Some Embeddings into the Multiplier Spaces Associated to Besov and Lizorkin-Triebel Spaces

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Abstract. We study the set of pointwise multipliers in the Lizorkin-Triebel space $F^{s,q}_p$ and of the corresponding multiplier set in the Besov space $B^{s,q}_p$, where we give sufficient conditions on the parameters $s$, $p$ and $p_1$ such that the embeddings $F^{n/p,1}_p \cap L^\infty \hookrightarrow M(F^{s,q}_p)$ and $B^{n/p,1}_p \hookrightarrow M(B^{s,q}_p)$ hold.

Keywords: Besov spaces, Lizorkin-Triebel spaces, pointwise multipliers

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1. Introduction

We propose a study of set $M(F^{s,q}_p)$ of pointwise multipliers in the Lizorkin-Triebel space $F^{s,q}_p$ and of the corresponding multiplier set in the Besov space $B^{s,q}_p$. Let us recall that

- $M(F^{s,2}_p) = F^{s,2}_{p,unif}(1 < p < \infty, s > \frac{n}{p})$ (Strichartz [9]).
- $M(B^{s,p}_p) = B^{s,p}_{p,unif}(1 \leq p \leq \infty, s > \frac{n}{p})$ (Peetre [6]).
- $M(B^{s,q}_p)$ ($1 \leq p \leq \infty, s > 0$) has been characterized in terms of capacities by Maz’ya and Shaposnikova [5].
- $M(F^{s,q}_p) = F^{s,q}_{p,unif}(1 \leq p < \infty, 1 \leq q \leq \infty, s > \frac{n}{p})$ (Franke [2]).
- $M(B^{s,q}_p) \neq B^{s,q}_{p,unif}(1 \leq q < \infty, s > \frac{n}{p})$ (Bourdaud [1]).
- $M(B^{s,q}_p)$ ($1 \leq q \leq \infty, s > 0$) has been characterized in Fourier analytic terms by Netrusov (see, for example, [7]).
- $M(B^{s,q}_p) = B^{s,q}_{p,unif}(1 \leq p \leq q \leq \infty, s > \frac{n}{p})$ (Sickel and Smirnov [9]).

In this paper we consider essentially the case $s = \frac{n}{p}$ and this contribution is the continuation of Runst and Sickel’s work [7].

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2. Preliminaries

All functions, spaces etc. are defined on the Euclidean space $\mathbb{R}^n$. We set $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$, $L^p = L^p(\mathbb{R}^n)$ etc. If $f \in \mathcal{S}$, then

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x) \exp(-ix \cdot \xi) \, dx \quad (\xi \in \mathbb{R}^n)$$

denotes the Fourier transform of $f$ and $\mathcal{F}^{-1}f$ its inverse transform.

Let $\phi \in \mathcal{D}$ such that $\phi \geq 0$, supp $\phi \subset \{ \xi \in \mathbb{R}^n : 1 \leq |\xi| \leq 3 \}$ and $\sum_{j \in \mathbb{Z}} \phi(2^{-j} \xi) = 1$. It follows that the function $\xi \to \varphi(\xi) = 1 - \sum_{j \geq 1} \phi(2^{-j} \xi)$ is in $C^\infty$ with support in the ball $|\xi| \leq 3$ and one has $\varphi(\xi) + \sum_{j \geq 1} \phi(2^{-j} \xi) = 1$ ($\xi \in \mathbb{R}^n$). To this partition of unity we associate the convolution operators $\Delta_k$ ($k \in \mathbb{N}$) and $Q_j$ ($j \in \mathbb{N} \cup \{0\}$) defined by

$$\Delta_k f(\xi) = \phi(2^{-k} \xi) \hat{f}(\xi) \quad \text{and} \quad (Q_j f)^\wedge(\xi) = \varphi(2^{-j} \xi) \hat{f}(\xi).$$

We set $\Delta_0 = Q_0$. The Littlewood-Paley decomposition is the identity

$$f = Q_k f + \sum_{j \geq k+1} \Delta_j f \quad \left( Q_k f = \sum_{j \leq k} \Delta_j f \right)$$

of all $f \in \mathcal{S}'$. The series converges in $\mathcal{S}'$.

The support of $\Delta_k(\Delta_j f \Delta_l g)$ is not empty in one of the following cases:

- $l \leq k + 1$ and $k - 2 \leq j \leq k + 4$
- $j \leq k + 1$ and $k - 2 \leq l \leq k + 4$
- $l, j \geq k$ and $|l - 1| \leq 1$.

Then we can write the product

$$fg = \sum_{k \geq 0} \left( \Delta_{k(1)} + \Delta_{k(2)} + \Delta_{k(3)} \right)(fg) \quad (1)$$

where

$$\Delta_{k(1)}(fg) = \Delta_k(\tilde{\Delta}_k f \cdot Q_{k+1} g)$$
$$\Delta_{k(2)}(fg) = \Delta_k(Q_{k+1} f \cdot \tilde{\Delta}_k g)$$
$$\Delta_{k(3)}(fg) = \sum_{j \geq k} \Delta_k(\Delta_j f \cdot \Delta_j g)$$

with $\tilde{\Delta}_k = \sum_{j=k-2}^{k+4} \Delta_j$ and $\overline{\Delta}_k = \sum_{j=k-1}^{k+1} \Delta_j$.

Now, we recall the definition of Besov and Lizorkin-Triebel spaces. For more details about equivalent norms, embeddings etc. see [6, 7, 10].
Definition 2.1. For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$ the Besov space is

$$B_{p,q}^s = \left\{ f \in S' : \left( \sum_{j \geq 0} 2^{sj} \| \Delta_j f \|_p^q \right)^{\frac{1}{q}} < \infty \right\}.$$  

For $s \in \mathbb{R}, 1 \leq p < \infty$ and $1 \leq q \leq \infty$ the Lizorkin-Triebel space is

$$F_{p,q}^s = \left\{ f \in S' : \left\| \left( \sum_{j \geq 0} 2^{sj} |\Delta_j f|^q \right)^{\frac{1}{q}} \right\|_p < \infty \right\}.$$  

We will use the following assertions throughout the paper.

Lemma 2.1. If $0 < \delta < 1$ and $1 \leq q \leq \infty$, then for every sequence $(\varepsilon_j)_{j \in \mathbb{N}} \in \ell^q$ of positive numbers one has

$$\left\| \left( \delta^j \sum_{k \leq j} \delta^{-k} \varepsilon_k \right) \right\|_{\ell^q} + \left\| \left( \delta^{-j} \sum_{k \geq j} \delta^k \varepsilon_k \right) \right\|_{\ell^q} \leq \frac{2}{1 - \delta \| (\varepsilon_j)_{j \in \mathbb{N}} \|_{\ell^q}}. \quad (2)$$

Lemma 2.2 (Bernstein’s inequality). If $1 \leq p \leq q \leq \infty$ and $\alpha \in \mathbb{N}^n$, then there exists a constant $C > 0$ such that

$$\| f^{(\alpha)} \|_q \leq C R^{(1 + n(\frac{1}{p} - \frac{1}{q}))} \| f \|_p \quad (3)$$

for all $f \in L^p$ with $\text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq R \}$.

Inequality (2) follows by using Young’s inequality in $\ell^q$. Similarly, for (3) we apply Young’s inequality to $f^{(\alpha)} = \theta_R^\alpha * f$ where $\theta_R(x) = R^n \theta(Rx)$ ($x \in \mathbb{R}^n, R > 0$) such that $\theta \in C^\infty$ and $\hat{\theta}(\xi) = 1$ if $|\xi| \leq 1$.

We finish now this section by recalling the definition of the pointwise multipliers space of a Banach space $E$ (in our work $E = F_{p,q}^s$ or $E = B_{p,q}^s$), denoted by $M(E)$. This is the set of all functions $m$ such that $\| mf \|_E \leq C \| f \|_E$ ($f \in E$). $M(E)$ is a Banach space with the norm equal to the infimum of the above constants $C$. Concerning the properties of $M(F_{p,q}^s)$ and $M(B_{p,q}^s)$ we do not go into details, referring the reader to [2, 6, 7].
3. Embedding into $\mathcal{M}(F^s_{p,q})$

In this section we shall formulate the result for Lizorkin-Triebel space.

**Theorem 3.1.** Let $s \in \mathbb{R}$, $1 \leq p \leq p_1 < \infty$, $1 \leq q \leq \infty$, $r \geq \frac{n}{p_1}$ and $\frac{n}{p_1} - r + \frac{n}{p} - n < s < r$. Then

$$F^r_{p_1,\infty} \cap L^\infty \hookrightarrow \mathcal{M}(F^s_{p,q})$$

**Proof.** We treat only the case $r = \frac{n}{p_1}$. The case $r > \frac{n}{p_1}$ is given in [7: Subsections 4.4.3 and 4.4.4] and the papers of Marschall [3, 4]. Let $f \in F^s_{p,q}$ and $g \in F^\infty_{p_1,1} \cap L^\infty$. For the estimate $\|fg\|_{F^s_{p,q}}$ we need decomposition (1) and the maximal inequality

$$\left\| \left( \sum_{k \geq 0} (\Delta_{k}(f)q)^{q} \right)^{\frac{1}{q}} \right\|_p \leq C \left\| \left( \sum_{k \geq 0} |\Delta_k f|^q \right)^{\frac{1}{q}} \right\|_p$$

satisfied for all $f \in S'$ and $a > \frac{n}{\min(p,q)}$, where $(\Delta_{k}^{*,a} f)(x) = \sup_{y \in \mathbb{R}^n} \frac{|(\Delta_k f)(x-y)|}{(1+|2^k y|)^a}$ (see [10: Theorem 2.3.6] or [7]).

**Estimate of $\Delta_{k(1)}(fg)$.** Let us set

$$C = \int_{\mathbb{R}^n} |(\mathcal{F}^{-1} \phi)(y)|(1+|y|)^a dy.$$ 

Since

$$\|Q_{k+1}g\|_\infty \leq C \|g\|_\infty$$

we obtain $|\Delta_{k(1)}(fg)| \leq C \|g\|_\infty (\Delta_{k}^{*,a} f)$. Taking $a > \frac{n}{\min(p,q)}$ and applying (4) yield

$$\left\| \left( \sum_{k \geq 0} 2^{sqk |\Delta_{k(1)}(fg)|q} \right)^{\frac{1}{q}} \right\|_p \leq C \|g\|_\infty \|f\|_{F^s_{p,q}}.$$ 

**Estimate of $\Delta_{k(2)}(fg)$.** We consider the case $p < p_1$. Let $a_1$ and $a_2$ in $\mathbb{R}^+$ such that

$$|\Delta_{k(2)}(fg)| \leq C (\Delta_{k}^{*,a_1} g) \sum_{j \leq k+1} \Delta_{j}^{*,a_2} f.$$ 

By Lemma 2.1 we have

$$\left( \sum_{k \geq 0} 2^{sqk |\Delta_{k(2)}(fg)|q} \right)^{\frac{1}{q}} \leq C \left( \sum_{k \geq 0} 2^{kq(s-n/p)} (\Delta_{k}^{*,a_2} f)^q \right)^{\frac{1}{q}} \sup_{l \geq 0} \left( 2^{l \frac{n}{p_1}} \Delta_{l}^{*,a_1} g \right).$$

(6)
Combining Hölder’s inequality (where \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \)) with (4) by taking \( a_1 > \frac{n}{p_1} \) and \( a_2 > \frac{n}{\min(p_2, q)} \) shows that the left-hand side of (6) in \( L^p \)-norm is bounded by

\[
C\|g\|_{F_{p_1}^{\frac{n}{p_1}, \infty}} \|f\|_{F_{p_2}^{\frac{n}{p_2}, q}}
\]

and it remains to use the inclusion \( F_{p_2}^{s_1, q} \hookrightarrow F_{p_2}^{\frac{n}{p_1}, q} \).

Now we study the case \( 1 \leq p = p_1 < \infty \). Let \( v > 1 \) such that \( 1 < p < v < \infty \) and \( s - \frac{n}{p} + \frac{n}{v} = s_1 < 0 \). Set \( \frac{1}{u} = \frac{1}{p} + \frac{1}{v} \). Then by Hölder’s inequality

\[
2^{s+\frac{n}{v}} k \| \Delta_{k(2)}(fg) \|_u \leq C 2^{s+\frac{n}{v}} k \| \tilde{\Delta}_k g \|_p \sum_{j \leq k+1} \| \Delta_j f \|_v
\]

\[
\leq C \|g\|_{B_{p_1}^{s+\frac{n}{p}, \infty}} 2^{s_1 k} \sum_{j \leq k+1} 2^{-s_1 j} (2^{s_1 j} \| \Delta_j f \|_v).
\]

By applying Lemma 2.1 we obtain

\[
\left( \sum_{k \geq 0} 2^{s+\frac{n}{v}} k \| \Delta_{k(2)}(fg) \|_u^p \right)^{\frac{1}{p}} \leq C \|g\|_{B_{p_1}^{s+\frac{n}{p}, \infty}} \|f\|_{B_{v_1}^{s_1, p}}^p.
\]

Since \( B_{u_1}^{s_1, p} \hookrightarrow F_{p}^{s, q} \hookrightarrow B_{v_1}^{s_1, p} \) and \( F_{p_1}^{\frac{n}{p}, \infty} \hookrightarrow B_{p_1}^{\frac{n}{p}, \infty} \) we obtain the desired estimation.

**Estimate of** \( \Delta_{k(3)}(fg) \). The difficult part of the product is given by \( \sum_{k \geq 0} \Delta_{k(3)} f g \). To get a bound for the norm of this expression one may use [7: Proposition 4.4.2/4(i)]:

\[
\left\| \sum_{k \geq 0} \Delta_{k(3)}(fg) \right\|_{F_{s+\frac{n}{p_1}, \infty}^{p, q}} \leq C \|g\|_{F_{p_1}^{\frac{n}{p_1}, \infty}} \|f\|_{F_{p}^{s, q}}
\]  

(7)

where \( \frac{1}{t} = \frac{1}{p} + \frac{1}{p_1} \) and \( s + \frac{n}{p_1} > n \max(0, \frac{1}{t} - 1) \) is needed. This gives the correct bound for \( s \) (see the necessary conditions in [7: Section 4.3]) in view of the embedding \( F_1^{\frac{n}{p_1}, \infty} \hookrightarrow F_{p_1}^{\frac{n}{p_1}, \infty} \). Observe that \( F_{t_1}^{s+\frac{n}{p_1}, \infty} \hookrightarrow F_{p}^{s, q} \).

**Remark 3.1.** It is well known that the Hölder-Zygmund space \( C^p \) is not included in \( M(F_{p_1}^{s, q}) \) for \( 0 < \rho < |s| \) (see [10: p. 143]). Hence, if \( 1 \leq p \leq p_1 < \infty \), \( r \geq \frac{n}{p_1} \) and \( \frac{n}{p_1} - r + \frac{n}{p} - n < s < r \), then \( C^p \not\subseteq M(F_{p_1}^{r, \infty}) \).
4. Embedding into $M(B_p^{s,q})$

We give now the corresponding result for $B_p^{s,q}$, where the following theorem improves the previous results obtained in [6: p. 146], [7: p. 173] and [10: p. 154].

**Theorem 4.1.** Let $s \in \mathbb{R}, 1 \leq p \leq p_1 \leq \infty, 1 \leq q \leq \infty, r \geq \frac{n}{p_1}$ and $\frac{n}{p_1} - r + \frac{n}{p} - n < s < r$. Then

$$B_r^{r,\infty} \cap L^\infty \hookrightarrow M(B_p^{s,q}).$$

**Proof.** As in the proof of Theorem 3.1, we consider only the case $r = \frac{n}{p_1}$.

Let $f \in B_p^{s,q}$ and $g \in B_{p_1}^{n,\infty} \cap L^\infty$. We will estimate $\|fg\|_{B_p^{s,q}}$ by using (1). We systematically use the fact that $\Delta_k$ and $Q_k$ are bounded operators in $L(L^p, L^p)$.

**Estimate of $\Delta_k(1)(fg)$.** We begin by

$$\|\Delta_k(1)(fg)\|_p \leq C\|\tilde{\Delta}_k f\|_{p} \|Q_{k+1} g\|_\infty. \quad (8)$$

Then (5) and (8) give the desired estimation.

**Estimate of $\Delta_k(2)(fg)$.** The fact that $\|\Delta_j f\|_{p_2} \leq C2^{j(n(\frac{1}{p} - \frac{1}{p_2})} \|\Delta_j f\|_p$ (Lemma (2.2)) where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and Hölder’s inequality imply

$$2^s\|\Delta_k(2)(fg)\|_p \leq C\|g\|_{B_{p_1}^{n,\infty}} 2^{k(s - \frac{n}{p_1})} \sum_{j \leq k+1} 2^{j\frac{n}{p_1}} \|\Delta_j f\|_p$$

$$\leq C\|g\|_{B_{p_1}^{n,\infty}} 2^{k(s - \frac{n}{p_1})} \sum_{j \leq k+1} 2^{j(\frac{n}{p_1} - s)}(2^j\|\Delta_j f\|_p).$$

We conclude by Lemma 2.1 (since $s < \frac{n}{p_1}$).

**Estimate of $\Delta_k(3)(fg)$.** As in (7) we have

$$\left\|\sum_{k \geq 0} \Delta_k(3)(fg)\right\|_{B_{s+\frac{n}{p_1},\infty}} \leq C\|g\|_{B_{p_1}^{n,\infty}} \|f\|_{B_p^{s,q}} \quad (9)$$

where $\frac{1}{t} = \frac{1}{p} + \frac{1}{p_1}$ and $s + \frac{n}{p_1} > n \max(0, \frac{1}{t} - 1)$ is needed. In [7] only the limit case $s + \frac{n}{p_1} = n \max(0, \frac{1}{t} - 1)$ is considered, but (9) is in the same spirit.

**Remark 4.1.** As in Remark 3.1, if $0 < \rho < |s|, 1 \leq p \leq p_1 \leq \infty, r \geq \frac{n}{p_1}$ and $\frac{n}{p_1} - r + \frac{n}{p} - n < s < r$, then $C^\rho \setminus B_{p_1}^{r,\infty} \not\subseteq M(B_p^{s,q})$.

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