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A Certain Series Associated with Catalan's Constant

V. S. Adamchik

Abstract. A parametric class of series generated by integration of complete elliptic integrals $\sum_{-r\neq k=0}^{\infty} \frac{\binom{2k}{k}}{(k+r)16^k}$ is valuated in closed form. Alternative proofs to results of Ramanujan and others are given. Also, a particular case of the Saalschützian hypergeometric series ${}_4F_3(1)$ is derived.

Keywords: *Summation of series, elliptic functions, hypergeometric functions, Catalan's constant*

AMS subject classification: Primary 33C, secondary 33E,11Y

1. Introduction

The subject of our interest is the hypergeometric series generated by elliptic integrals

$$S(r) = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2}{(k+r)16^k} = \frac{1}{r}\,{}_4F_3\left(\tfrac{1}{2}, \tfrac{1}{2}, r; 1, r+1; 1\right). \tag{1}$$

This series has a long and interesting story. About a century ago Ramanujan (see [8: p. 351] and [3: p. 39]) in his first letter to Hardy stated without proof a particular case of (1), when the parameter $r = n$ is a positive integer, namely

$$S(r) = \frac{16^n}{\pi n^2 \binom{2n}{n}^2} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{16^k}. \tag{2}$$

In 1927, when Ramanujan's collected papers were published and result (2) became publicly known, it attracted a great deal of attention. Different proofs were given by Watson [13] and Darling [4], later Bailey [2] and Hodgkinson [9] generalized (2) to

$${}_3F_2\big(a, b, c+n-1; c, a+b+n; 1\big) = \frac{\Gamma(n)\Gamma(a+b+n)}{\Gamma(a+n)\Gamma(b+n)} \sum_{k=0}^{n-1} \frac{(a)_k (b)_k}{(c)_k k!}$$

V. S. Adamchik: Carnegie Mellon Univ., Dept. Comp. Sci., Pittsburgh, PA 15213-3891, USA; adamchik@cs.cmu.edu

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which gives Ramanujan's result when $a = b = \frac{1}{2}$ and $c = 1$. Ramanujan (see [11: pp. 237 - 239] and [3: p. 45]) also stated a complementary formula to (2), when the parameter $r = n + \frac{1}{2}$ is a half integer, namely

$$S(n+\tfrac{1}{2}) = \frac{4}{\pi}\frac{\binom{2n}{n}^2}{16^n}\left(2G + \sum_{k=0}^{n-1}\frac{16^k}{\binom{2k}{k}^2(2k+1)^2}\right). \tag{3}$$

Here G is Catalan's constant defined by

$$G = \tfrac{1}{2}\int_0^1 \mathbf{K}(k)\,dk = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}$$

and $\mathbf{K}$ is the complete elliptic integral of the first kind, given by

$$\mathbf{K}(k) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-k^2\sin^2 t}}.$$

As mentioned in [3: p. 47], Ramanujan's proofs of formulas (2) and (3) most likely were based on the recurrence equation

$$(r+\tfrac{1}{2})^2 S(r+1) - r^2 S(r) = \tfrac{1}{\pi} \tag{4}$$

subject to initial conditions. This equation is derived from the fact that $S(r)$ is generated by integration of complete elliptic integrals as

$$S(r) = \tfrac{2}{\pi}\int_0^1 z^{r-1}\mathbf{K}(z)\,dz \qquad (\Re(r) > 0). \tag{5}$$

In 1981, unawared of Ramanujan's equation (4), Dutka [5] employed (5) to rediscover formulas (2) and (3). In Section 2 we outline the derivation of equation (4), as well as its solution. In view of (4), it is pretty straightforward to see that for any rational $r = n + p$, where n is a positive integer and $0 < p \le 1$, series (1) has a closed form representation

$$S(n+p) = \frac{(p)_n^2}{(p+\frac{1}{2})_n^2}\left(S(p) + \frac{1}{\pi p^2}\sum_{k=0}^{n-1}\frac{(p+\frac{1}{2})_k^2}{(p+1)_k^2}\right).$$

Here $(p)_n = p(p+1)\cdots(p+n-1)$ is the Pochhammer symbol. There are only three known cases when the function $S(p)$is expressible in terms other than hypergeometric functions, namely $p \in \{1, \frac{1}{2}, \frac{1}{4}\}$ with

$$S(1) = {}_3F_2(\tfrac{1}{2},\tfrac{1}{2},1;1,2;1) = {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};2;1) = \tfrac{4}{\pi}$$
$$S(\tfrac{1}{2}) = 2{}_3F_2(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};1,\tfrac{3}{2};1) = \tfrac{8G}{\pi}$$
$$S(\tfrac{1}{4}) = 4{}_3F_2(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{4};1,\tfrac{5}{4};1) = \tfrac{\Gamma(\frac{1}{4})^4}{4\pi^2}$$

where $\Gamma(z)$ is the Euler gamma function. All these cases are due to Ramanujan (see [3]). Glasser [6] made a conjecture that it is possible to express $S(\frac{1}{2^k})$ for $k \geq 3$ in finite terms, however that is remained to be seen.

It does not appear to have been previously studied the case when the parameter r in (1) is a negative integer (assuming that the term $r = -k$ is dropped from summation):

$$S(r) = \sum_{-r \neq k=0}^{\infty} \frac{\binom{2k}{k}^2}{(k+r)16^k}. \tag{6}$$

A few particular cases of (6) appeared in the handbooks by Adams and Hippisley [1] and by Hansen [7]:

$$S(-1) = -\tfrac{2G+1}{\pi} + \log 2 - \tfrac{1}{2}$$
$$S(-2) = -\tfrac{18G+13}{16\pi} + \tfrac{9}{16}\log 2 - \tfrac{21}{64}.$$

In the present paper, using contour integration technique, we will show that for negative integer r sum (6) is solvable in closed form by

$$S(r) = -S(\tfrac{1}{2} - r) + \tfrac{4}{16^{-r}}\binom{-2r}{-r}^2\big(H_{-r} - H_{-2r} + \log 2\big)$$

where H_n are the harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

As a consequence of this result, in Section 3 we derive the new representation for Saalschützian ${}_4F_3(1)$ series with a special set of the parameters

$$(n - \tfrac{1}{2}){}_4F_3(1, 1, n + \tfrac{1}{2}, n + \tfrac{1}{2}; 2, n + 1, n + 1; 1)$$
$$= \tfrac{4n^2}{2n-1}\,(H_{n-1} + \log 4) - \tfrac{16^n}{\binom{2n}{n}^2}{}_3F_2(\tfrac{1}{2}, \tfrac{1}{2}, n - \tfrac{1}{2}; 1, n + \tfrac{1}{2}; 1).$$

2. Evaluation

We consider two cases, namely when r is positive and negative. We denote

$$S^+(r) = S(r) \qquad (\Re(r) > 0)$$
$$S^-(r) = S(r) \qquad (\Re(r) \leq 0).$$

Let r be a positive integer. We transform series (1) to a definite integral involving complete elliptic integrals. Multiplying the summand by x^{k+r} and differentiating it with respect to x, we get

$$g(r, x) = x^{r-1} \sum_{k=0}^{\infty} \binom{2k}{k}^2 \frac{x^k}{16^k} = \frac{2}{\pi} x^{r-1}\mathbf{K}(x) \tag{7}$$

for $|x| < 1$ where $\mathbf{K}(x)$ is the elliptic integral. Integrating both sides of (7), we arrive at

$$S^+(r) = \int_0^1 g(r, x)\, dx = \tfrac{2}{\pi} \int_0^1 x^{r-1}\mathbf{K}(x)\, dx \qquad (\Re(r) > 0). \tag{8}$$

In the next subsections we evaluate $S^+(r)$ by first developing a recurrent equation for $S^+(r)$ and then solving it by iteration. The result depends on the disparity of r.

Now let us consider the second case when r is a negative integer. We split the series $S(r)$ into two sums as

$$S^-(r) = \sum_{-r\neq k=0}^{\infty} \frac{\binom{2k}{k}^2}{(k+r)16^k} = \left(\sum_{k=0}^{-r-1} + \sum_{k=-r+1}^{\infty} \right) \frac{\binom{2k}{k}^2}{(k+r)16^k}.$$

Leaving the first sum unchanged, and converting the second sum into an elliptic integral (by applying the same reasoning as above), we obtain

$$S^-(r) = \sum_{k=0}^{-r-1} \frac{\binom{2k}{k}^2}{(k+r)16^k} + \int_0^1 x^{r-1} \left(\frac{2}{\pi}\mathbf{K}(x) - \sum_{k=0}^{-r} \binom{2k}{k}^2 \frac{x^k}{16^k} \right) dx \tag{9}$$

for $\Re(r) \leq 0$. In Subsection 2.3, using contour integration technique, we establish a functional relation transforming $S^-(r)$ into $S^+(r)$.

2.1 $S^+(r)$ for r a non-negative integer. Consider the system of indefinite integrals

$$\left.\begin{aligned} k_p(x) &= \int x^p \mathbf{K}(x)\, dx \\ e_p(x) &= \int x^p \mathbf{E}(x)\, dx \end{aligned}\right\} \tag{10}$$

where the parameter p is a positive integer or zero, and $\mathbf{E}(x)$ and $\mathbf{K}(x)$ are complete elliptic integrals. Using integration by parts, the above integral system can be reduced to the system of coupled recurrent equations

$$\left.\begin{aligned} k_p(x) &= x^p k_0(x) - 2p\big(k_p(x) - k_{p-1}(x) + e_{p-1}(x)\big) \\ e_p(x) &= x^p e_0(x) - \tfrac{2}{3}p\big(e_{p-1}(x) + e_p(x) + k_p(x) - k_{p-1}(x)\big) \end{aligned}\right\}$$

with initial conditions

$$\begin{aligned} 2k_0(x) &= \mathbf{E}(x) + (x-1)\mathbf{K}(x) \\ \tfrac{3}{2}e_0(x) &= (x+1)\mathbf{E}(x) + (x-1)\mathbf{K}(x). \end{aligned}$$

Eliminating $e_{p-1}(x)$ from the first equation, and $k_{p-1}(x)$ and $k_p(x)$ from the second, the system is simplified to

$$\left.\begin{aligned} k_p(x) &= \frac{4p^2}{(2p+1)^2} k_{p-1}(x) + \frac{2x^p\mathbf{E}(x) + 2(2p+1)(x-1)x^p\mathbf{K}(x)}{(2p+1)^2} \\ e_p(x) &= \frac{4p^2}{(2p+1)(2p+3)} e_{p-1}(x) \\ &\quad + \frac{2\big(1-2p+(2p+1)x\big)x^p\mathbf{E}(x) + 2(x-1)x^p\mathbf{K}(x)}{(2p+1)(2p+3)} \end{aligned}\right\}.$$

Now we compute the values of $k_p(x)$ and $e_p(x)$ at the limiting points $x = 0$ and $x = 1$. We get two recurrent equations

$$\left.\begin{aligned} &k_p(0) = 0 \quad (p \geq 0) \\ &k_0(1) = 2 \\ &k_p(1) = \frac{4p^2}{(2p+1)^2} k_{p-1}(1) + \frac{2}{(2p+1)^2} \quad (p \geq 1) \end{aligned}\right\} \tag{11}$$

and

$$\begin{aligned} &e_p(0) = 0 \quad (p \geq 0) \\ &e_p(1) = \frac{4p^2}{(2p+1)(2p+3)} e_{p-1}(1) + \frac{4}{(2p+1)(2p+3)} \quad (p \geq 1). \end{aligned}$$

In view of formulas (8) and (11) we conclude that

$$S^+(r) = \tfrac{2}{\pi}\big(k_{r-1}(1) - k_{r-1}(0)\big) = \tfrac{2}{\pi}\, k_{r-1}(1)$$

where $S^+(r)$ satisfies the recurrence relation

$$\left.\begin{aligned} S^+(1) &= \tfrac{4}{\pi} \\ (r+\tfrac{1}{2})^2 S^+(r+1) - r^2 S^+(r) &= \tfrac{1}{\pi} \quad (r \geq 1) \end{aligned}\right\}. \tag{12}$$

This recurrence equation can be solved by iteration (see Section 4 for details).

We have proven

Proposition 2.1. *Let n be a positive even. Then $S(n)$ defined by (1) evaluates to*

$$S(n) = \frac{16^n}{\pi n^2 \binom{2n}{n}^2} \sum_{k=0}^{n-1} \binom{2k}{k}^2 \frac{1}{16^k}.$$

2.2 $S^+(r)$ for r a positive half-integer. Consider slightly different (than (10)) system of indefinite integrals

$$\left.\begin{aligned} \widehat{k}_p(x) &= \int x^{p-\frac{1}{2}} \mathbf{K}(x)\, dx \\ \widehat{e}_p(x) &= \int x^{p-\frac{1}{2}} \mathbf{E}(x)\, dx \end{aligned}\right\} \tag{13}$$

where the parameter p is a positive integer or zero, and $\mathbf{E}(x)$ and $\mathbf{K}(x)$ are complete elliptic integrals. Using integration by parts, we transform (13) to the system of recurrent equations

$$\begin{aligned} p^2 \widehat{k}_r(x) &= \big(p - \tfrac{1}{2}\big)^2 \widehat{k}_{p-1}(x) + \tfrac{1}{2} x^{p-\frac{1}{2}} \big(\mathbf{E}(x) + 2p(x-1)\mathbf{K}(x)\big) \\ p(p+1)\widehat{e}_r(x) &= \big(p - \tfrac{1}{2}\big)^2 \widehat{e}_{p-1}(x) + x^{p-\frac{1}{2}} \big((p(x-1)+1)\mathbf{E}(x) + \tfrac{x-1}{2}\mathbf{K}(x)\big) \end{aligned} \tag{14}$$

where

$$\widehat{k}_0(x) = \pi\sqrt{x}\,{}_3F_2\big(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};1,\tfrac{3}{2};x\big)$$
$$\widehat{e}_0(x) = \pi\sqrt{x}{}_3F_2\big(-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};1,\tfrac{3}{2};x\big)$$

and ${}_3F_2(x)$ is the hypergeometric function. By computing the limits at $x = 0$ and $x = 1$, system (14) yields

$$\widehat{k}_p(0) = 0 \quad (p \geq 0)$$
$$\widehat{k}_0(1) = 4G$$
$$\widehat{k}_p(1) = \frac{(p-\frac{1}{2})^2}{p^2}\widehat{k}_{p-1}(1) + \frac{1}{2p^2} \quad (p \geq 1)$$

where G is Catalan's constant. Therefore, $S^+(p+\frac{1}{2}) = \frac{2}{\pi}\widehat{k}_p(1)$ $(p \geq 0)$. The sequence $S^+(r)$, where r is a positive half integer, satisfies the same recurrence equation (12), but with a different initial condition

$$\begin{aligned} S^+(\tfrac{1}{2}) &= \tfrac{8G}{\pi} \\ (r+\tfrac{1}{2})^2 S^+(r+1) - r^2 S^+(r) &= \tfrac{1}{\pi}. \end{aligned} \tag{15}$$

Solving this recurrence by iteration (see Section 4 for details), we have proven

Proposition 2.2. *Let n be a positive integer. Then $S(n+\frac{1}{2})$ defined by* (1) *evaluates to*

$$S(n+\tfrac{1}{2}) = \frac{4}{\pi}\frac{\binom{2n}{n}^2}{16^n}\left(2G + \sum_{k=0}^{n-1}\frac{16^k}{\binom{2k}{k}^2(2k+1)^2}\right). \tag{16}$$

2.3 $S^-(r)$ for r a negative integer. Recall formula (9). Observing that the finite sum inside of the integrand $\sum_{k=0}^{-r}\binom{2k}{k}^2\frac{x^k}{16^k}$ is the Taylor expansion of $\frac{2}{\pi}\mathbf{K}(x)$ at $x = 0$, we pull that sum out of integration, by understanding integration in the Hadamard sense (*finite part*). Computing limits at the end points and obliterating logarithmic and polynomial order singularities, we get

$$S^-(r) = \text{f.p.}\frac{2}{\pi}\int_0^1 x^{-r-1}\mathbf{K}(x)\,dx.$$

Comparing this integral with formula (8) immediately implies that

$$S^-(r) = S^+(r) + F(r)$$

where $F(r)$ is an unknown function. The necessity of F becomes obvious once we recall that in the original series we skip the term $k = -r$, when r is a negative integer. In order to find F, we derive a contour integral representation for the sum $S(r)$ as

$$S(r) = \tfrac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s)\Gamma(\frac{1}{2}-s)}{\Gamma(1-s)\Gamma(\frac{1}{2}+s)}\frac{ds}{r-s}. \tag{17}$$

The contour $(\gamma - i\infty, \gamma + i\infty)$ is a straight line lying in the strip $0 < \gamma = \Re(s) < \frac{1}{2}$. In fact, evaluating integral (17) by residues at single poles $s = 0, -1, -2, ...$, lying to the left of the contour, we arrive at series (1). However, if r is a negative integer, the integrand in (17) has a double pole at $s = r$. According to the definition of $S^-(r)$ we must skip this pole. Thus, we have

$$S^-(r) = \tfrac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(\frac{1}{2}-s)}{\Gamma(1-s)\Gamma(\frac{1}{2}+s)}\frac{ds}{r-s} - \mathrm{res}_{s=r}\Big(\frac{\Gamma(s)\Gamma(\frac{1}{2}-s)}{\Gamma(1-s)\Gamma(\frac{1}{2}+s)}\frac{1}{r-s}\Big).$$

As a matter of fact, the contour integral herein can also be computed via residues at the poles $s = \frac{1}{2}, \frac{3}{2}, ...$, lying to the right of the contour. Evaluating the integral via those poles allows us to avoid the double pole at $s = r$. This yields

$$\tfrac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(\frac{1}{2}-s)}{\Gamma(1-s)\Gamma(\frac{1}{2}+s)}\frac{ds}{r-s} = -\sum_{k=0}^{\infty}\frac{(2k)!^2}{k!^4(k-r+\frac{1}{2})16^k} = -S^+(\tfrac{1}{2}-r).$$

Finally, computing the residue

$$\mathrm{res}_{s=r}\Big(\frac{\Gamma(s)\Gamma(\frac{1}{2}-s)}{\Gamma(1-s)\Gamma(\frac{1}{2}+s)}\frac{1}{r-s}\Big) = \frac{4}{16^{-r}}\binom{-2r}{-r}^2(H_{-2r} - H_{-r} - \log 2)$$

we establish

Proposition 2.3. Let r be a negative integer or zero. Then

$$S^-(r) = -S^+(\tfrac{1}{2}-r) - \frac{4}{16^{-r}}\tbinom{-2r}{-r}^2(H_{-r} - H_{-2r} + \log 2) \tag{18}$$

where $S^+(\frac{1}{2}-r)$ is defined in Proposition 2.2.

2.4 $S^-(r)$ for r a negative half integer. This case immediately follows from the previous subsection, taking into consideration that the integrand in (17) has only a single pole at $s = r$.

Proposition 2.4. *Let n be a positive integer. Then $S^-(-n+\frac{1}{2}) = -S^+(n)$.*

3. Special cases of hypergeometric functions

In this section we derive a particular case of the Saalschützian hypergeometric series ${}_4F_3(1)$. We begin by recalling that the hypergeometric series

$$ {}_{p+1}F_p\big(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; 1\big) $$

is called Saalschützian if the parameters a_i and b_i satisfy the relation

$$ 1 + a_1 + \ldots + a_{p+1} = b_1 + \ldots + b_p. $$

Proposition 3.1. *Let n be a positive integer. Then*

$$ \begin{aligned} &\frac{(2n-1)^2}{8n^2}\, {}_4F_3\big(1, 1, n+\tfrac{1}{2}, n+\tfrac{1}{2}; 2, n+1, n+1; 1\big) \\ &\qquad = -\frac{4G}{\pi} + H_{n-1} + \log 4 - \frac{2}{\pi}\sum_{k=0}^{n-2} \frac{16^k}{(2k+1)^2\binom{2k}{k}^2} \end{aligned} \tag{19} $$

where G is Catalan's constant and H_n are harmonic numbers.

Proof. In view of formula (18) with $r = -n \ \ (n \in \mathbb{N}_0)$ we have

$$ S^-(-n) = -S^+(n+\tfrac{1}{2}) - \tfrac{4}{16^n}\tbinom{2n}{n}^2 (H_n - H_{2n} + \log 2) \tag{20} $$

where $S^+(n+\frac{1}{2})$ is defined in (16). On the other hand, if we evaluate the original sum (6) by means of the hypergeometric function, we obtain

$$ \begin{aligned} S^-(-n) = &\sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{(k-n)16^k} \\ &+ \frac{\binom{2n+2}{n+1}^2}{16^{n+1}}\, {}_4F_3\big(1, 1, n+\tfrac{3}{2}, n+\tfrac{3}{2}; 2, n+2, n+2; 1\big). \end{aligned} \tag{21} $$

The finite sum in the right-hand side herein can be evaluated in terms of harmonic numbers (see Proposition 4.2) as

$$ 16^n \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{16^k(n-k)} = 4\binom{2n}{n}^2 \sum_{k=0}^{n-1} \frac{1}{2k+1} = 2\binom{2n}{n}^2 (2H_{2n-1} - H_{n-1}). $$

Combining formulas (20) and (21), and replacing n by $n-1$, we arrive at (19) ∎

Remark 3.2. By using different ideas, formula (19) was first proved in [10].

4. Addendum

In this section we provide a solution to equations (12) and (15).

Proposition 4.1. *The solution to the recurrence relation*

$$\left.\begin{aligned} x_1 &= b \\ (2n+1)^2 x_{n+1} - (2n)^2 x_n &= a \quad (n \geq 1) \end{aligned}\right\}$$

is

$$x_n = \frac{16^n}{4n^2 \binom{2n}{n}^2} \left(b + a \sum_{k=1}^{n-1} \frac{\binom{2k}{k}^2}{16^k} \right).$$

Proof. We solve the recurrence by iteration. Iterating it $n-1$ times, we get

$$x_{n+1} = b \prod_{j=0}^{n-1} \frac{(2n-2j)^2}{(2n-2j+1)^2} + a \sum_{k=0}^{n-1} \frac{\prod_{j=0}^{k-1}(2n-2j)^2}{\prod_{j=0}^{k}(2n+1-2j)^2}. \tag{22}$$

In pretty straightforward manner the finite products herein can be converted to the binomial coefficients by using Euler's product representation for the Gamma function. We obtain

$$\prod_{j=0}^{n-1} \frac{(2n-2j)}{(2n-2j+1)} = \frac{4^{n+1}}{2(n+1)\binom{2n+2}{n+1}}$$

and

$$\sum_{k=0}^{n-1} \frac{\prod_{j=0}^{k-1}(2n-2j)^2}{\prod_{j=0}^{k}(2n-2j+1)^2} = \frac{16^{n+1}}{4(n+1)^2\binom{2n+2}{n+1}^2} \sum_{k=1}^{n} \frac{\binom{2k}{k}^2}{16^k}.$$

Substituting them into (22) yields the desired result ∎

Proposition 4.2. *Let n be a positive integer. Then*

$$\frac{16^n}{4\binom{2n}{n}^2} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{16^k(n-k)} = \sum_{k=0}^{n-1} \frac{1}{2k+1}. \tag{23}$$

Proof. We rearrange the terms in the sum in the left-hand side of (23) by summing them in the opposite order from $n-1$ to 0. We get

$$\sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{(n-k)16^k} = \sum_{k=1}^{n} \frac{\binom{2n-2k}{n-k}^2}{k\,16^{n-k}}.$$

Since the summand evaluates to zero for $k > n$, we extend the range of summation to infinity. Using the definition of the hypergeometric series, we rewrite that sum in terms of ${}_4F_3$ as

$$\frac{16^n}{4\binom{2n}{n}^2} \sum_{k=1}^{\infty} \frac{\binom{2n-2k}{n-k}^2}{k\,16^{n-k}} = \frac{n^2}{(2n-1)^2}\, {}_4F_3\big(1,1,1-n,1-n;2,\tfrac{3}{2}-n,\tfrac{3}{2}-n;1\big).$$

The latter further simplifies to polygamma functions by [13: Formula 7.5.3.43] as

$$\frac{2n^2}{(2n-1)^2}\,{}_4F_3\big(1,1,1-n,1-n;2,\tfrac{3}{2}-n,\tfrac{3}{2}-n;1\big)=\psi(n+\tfrac{1}{2})-\psi(\tfrac{1}{2})$$
$$=\sum_{k=0}^{n-1}\frac{2}{2k+1}$$

and the statement is proven ∎

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