Quadratic Forms and Nonlinear Non-Resonant Singular Second Order Boundary Value Problems of Limit Circle Type

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Abstract. New existence results are presented for non-resonant second order singular boundary value problems

\[
\begin{align*}
\frac{1}{p(t)}(p(t)y'(t))' + \tau(t)y(t) &= \lambda f(t, y(t)) \quad \text{a.e. on } [0, 1] \\
\lim_{t \to 0^+} p(t)y'(t) &= y(1) = 0
\end{align*}
\]

where one of the endpoints is regular and the other may be singular or of limit circle type.

Keywords: Singular and non-resonant problems, points of limit circle type, existence criteria for solutions

AMS subject classification: 34B15

1. Introduction

In this paper we develop an existence theory for

\[
\begin{align*}
\frac{1}{p(t)}(p(t)y'(t))' + \tau(t)y(t) &= \lambda f(t, y(t)) \quad \text{a.e. on } [0, 1]
\end{align*}
\]

which makes use of the relationship between the asymptotic behavior of the non-linearity \( \frac{f(t, y)}{y} \) and the spectrum of the differential operator. In particular, we examine the non-resonant second order singular boundary value problem

\[
\begin{align*}
\frac{1}{p(t)}(p(t)y'(t))' + \tau(t)y(t) &= \lambda f(t, y(t)) \quad \text{a.e. on } [0, 1] \\
\lim_{t \to 0^+} p(t)y'(t) &= y(1) = 0
\end{align*}
\]

\( (P_\lambda) \)
Throughout $p \in C[0,1] \cap C^1(0,1)$ together with $p > 0$ on $(0,1)$, $\tau$ is measurable with $\tau > 0$ a.e. on $[0,1]$ and $\int_0^1 p(x) \tau(x)\,dx < \infty$, and $\lambda \in \mathbb{R}$ is some parameter. We do not assume $\int_0^1 \frac{ds}{p(s)} < \infty$ but rather $\int_0^1 \frac{1}{p(s)} \left( \int_0^s p(x) \tau(x)\,dx \right)^{1/2} \,ds < \infty$. As a result for the eigenvalue problem

$$Lu = \lambda u \quad \text{a.e. on } [0,1]$$
$$\lim_{t \to 0^+} p(t)u'(t) = u(1) = 0$$

(1.1)

where $Lu = -\frac{1}{pq}(pu')'$, one of the endpoints, $t = 1$, will be regular and the other, $t = 0$, may be singular or of limit circle type [6, 7]. For nonlinear non-resonant problems of limit circle type only a handful of papers have appeared in the literature (see [1, 3, 6]). All other papers, to our knowledge, concerning nonlinear non-resonant problems discuss the case when $t = 0$ and $t = 1$ are regular points (see [2, 4, 5, 7] and the references therein). In [6], Fonda and Mawhin presented a technique for discussing non-resonant problems (i.e. (1.1) with $p \equiv 1$) based on quadratic forms. We will use part of this technique in this paper. However, as we will see, many extra steps will be needed to discuss non-resonant problems when one of the endpoints is of limit circle type.

For notational purposes let $w$ be a weight function. By $L^2_w[0,1]$ we mean the space of functions $u$ such that $\int_0^1 w(t)|u(t)|^2\,dt < \infty$ (also, if $u \in L^2_w[0,1]$, we define $\|u\|_w = (\int_0^1 w(t)|u(t)|^2\,dt)^{1/2}$). Let $AC[0,1]$ be the space of functions which are absolutely continuous on $[0,1]$.

The following well known existence principle [6, 7] (which is a special case of the Leray-Schauder continuation theorem), due to O’Regan, will be needed in Section 2.

**Theorem 1.1.** Suppose the following conditions are satisfied:

(i) $p \in C[0,1] \cap C^1(0,1)$ with $p > 0$ on $(0,1)$.

(ii) $\tau \in L^1_p[0,1]$ with $\tau > 0$ a.e. on $[0,1]$.

(iii) $\int_0^1 \frac{1}{p(s)} \left( \int_0^s p(x) \tau(x)\,dx \right)^{1/2} \,ds < \infty$.

(iv) $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e.

(i) $t \mapsto f(t, y)$ is measurable for all $y \in \mathbb{R}$

(ii) $y \mapsto f(t, y)$ is continuous for a.e. $t \in [0,1]$.

(v) $\frac{f(t,y(t))}{\tau(t)} \in L^2_{pr}[0,1]$ whenever $y \in L^2_{pr}[0,1]$.

In addition, assume that problem $(P_0)$ has only the trivial solution. Further, suppose there is a constant $M_0$, independent of $\lambda$, with

$$\|y\|_{pr} = \left( \int_0^1 p(t)\tau(t)|y(t)|^2\,dt \right)^{1/2} \neq M_0$$

for any solution $y$ (here $y \in L^2_{pr}[0,1]$ with $y \in C(0,1] \cap C^1(0,1)$ and $py' \in AC[0,1]$) to problem $(P_\lambda)$, for each $\lambda \in (0,1)$. Then problem $(P_1)$ has at least one solution.
Finally, we remark that problems of type \((P_\lambda)\) occur in many applications in the physical sciences, for example in radially symmetric nonlinear diffusion in the \(n\)-dimensional sphere we have \(p(t) = t^{n-1}\); these problems involve a homogeneous Neumann condition at zero, i.e. \(\lim_{t \to 0^+} r^{n-1} u'(t) = 0\). Another example is the Poisson-Boltzmann equation

\[
\begin{aligned}
\frac{d^2 y}{dt^2} + \frac{\alpha}{t} y' &= f(t, y) \quad (0 < t < 1) \\
y'(0^+) &= y(1) = 0 \quad (\alpha \geq 1)
\end{aligned}
\]

(1.2)

which occurs in the theory of thermal explosions and in the theory of electrohydrodynamics. The results related to problem (1.2) in the literature [1, 3] usually consider the situation when \(\inf \frac{\partial f}{\partial y}\) and \(\sup \frac{\partial f}{\partial y}\) are bounded and satisfy a “non-resonant” condition. In this paper we improve the above existence result (in fact, in our theory the existence of \(\frac{\partial f}{\partial y}\) is not assumed).

We also note that the results in [6] are a special case of Theorems 2.1 and 2.2 in this paper (see the special example after the proof of Theorem 2.1).

2. Non-resonance type problems

In this section we present two existence results for singular boundary value problem \((P_1)\). Conditions (i) - (v) of Theorem 1.1 will be assumed throughout this section. Notice condition (iii) implies (see [7]) \(\int_0^1 p(x) \tau(x) \left( \int_x^1 \frac{ds}{p(s)} \right)^2 dx < \infty\).

Our first result establishes existence if a certain integral inequality is satisfied.

**Theorem 2.1.** Suppose conditions (i) - (v) of Theorem 1.1 hold and suppose problem \((P_6)\) has only the trivial solution. In addition, assume \(f\) has the decomposition

\[
f(t, u) = g_1(t, u) u + g_2(t, u) + h(t, u)
\]

where \(g_1, g_2, h : [0, 1] \times \mathbb{R} \to \mathbb{R}\) are Carathéodory functions and the following conditions are satisfied:

\[
\begin{align*}
\text{(i)} \quad & u g_2(t, u) \geq 0 \text{ for a.e. } t \in [0, 1] \text{ and } u \in \mathbb{R}, \\
\text{(ii)} \quad & \exists \tau_1 \in C[0, 1] \text{ with } \tau_1(t) \tau(t) \leq g_1(t, u) \leq 0 \text{ for a.e. } t \in [0, 1] \text{ and } u \in \mathbb{R}, \\
\text{(iii)} \quad & |h(t, u)| \leq \phi_1(t) + \phi_2(t)|u|^\gamma \text{ for a.e. } t \in [0, 1], \text{ with } 0 \leq \gamma < 1, \\
\text{(iv)} \quad & \int_0^1 p(t) \phi_1(t) \left( \int_t^1 \frac{ds}{p(s)} \right)^{1/2} dt < \infty \text{ and } \int_0^1 p(t) \phi_2(t) \left( \int_t^1 \frac{ds}{p(s)} \right)^{\gamma+1/2} dt < \infty, \\
\text{(v)} \quad & \int_0^1 \left[ p(u')^2 - (\tau - \tau_1 \tau) p u^2 \right] dt > 0 \text{ for any } 0 \neq u \in K^*,
\end{align*}
\]

where

\[
K^* = \left\{ w : [0, 1] \to \mathbb{R} \mid w \in L^2_{p\tau}[0, 1] \text{ with } w \in C(0, 1) \quad \text{ and } \quad w'(0) = 0 \right\}.
\]

Then problem \((P_1)\) has a solution \(y \in L^2_{p\tau}[0, 1] \text{ with } y \in C(0, 1) \cap C^1(0, 1) \text{ and } py' \in AC[0, 1].\)
Proof. We first show that there exists \( \varepsilon > 0 \) with
\[
\int_0^1 \left[ p(y')^2 - (\tau - \tau_1 \tau)py^2 \right] dt \geq \varepsilon (\|y\|_{p\tau}^2 + \|y'\|_p^2)
\] (2.1)
for any \( y \in K^* \). If this is not the case, then there exists a sequence \( \{y_n\} \subset K^* \) with
\[
\|y_n\|_{p\tau}^2 + \|y'_n\|_p^2 = 1 \quad (2.2)
\]
\[
\int_0^1 \left[ p(y'_n)^2 - (\tau - \tau_1 \tau)py^2_n \right] dt \to 0 \quad \text{as} \quad n \to \infty. \quad (2.3)
\]
The Riesz compactness criteria together with a standard result in functional analysis (if \( E \) is a reflexive Banach space, then any norm bounded sequence in \( E \) has a weakly convergent subsequence) implies that there is a subsequence \( S \) of integers with
\[
y_n \to y \quad \text{in} \quad L^2_{p\tau}[0,1] \quad \text{and} \quad y'_n \rightharpoonup y' \quad \text{in} \quad L^2_p[0,1] \quad (2.4)
\]
as \( n \to \infty \) in \( S \) where \( \rightharpoonup \) denotes weak convergence.

Note \( \{y_n\} \) is bounded in \( L^2_{p\tau}[0,1] \) (see (2.2)) and, for \( r > 0 \), Hölder’s inequality yields
\[
\int_0^1 p(t)\tau(t)|y_n(t + r) - y_n(t)|^2 dt = \int_0^1 p\tau \int_t^{t+r} y_n'(s) ds \quad \text{dt} 
\leq \|y'_n\|_p^2 \int_0^1 p\tau \int_t^{t+r} \frac{ds}{p(s)} dt 
\leq \int_0^1 p\tau \int_t^{t+r} \frac{ds}{p(s)} dt - \int_0^1 p\tau \int_{t+r}^{t+1} \frac{ds}{p(s)} dt 
\to 0 \quad \text{as} \quad r \to 0^+
\]
by the Lebesgue dominated convergence theorem and assumption (iii) of Theorem 1.1. Thus \( \{y_n\} \) is relatively compact in \( L^2_{p\tau}[0,1] \).

Next, a standard result in functional analysis [7] yields
\[
\int_0^1 p[y']^2 dt \leq \lim inf \int_0^1 p[y'_n]^2 dt. \quad (2.5)
\]
Now (2.3) - (2.5) and the fact that \( \lim inf |s_n + t_n| \geq \lim inf s_n + \lim inf t_n \) for sequences \( \{s_n\} \) and \( \{t_n\} \) imply
\[
\int_0^1 \left[ p(y')^2 - (\tau - \tau_1 \tau)py^2 \right] dt \leq 0 \quad (2.6)
\]
since
\[
\lim inf \int_0^1 (\tau - \tau_1 \tau)py^2_n dt = \int_0^1 (\tau - \tau_1 \tau)py^2 dt.
\]
Note $y(1) = 0$ since in fact $y_n \to y$ in $C[\varepsilon, 1]$ ($\varepsilon > 0$) by the Arzela-Ascoli theorem. By assumption (v) we have $y \equiv 0$. However,

$$\|y_n\|_{p\tau}^2 + \|y'_n\|_p^2 = \int_0^1 p\tau y_n^2 dt + \int_0^1 (\tau - \tau_1 \tau)py_n^2 dt$$

$$+ \int_0^1 [p(y'_n)^2 - (\tau - \tau_1 \tau)py_n^2] dt$$

$$\to 0 \quad \text{as } n \to \infty \text{ in } S$$

which is impossible. Thus (2.1) holds for some $\varepsilon > 0$.

Let $y$ be a solution to problem $(P_\lambda)$ for some $0 < \lambda < 1$. Note, in particular, $y \in K^*$. Multiply the differential equation by $y$ and integrate from 0 to 1 to obtain

$$\int_0^1 [p(y')^2 - \tau py^2] dt = -\lambda \int_0^1 py^2 g_1(t,y) dt - \lambda \int_0^1 pyg_2(t,y) dt - \lambda \int_0^1 ph(t,y) dt$$

and so (use assumptions (i) - (ii))

$$\int_0^1 [p(y')^2 - (\tau - \tau_1 \tau)py^2] dt \leq \int_0^1 p|yh(t,y)| dt.$$

This together with assumption (iii) and (2.1) imply that there exists $\varepsilon > 0$ (fix it) with

$$\varepsilon(\|y\|_{p\tau}^2 + \|y'_p\|^2) \leq \int_0^1 p\phi_1 |y| dt + \int_0^1 p\phi_2 |y|^\gamma+1 dt.$$

Since $y(1) = 0$, we have from Hölder’s inequality

$$|y(t)| = \left| \int_t^1 y'(s) ds \right| \leq \|y'\|_p \left( \int_t^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}}$$

for $t \in (0,1)$, and so

$$\varepsilon(\|y\|_{p\tau}^2 + \|y'_p\|^2) \leq K_0 \|y\|_p + K_1 \|y'_p\|^\gamma+1$$

(2.7)

where

$$K_0 = \int_0^1 p(t)\phi_1(t) \left( \int_t^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} dt \quad \text{and} \quad K_1 = \int_0^1 p(t)\phi_2(t) \left( \int_t^1 \frac{ds}{p(s)} \right)^{\frac{2\gamma+1}{2}} dt.$$

Now (2.7) guarantees that there is a constant $M > 0$, independent of $\lambda$, with $\|y'_p\|_p \leq M$. This together with (2.7) guarantees the existence of a constant $M_0 > 0$, independent of $\lambda$, with $\|y\|_{p\tau} \leq M_0$. The result now follows from Theorem 1.1.
We now discuss briefly assumption (v) of Theorem 2.1. Inequalities of this type play a major role in the literature of calculus of variation. We illustrate the ideas involved with a simple example. Consider the problem

\[
\begin{aligned}
\frac{1}{p}(py')' + \mu qy &= f(t, y) \quad \text{a.e. on } [0, 1] \\
\lim_{t \to 0^+} p(t)y'(t) &= y(1) = 0
\end{aligned}
\]  

(2.8)

with \( q \in L^1_p[0, 1] \), \( q > 0 \) a.e. on \([0, 1] \), and

\[\mu(1 - \tau(t)) < \lambda_0 \text{ for } t \in [0, 1], \]  

(2.9)

\( \lambda_0 \) being the first eigenvalue of problem (1.1) with \( Lu = -\frac{1}{pq}(pu')' \). Let also assumptions (i), (iii) - (v) of Theorem 1.1 and assumptions (i) - (iv) of Theorem 2.1 hold, with \( \tau(t) = \mu q(t) \). Recall (see [7: Chapter 11], limit circle case) that \( L \) has a countable number of real eigenvalues \( \lambda_i > 0 \) (arranged so that \( \lambda_0 < \lambda_1 < \lambda_2 < \ldots \)) with corresponding (orthonormal) eigenfunctions \( \psi_i \). The set \( \{\psi_i\} \) form a basis of \( L^2_{pq}[0, 1] \), and so for any \( u \in K^* \) we have

\[u(t) = \sum_{i=0}^{\infty} \eta_i \psi_i(t), \quad \eta_i = \langle u, \psi_i \rangle_{pq}\]

where \( \langle u, v \rangle_{pq} = \int_0^1 pqu\bar{v}dt \).

We claim that problem (2.8) has at least one solution. This follows immediately from Theorem 2.1 once we show its condition (v) is satisfied. First notice from (2.9) (note \( \tau_1 \in C[0, 1] \)) that there exists \( \delta > 0 \) with \( \mu(1 - \tau_1(t)) \leq \lambda_0 - \delta \) for \( t \in [0, 1] \). Now for \( u \in K^* \) we have

\[\int_0^1 [p(u')^2 - (\tau - \tau_1 \tau)pu^2] dt \geq \int_0^1 [p(u')^2 - (\lambda_0 - \delta)pqu^2] dt = \sum_{i=0}^{\infty} \eta_i^2 [\lambda_i - (\lambda_0 - \delta)] \int_0^1 pq\bar{\psi}_i^2 dt \]

since \( (p\psi_i')' + \lambda_i pq\psi_i = 0 \) a.e. on \([0, 1] \) and \( \lim_{t \to 0^+} p(t)\psi_i(t) = \psi_i(1) = 0 \). Consequently,

\[\int_0^1 [p(u')^2 - (\tau - \tau_1 \tau)pu^2] dt \geq \delta \sum_{i=0}^{\infty} \eta_i^2 \int_0^1 pq\bar{\psi}_i^2 dt = \delta \int_0^1 pq|u|^2 dt > 0 \]

for \( u \neq 0 \). Thus condition (v) of Theorem 2.1 holds, so our claim is established.

For the remainder of this paper let

\[E = \left\{ y \in L^2_{pq}[0, 1] : y' \in L^2_{p}[0, 1] \text{ and } y(1) = 0 \right\} \]

For \( u, v \in E \) we define

\[\langle u, v \rangle = \int_0^1 p\tau u\bar{v}dt + \int_0^1 pqu'\bar{v}'dt.\]
We show $E$ is complete. Let $\{y_n\}$ be a Cauchy sequence in $E$. Then there exist functions $y \in L^2_{\tau}[0, 1]$ and $u \in L^2_p[0, 1]$ with $y_n \to y$ in $L^2_{\tau}[0, 1]$ and $y'_n \to u$ in $L^2_p[0, 1]$ as $n \to \infty$. Let

$$v(t) = - \int_t^1 u(s) \, ds.$$  

Note $v(1) = 0$. Also, notice since $y_n \in E$ (so $y_n(1) = 0$) that

$$\int_0^1 p(t) \tau(t) |y_n(t) - v(t)|^2 \, dt$$

$$= \int_0^1 p(t) \tau(t) \left( \int_t^1 (y_n - v)'(s) \, ds \right)^2 \, dt$$

$$\leq \left( \int_0^1 p(t) \tau(t) \int_t^1 \frac{ds}{p(s)} \, dt \right) \left( \int_0^1 p(s) (y_n - v)'(s)^2 \, ds \right)$$

$$= \left( \int_0^1 p(t) \tau(t) \int_t^1 \frac{ds}{p(s)} \, dt \right) \left( \int_0^1 p(s) |y_n'(s) - u(s)|^2 \, ds \right)$$

and the right-hand side goes to zero as $n \to \infty$. Thus $y_n \to v$ in $L^2_{\tau}[0, 1]$ as $n \to \infty$, and so $y = v$ a.e. on $[0, 1]$. As a result, $y_n \to v$ in $E$, so $E$ is complete. [In fact, in the following theorem, we could let $E$ be the space of functions $y \in L^2_{\tau}[0, 1]$ with $y' \in L^2_p[0, 1]$.]

**Theorem 2.2.** Suppose conditions (i) - (v) of Theorem 1.1 hold and assume problem $(P_0)$ has only the trivial solution. In addition, assume $f$ has the decomposition

$$f(t, u) = g(t, u)u + h(t, u)$$

where $g, h : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying conditions (iii) - (iv) of Theorem 2.1. Also, suppose the following conditions are satisfied:

(i) There exist $0 \leq -\tau_1, \tau_2 \in C[0, 1]$ with $\tau_1(t) \tau(t) \leq g_1(t, u) \leq \tau_2(t) \tau(t)$ for a.e. $t \in [0, 1]$ and $u \in \mathbb{R}$.

(ii) $E = \Omega \oplus \Gamma$ where $\Omega \subseteq K^*$ is finite-dimensional and for every $0 \neq y = u + v \in K^*$ with $u \in \Omega, v \in \Gamma$ we have $R(y) > 0$

where

$$R(y) = \int_0^1 [p(v')^2 - (\tau - \tau_1) pv^2] \, dt - \int_0^1 [p(u')^2 - (\tau - \tau_2) pu^2] \, dt.$$  

Then problem $(P_1)$ has at least one solution.

**Remark 2.1.** The set $K^*$ in condition (ii) here is as defined in condition (v) of Theorem 2.1. In (ii) we have $y = u + v$ with $u \in \Omega$ and $v \in \Gamma$, so $\int_0^1 p \tau uv \, dt + \int_0^1 pu'v' \, dt = 0$.

**Proof of Theorem 2.2.** We first show that there exists $\varepsilon > 0$ with

$$R(y) \geq \varepsilon (\|y\|^2_{L^2_{\tau}} + \|y'\|^2_p)$$  

(2.10)
Similarly, \( \varepsilon > 0 \); here \( y = u + v \) with \( u \in \Omega \) and \( v \in \Gamma \). If this is false, then there exists a sequence \( \{y_n\} \subset K^* \) with \( \|y_n\|_{p^*} + \|y'_n\|_p^* = 1 \) and
\[
R(y_n) \to 0 \quad \text{as } n \to \infty. \tag{2.11}
\]

Note \( y_n = u_n + v_n \) with \( u_n \in \Omega \) and \( v_n \in \Gamma \). Now there is a subsequence \( S \) of integers with
\[
y_n \to y \text{ in } L^2_{p^*}[0,1] \quad \text{and} \quad y'_n \to y' \text{ in } L^2_p[0,1] \tag{2.12}
\]
as \( n \to \infty \) in \( S \). Also, since strong and weak convergence are the same in finite-dimensional spaces we have
\[
u'_n \to u' \quad \text{in } L^2_p[0,1] \quad \text{as } n \to \infty \text{ in } S. \tag{2.13}
\]

We also have
\[
\int_0^1 p|v'|^2dt \leq \liminf \int_0^1 p|v'|^2dt. \tag{2.14}
\]

Now (2.11) - (2.14) imply that \( R(y) \leq 0 \). From assumption (ii) we have \( y \equiv 0 \). Finally (note \( E = \Omega \oplus \Gamma \), so \( \int_0^1 p\tau u_n v_n dt + \int_0^1 pu_n'v_n' dt = 0 \),
\[
\|y_n\|^2_{p^*} + \|y'_n\|^2_p = R(y_n) + \int_0^1 p\tau \|v_n^2 + u_n^2\| dt + 2\int_0^1 p|u_n'|^2dt
+ \int_0^1 ([\tau - \tau_1\tau]pv_n^2 - [\tau - \tau_2\tau]pu_n^2)dt \to 0 \quad \text{as } n \to \infty \text{ in } S
\]
which is impossible. Thus (2.10) holds for some \( \varepsilon > 0 \).

Let \( y (= u + v) \) be a solution to problem \((P_\lambda)\) for some \( 0 < \lambda < 1 \). Then
\[
-\int_0^1 (v - u)(py')' + p\tau ydt = -\lambda \int_0^1 p(v - u)y g(t,y) dt - \lambda \int_0^1 p(v - u)h(t,y) dt
\]
and so integration by parts yield
\[
\int_0^1 [p|v'|^2 + pv^2(-\tau + \lambda g(t,y))] dt - \int_0^1 [p|u'|^2 + pu^2(-\tau + \lambda g(t,y))] dt
\leq \int_0^1 p|v - u||h(t,y)| dt. \tag{2.15}
\]

Now
\[
pv^2[ - \tau + \lambda g(t,y)] = pv^2[ - (\tau - \tau_1\tau) + \lambda g(t,y) - \tau_1\tau] 
\geq pv^2[ - (\tau - \tau_1\tau) + (\lambda - 1)\tau_1\tau]
\geq -p(\tau - \tau_1\tau)v^2 \quad \text{a.e. on } [0,1].
\]

Similarly,
\[
pv^2[ - \tau + \lambda g(t,y)] \leq -p(\tau - \tau_2\tau)v^2 \quad \text{a.e. on } [0,1].
\]
Putting these into (2.15) yields

\[ R(y) \leq \int_0^1 p|v - u|h(t, y)\,dt. \]

This together with (2.10) implies that there is an \( \varepsilon > 0 \) with

\[ \varepsilon (\|y\|_{p\tau}^2 + \|y'\|_p^2) \leq \int_0^1 p|v - u|h(t, y)\,dt. \tag{2.16} \]

Next, notice that for \( t \in (0, 1) \) we have

\[ |v(1) - u(1)| \leq |v(t) - u(t)| + \int_t^1 |(v - u)'(s)|\,ds \]

and so for \( t \in (0, 1) \)

\[ |v(1) - u(1)| \leq |v(t) - u(t)| + \|v' - u'\|_p \left( \int_t^1 \frac{ds}{p(s)} \right)^\frac{1}{2}. \tag{2.17} \]

Note also that

\[ \|v - u\|_{p\tau}^2 + \|v' - u'\|_p^2 = \|y\|_{p\tau}^2 + \|y'\|_p^2 \tag{2.18} \]

and this together with (2.17) yields for \( t \in (0, 1) \)

\[ |v(1) - u(1)| \leq |v(t) - u(t)| + \left( \|y\|_{p\tau}^2 + \|y'\|_p^2 \right) \left( \int_t^1 \frac{ds}{p(s)} \right)^\frac{1}{2}. \]

Multiply this by \( \sqrt{p(t)\tau(t)} \) and integrate from 0 to 1 (using Hölder’s inequality) to obtain

\[ |v(1) - u(1)| \int_0^1 \sqrt{p(t)\tau(t)}\,dt \leq \|v - u\|_{p\tau} + \left( \|y\|_{p\tau}^2 + \|y'\|_p^2 \right) \left( \int_0^1 \frac{ds}{p(s)} \right)^\frac{1}{2}. \]

This together with (2.18) yields

\[ |v(1) - u(1)| \leq K_2 \left( \|y\|_{p\tau}^2 + \|y'\|_p^2 \right)^\frac{1}{2} \tag{2.19} \]

where

\[ K_2 = \frac{1 + \left( \int_0^1 p(t)\tau(t) \int_t^1 \frac{ds}{p(s)} dt \right)^\frac{1}{2}}{\int_0^1 \sqrt{p(t)\tau(t)}\,dt}. \]

Also, for \( t \in (0, 1) \) we have

\[ |v(t) - u(t)| \leq |v(1) - u(1)| + \|v' - u'\|_p \left( \int_t^1 \frac{ds}{p(s)} \right)^\frac{1}{2} \]
and so (use (2.18) and (2.19)) for $t \in (0,1)$

$$|v(t) - u(t)| \leq \left( \|y\|_{p^r}^2 + \|y'\|_p^2 \right)^{\frac{1}{2}} \left\{ K_2 + \left( \int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} \right\}. \tag{2.20}$$

In addition, since $y(1) = 0$ we have $|y(t)| \leq \int_t^1 |y'(s)| \, ds$ for $t \in (0,1)$ and so

$$|y(t)| \leq \left( \|y\|_{p^r}^2 + \|y'\|_p^2 \right)^{\frac{1}{2}} \left( \int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} \tag{2.21}$$

for $t \in (0,1)$. Put condition (iii) of Theorem 2.1 into (2.16) to obtain

$$\varepsilon \left( \|y\|_{p^r}^2 + \|y'\|_p^2 \right) \leq \int_0^1 p(t)|v(t) - u(t)| \phi_1(t) \, dt + \int_0^1 p(t)|v(t) - u(t)| \gamma \phi_2(t) \, dt.$$

This together with (2.20) - (2.21) gives

$$\varepsilon \left( \|y\|_{p^r}^2 + \|y'\|_p^2 \right) \leq \left( \|y\|_{p^r}^2 + \|y'\|_p^2 \right)^{\frac{1}{2}} \left[ K_2 \int_0^1 p(t) \phi_1(t) \, dt + K_0 \right]$$

$$+ \left( \|y\|_{p^r}^2 + \|y'\|_p^2 \right)^{\frac{\gamma + 1}{2}} \left[ K_2 \int_0^1 p(t) \phi_2(t) \left( \int_t^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} \, dt + K_1 \right]$$

where

$$K_0 = \int_0^1 p(t) \phi_1(t) \left( \int_t^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} \, dt \quad \text{and} \quad K_1 = \int_0^1 p(t) \phi_2(t) \left( \int_t^1 \frac{ds}{p(s)} \right)^{\frac{\gamma + 1}{2}} \, dt.$$

Now since $0 \leq \gamma < 1$, there exists a constant $M > 0$, independent of $\lambda$, with $\|y\|_{p^r}^2 + \|y'\|_p \leq M$. The result now follows from Theorem 1.1.

**References**


Received 12.01.2001; in revised form 22.05.2001