On the other hand, if we evaluate the original sum (6) by means of the hypergeometric function, we obtain

\[
S^-(n) = \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{(k - n)16^k}
\]

\[
+ \frac{\binom{2n+2}{n+1}}{16^{n+1}} _4F_3(1, 1, n + \frac{3}{2}, n + \frac{3}{2}; 2, n + 2, n + 2; 1). \tag{21}
\]

The finite sum in the right-hand side herein can be evaluated in terms of harmonic numbers (see Proposition 4.2) as

\[
16^n \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{16^k(n - k)} = 4 \left(\frac{2n}{n}\right)^2 \sum_{k=0}^{n-1} \frac{1}{2k + 1} = 2 \left(\frac{2n}{n}\right)^2 \left(2H_{2n-1} - H_{n-1}\right).
\]

Combining formulas (20) and (21), and replacing \( n \) by \( n - 1 \), we arrive at (19) ■

**Remark 3.2.** By using different ideas, formula (19) was first proved in [10].
4. Addendum

In this section we provide a solution to equations (12) and (15).

**Proposition 4.1.** The solution to the recurrence relation

\[
\begin{align*}
&x_1 = b \\
&(2n + 1)^2 x_{n+1} - (2n)^2 x_n = a \quad (n \geq 1)
\end{align*}
\]

is

\[
x_n = \frac{16^n}{4n^2(2n)^2} \left( b + a \sum_{k=1}^{n-1} \frac{(2k)^2}{16^k} \right).
\]

**Proof.** We solve the recurrence by iteration. Iterating it \(n - 1\) times, we get

\[
x_{n+1} = b \prod_{j=0}^{n-1} \frac{(2n - 2j)^2}{(2n - 2j + 1)^2} + a \sum_{k=0}^{n-1} \frac{\prod_{j=0}^{k-1} (2n - 2j)^2}{\prod_{j=0}^{k} (2n + 1 - 2j)^2}.
\]

(22)

In pretty straightforward manner the finite products herein can be converted to the binomial coefficients by using Euler’s product representation for the Gamma function. We obtain

\[
\prod_{j=0}^{n-1} \frac{(2n - 2j)}{(2n - 2j + 1)} = \frac{4^{n+1}}{2(n+1)(2n+2)}
\]

and

\[
\sum_{k=0}^{n-1} \frac{\prod_{j=0}^{k-1} (2n - 2j)^2}{\prod_{j=0}^{k} (2n + 1 - 2j)^2} = \frac{16^{n+1}}{4(n+1)^2(2n+2)^2} \sum_{k=1}^{n} \frac{(2k)^2}{16^k}.
\]

Substituting them into (22) yields the desired result.

**Proposition 4.2.** Let \(n\) be a positive integer. Then

\[
\frac{16^n}{4(2n)^2} \sum_{k=0}^{n-1} \frac{(2k)^2}{16^k(n-k)} = \sum_{k=0}^{n-1} \frac{1}{2k+1}.
\]

(23)

**Proof.** We rearrange the terms in the sum in the left-hand side of (23) by summing them in the opposite order from \(n - 1\) to 0. We get

\[
\sum_{k=0}^{n-1} \frac{(2k)^2}{(n-k)16^k} = \sum_{k=1}^{n} \frac{(2n-2k)^2}{k(16)^{n-k}}.
\]

Since the summand evaluates to zero for \(k > n\), we extend the range of summation to infinity. Using the definition of the hypergeometric series, we rewrite that sum in terms of \(_4F_3\) as

\[
\frac{16^n}{4(2n)^2} \sum_{k=1}^{\infty} \frac{(2n-2k)^2}{k16^{n-k}} = \frac{n^2}{(2n-1)^2} _4F_3(1, 1, 1 - n, 1 - n; 2, 3/2 - n, 3/2 - n; 1).
\]
The latter further simplifies to polygamma functions by [13: Formula 7.5.3.43] as

$$\frac{2n^2}{(2n-1)^2}{}_4F_3(1,1,1-n,1-n;2,\frac{3}{2}-n,\frac{3}{2}-n;1) = \psi(n + \frac{1}{2}) - \psi(\frac{1}{2})$$

$$= \sum_{k=0}^{n-1} \frac{2}{2k+1}$$

and the statement is proven.

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References


