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# A Quaternionic Beltrami-Type Equation and the Existence of Local Homeomorphic Solutions

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**Abstract.** The paper deals with a quaternionic Beltrami-type equation, which is a very natural generalization of the complex Beltrami equation to higher dimensions. Special attention is paid to the systematic use of the embedding of the set of quaternions $\mathbb{H}$ into $\mathbb{C}^2$ and the corresponding application of matrix singular integral operators. The proof of the existence of local homeomorphic solutions is based on a necessary and sufficient criterion, which relates the Jacobian determinant of a mapping from $\mathbb{R}^4$ into $\mathbb{R}^4$ to the quaternionic derivative of a monogenic function.

**Keywords:** *Generalized Beltrami equation, quaternions, singular integral operators, homeomorphic solutions, monogenic functions*

**AMS subject classification:** Primary 30 G 35, secondary 35 J 45

## 1. Introduction

Let $G$ be a domain in $\mathbb{C}$ and $q = q(z)$ with $|q(z)| \leq q_0 < 1 \quad (z \in \mathbb{C})$ a given complex-valued function which links both formal complex partial derivatives

$$\frac{\partial w}{\partial z} = \frac{1}{2}\Big(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\Big)$$
$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\Big)$$

of an unknown complex-valued function $w = w(z)$ by means of the equation

$$\frac{\partial w}{\partial \bar{z}} = q(z)\frac{\partial w}{\partial z.} \tag{1}$$

This equation is usually called the (complex) Beltrami equation and represents the complex form of a first order uniformly elliptic Beltrami system for two unknown real functions $u = \operatorname{Re} w$ and $v = \operatorname{Im} w$.

The number of applications of the Beltrami system is very high. Besides its importance for the general theory of linear and quasilinear elliptic systems, it is related to

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problems of conformal and almost complex structures on general Riemannian surfaces and plays the central role in geometric function theory, particularly for quasi-conformal mappings, uniformisation and the theory of Teichmüller spaces. The last is essential for recently developed conformally invariant string theories in theoretical physics. Nowadays, the Beltrami equation appears in complex dynamics, too, and this list of subjects is far from being exhaustive (c.f. [2]).

There exist many methods for generalizing the theory of the complex Beltrami equation to higher dimensions, mainly by using function-theoretic methods in $\mathbb{C}^n$ (see, e.g., [4, 10 - 12, 14, 18, 19, 27, 29]). However, up to now there were only few attempts to invoke methods from quaternionic or, more generally, from Clifford analysis (e.g., [5, 15, 21, 22], and more recently [13]). This latter area of research (named after the title of the book [3] by Brackx, Delanghe and Sommen, which was published in 1982) started as a generalization of complex analysis by using Clifford algebras, but is meanwhile turning into an independent discipline.

There are a lot of papers in Mathematics and Mathematical Physics dealing with the development of Clifford-analysis methods for the treatment of special higher-dimensional differential equations (for an overview we refer the reader to [8 - 10]). However, until now there has been no attempt to develop in a systematic way a general theory of differential equations in the framework of quaternion-valued functions of a quaternionic variable and, of course, in general, in the framework of Clifford analysis. In fact, there may have been several reasons for such a situation. It would be enough to refer to the peculiarities of the non-commutative algebra as the algebraic basis for such a theory – the general opinion (particularly in engineering) relies more on other methods like matrix, vector or tensor analysis. But the treatment of problems in higher dimensions by several complex-variable methods is in many cases restricted to even dimensions. On the contrary, methods of function theories in algebras more general than the algebra of complex numbers, namely in Clifford algebras and the particular case of quaternions $\mathbb{H}$ are free from such restraint.

Of course, not all properties of holomorphic functions and other facts of classical complex function theory are valid or have an easy and obvious counterpart in Clifford analysis. But even some years after the publication of [3] the idea of an appropriate derivative of the corresponding class of regular functions (defined as the kernel of a certain generalized Cauchy-Riemann operator $D$, which the authors of [3] also called the set of "monogenic" functions) was still not clear. Surely, this has to be the first step towards a general theory of differential equations similar to that in $\mathbb{C}$.

Today, there are several arguments to accept that the derivative of a monogenic function $f$ is given by $\overline{D}f$, where $\overline{D}$ is the conjugate differential operator to $D$ mentioned above (see [7, 25]). Taking into account that the classical Beltrami equation in the form of (1) is nothing else than a linear combination of the complex partial derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$, this suggested the idea (cf. [22]) to consider a corresponding linear combination of $Df$ and $\overline{D}f$ as a natural generalization of (1) to a Beltrami-type equation in higher dimensions. That is why an equation of the same form

$$Df = Q(Z)\overline{D}f,$$

but now with a quaternion-valued unknown function $f = f(Z)$ of the quaternionic variable $Z$ and a quaternion-valued coefficient $Q(Z)$, is taken as our object of study. The restriction to the quaternionic case is related to the problem of proving the existence of locally homeomorphic solutions of this Beltrami-type equation. At least in the classical case such type of solutions are of special interest for all applications. But, therefore, the numbers of components of the independent and of the dependent variables have to be the same. In the special Clifford algebra of quaternions this is ensured, but a more general treatment would need some restrictions of algebraic nature to guarantee the same relation.

The paper is organised as follows. Section 2 prepares the algebraic background, including the embedding of the set of quaternions $\mathbb{H}$ into $\mathbb{C}^2$ and the corresponding representation of the involved differential operators in form of matrix operators. Then, in Section 3, a theorem will be proved which is of own interest in the theory of monogenic functions in quaternionic analysis. It relates the Jacobian determinant of a mapping from $\mathbb{R}^4$ into $\mathbb{R}^4$ to the quaternionic derivative of a monogenic function. Section 4 studies the general aspects concerning the solvability of the Beltrami-type equation in matrix form. Singular integral operators are introduced and applied to solve the Beltrami-type equation by fixed-point methods. Finally, by using the theorem of Section 3 as a criterion for the property of being a local homeomorphic mapping we show the existence of homeomorphic solutions of the considered Beltrami-type equation in Section 5.

## 2. Preliminaries

In what follows, we will work in the skew field of quaternions $\mathbb{H}$ which results from the algebraization of the vector space $\mathbb{R}^3$. Thereby, we write an arbitrary element $Z \in \mathbb{H}$ in the form

$$Z = x_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3,$$

where $x_0, \ldots, x_3$ are real and $1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ stand for the elements of the basis of $\mathbb{H}$, subject to the multiplication rules

$$\begin{aligned} &\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -1 \\ &\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_3 \\ &\mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_1 \\ &\mathbf{e}_3\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_2. \end{aligned}$$

In this way the quaternionic algebra arises as natural extension of the complex field $\mathbb{C}$. We denote by $\operatorname{Sc} Z = x_0$ the *scalar part* of $Z$ and by $\operatorname{Vec} Z = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ its *vector part*. Like in the complex case, the conjugate element $\overline{Z}$ of $Z$ is given by

$$\overline{Z} = \operatorname{Sc} Z - \operatorname{Vec} Z = x_0 - x_1\mathbf{e}_1 - x_2\mathbf{e}_2 - x_3\mathbf{e}_3$$

with the properties

$$Z\overline{Z} = \overline{Z}Z = |Z|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

In such a way $|Z|$ coincides with the Euclidean norm of $Z$ regarded as an element of the vectorial space $\mathbb{R}^4$. Also, each non-zero quaternion $Z$ has a unique inverse $Z^{-1} = \frac{\overline{Z}}{|Z|^2}$.

Then it is evident that an $\mathbb{H}$-valued function $W = W(Z)$ can be written as

$$W(Z) = w_0(Z) + w_1(Z)\mathbf{e}_1 + w_2(Z)\mathbf{e}_2 + w_3(Z)\mathbf{e}_3$$

where $w_0, \dots, w_3$ are real-valued functions. Properties such as continuity, differentiability, integrability, and so on, which are ascribed to the $\mathbb{H}$-valued function $W$ have to be fulfilled by all real-valued components $w_k$. Like usual, $\mathcal{C}^{k,\alpha}$, $\mathcal{L}_p$ and $\mathcal{W}_p^k$ denote the corresponding Hölder, Lebesgue and Sobolev spaces of those functions, respectively.

In the case of $p = 2$ we introduce in $\mathcal{L}_2(\Omega)$ the $\mathbb{H}$-valued inner product

$$(U, V) = \int_\Omega \overline{U}(Z)V(Z)\, d\Omega_Z.$$

The differential operator given by

$$D = \frac{\partial}{\partial x_0} + \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} \tag{2}$$

plays a central role in the sequel. It is easy to see that it is the quaternionic generalization of the complex Cauchy-Riemann operator $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. Note that

$$D\overline{D} = \overline{D}D = \Delta$$

where $\Delta$ is the Laplacian and

$$\overline{D} = \frac{\partial}{\partial x_0} - \mathbf{e}_1 \frac{\partial}{\partial x_1} - \mathbf{e}_2 \frac{\partial}{\partial x_2} - \mathbf{e}_3 \frac{\partial}{\partial x_3}$$

is the conjugate generalized Cauchy-Riemann operator. If $\Omega \subset \mathbb{R}^4 \cong \mathbb{H}$ is a domain, then a function $W : \Omega \mapsto \mathbb{H}$ is said to be *left-monogenic* in $\Omega$ if it satisfies the equation $(DW)(Z) = 0$ for each $Z \in \Omega$. The fundamental solution with respect to $D$ is given by

$$e(Z) = \frac{1}{\omega} \frac{\overline{Z}}{|Z|^4}$$

where $\omega$ is the surface area of the unit ball in $\mathbb{R}^4$. This function is left-monogenic (and right-monogenic) for $Z \neq 0$ and is used in the next section for constructing the generalized Cauchy kernel of the right inverse integral operator $T$ of $D$. We remark that the set of monogenic functions in $\mathbb{H}$ does not form an algebra which is a fact different from the complex case. For more information about these topics and general quaternionic analysis we refer to [8 - 10].

As it is well-known there exist several equivalent possibilities for the treatment of quaternions (cf. [9]). One of them is by using the one-to-one correspondence between quaternions and vectors of complex numbers. In order to show the specific technical tools, we will now present the treatment of the generalized Beltrami equation

$$DW(Z) = Q(Z)\overline{D}W(Z) \tag{3}$$

by means of this approach.

Identifying the complex imaginary unit $i$ with $\mathbf{e}_2$ we can write each quaternion $Z$ in the form $Z = z_1 + \mathbf{e}_1 z_2$, with $z_1 = x_0 + \mathbf{e}_2 x_2$ and $z_2 = x_1 + \mathbf{e}_2 x_3$ being elements of

$\mathbb{C} \cong \mathbb{C}_{(\mathbf{e}_2)}$. More explicitly, this means that we can use the correspondence $\mathbb{H} = \mathbb{C}_{(\mathbf{e}_2)} \oplus \mathbf{e}_1\mathbb{C}_{(\mathbf{e}_2)}$. Then we can write our generalized quaternionic Cauchy-Riemann operator in the form

$$D = 2\left(\overline{\partial}_{z_1} + \mathbf{e}_1\overline{\partial}_{z_2}\right) \tag{4}$$

where

$$\overline{\partial}_{z_1} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \mathbf{e}_2\frac{\partial}{\partial x_2}\right) \qquad \text{and} \qquad \overline{\partial}_{z_2} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_3}\right)$$

are the corresponding $\mathbb{C}_{(e_2)}$-Cauchy-Riemann operators with respect to $z_1$ and $z_2$. Analogously, the conjugate quaternionic Cauchy-Riemann operator can be written as

$$\overline{D} = 2\left(\partial_{z_1} - \mathbf{e}_1\overline{\partial}_{z_2}\right).$$

With the help of $W_1 := w_0 + \mathbf{e}_2 w_2$ and $W_2 := w_1 + \mathbf{e}_2 w_3$ we also get the representation of $W$ in the corresponding form

$$W(Z) = w_0(Z) + w_1(Z)\mathbf{e}_1 + w_2(Z)\mathbf{e}_2 + w_3(Z)\mathbf{e}_3 = W_1(Z) + \mathbf{e}_1 W_2(Z).$$

After having passed to the representation of a quaternion by two $\mathbb{C}_{(e_2)}$-components we will also use the relation to the corresponding $2\times 2$ matrix calculus as an auxiliary tool. For that purpose we consider the function $W = W_1(z) + \mathbf{e}_1 W_2(z)$ as a vector $W = (W_1, W_2)^T$ of two $\mathbb{C}_{(e_2)}$-valued functions $W_1$ and $W_2$ depending from the $\mathbb{C}_{(e_2)}$-variables $z_1$ and $z_2$.

Notice that for simplicity we are using in both cases the same symbols. In general it will follow from the context which form, the complex vector or the quaternionic one, has to be used.

Continuing towards our aim we obtain the generalized Cauchy-Riemann operator $D$ in the form of a matrix operator from $\mathbb{C}^2_{(e_2)}$ into $\mathbb{C}^2_{(e_2)}$, acting from the left on $W$ in the form

$$DW = 2\begin{pmatrix} \overline{\partial}_{z_1} & -\partial_{z_2} \\ \overline{\partial}_{z_2} & \partial_{z_1} \end{pmatrix}\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}.$$

It is easy to check that the formally adjoint matrix of $D$ is exactly the conjugate generalized Cauchy-Riemann operator $\overline{D}$ in matrix form, i.e.

$$\overline{D} = 2\begin{pmatrix} \partial_{z_1} & \partial_{z_2} \\ -\overline{\partial}_{z_2} & \overline{\partial}_{z_1} \end{pmatrix}.$$

Altogether, this allows us to rewrite the quaternionic Beltrami equation (3) in the corresponding $\mathbb{C}_{(e_2)}$-matrix form

$$DW = Q\overline{D}W \tag{5}$$

where $W = (W_1, W_2)^T$ and $Q = \begin{pmatrix} Q_1 & -\overline{Q_2} \\ Q_2 & \overline{Q_1} \end{pmatrix}$ is the coefficient matrix with entries related to $Q(Z) = q_0(Z) + q_1(Z)\mathbf{e}_1 + q_2(Z)\mathbf{e}_2 + q_3(Z)\mathbf{e}_3, q_k(Z) \in \mathbb{R}\ \ (k = 0, \ldots 3)$, by $Q_k = q_{k-1} + e_2 q_{k+1}\ \ (k = 1, 2)$.

## 3. The derivative of a monogenic function and geometrical properties of the mapping realized by a monogenic function

We will now characterize the quasiconformality of a monogenic function in terms of its derivative. It is shown in [7] that for all dimensions of the underlying real Euclidean space $\mathbb{R}^{n+1}$ the hypercomplex derivative of a monogenic function $W$ is the term $-\frac{1}{2}\overline{D}W$. For the case of quaternionic monogenic functions this was shown in [25] already in 1979 and later in [17]. We will see that the relations between derivative, Jacobian determinant, and quasiconformality are not so direct as in the complex case (c.f [1]). But, nevertheless, it is possible to characterize the Jacobian determinant at least in terms of the derivative.

First, we consider the problem of the existence of a homeomorphism. Assume that $W \in \ker D$. Furthermore, let $J = \left(\frac{\partial w_i}{\partial x_j}\right)_{i,j=0}^{3}$ be the Jacobian. The function $W$ realizes a local homeomorphism at the point $Z = 0$ if $\det J|_{Z=0} \neq 0$. As a first step we can prove the following result.

**Lemma 3.1.** $\det J|_{Z=0} \neq 0$ *implies* $\overline{D}W(0) \neq 0$.

This is due to the fact that otherwise from $D + \overline{D} = 2\partial_{x_0}$ and $DW = 0$ it follows $\partial_{x_0} f(0) = 0$. Therefore, $\det J|_{Z=0} = 0$ because $\partial_{x_0} W(0)$ represents the first row in the Jacobian $J$. However, this condition is not sufficient, take for example $W(Z) = x_1 - x_0\mathbf{e}_1$.

**Remark 3.1.** For each function $W \in \ker D \cap \ker\overline{D}$ it follows that $W$ is not a homeomorphism at $Z = 0$. This may look strange at first sight because this subspace of monogenic functions has an infinite dimension. But, remembering what a derivative is in the hypercomplex setting, the result seems to be natural. The set $\ker D \cap \ker\overline{D}$ defines the "constants" with respect to the introduced derivative and in this sense the result corresponds to the complex result.

Now, let $P \in \mathbb{H}$, $|P| = 1$ a constant quaternion. We consider the terms $DW(PZ)$ and $\overline{D}W(PZ)$. For the first term we have

$$DW(PZ) = \overline{P}DW(Z).$$

This implies $W(PZ) \in \ker D$ if $W \in \ker D$. Furthermore, for the second term we have

$$\begin{aligned}
\overline{D}W(PZ) &= \sum\nolimits_j \sum\nolimits_i \overline{\mathbf{e}}_i \partial_j W \frac{\partial (PZ)_j}{\partial x_i} \\
&= \big(p_0\partial_0 W + p_1\partial_1 W + p_2\partial_2 W + \partial_3 W\big) \\
&\quad + \mathbf{e}_1\big(p_1\partial_0 W - p_0\partial_1 W - p_3\partial_2 W + p_2\partial_3 W\big) \\
&\quad + \mathbf{e}_2\big(p_2\partial_0 W + p_3\partial_1 W - p_0\partial_2 W - p_1\partial_3 W\big) \\
&\quad + \mathbf{e}_3\big(p_3\partial_0 W - p_2\partial_1 W + p_1\partial_2 W - p_0\partial_3 W\big) \\
&= \big(p_0 + \mathbf{e}_1 p_1 + \mathbf{e}_2 p_2 + \mathbf{e}_3 p_3\big)\partial_0 W \\
&\quad + \big(p_1 - \mathbf{e}_1 p_0 + \mathbf{e}_2 p_3 - \mathbf{e}_3 p_2\big)\partial_1 W \\
&\quad + \big(p_2 - \mathbf{e}_1 p_3 - \mathbf{e}_2 p_0 + \mathbf{e}_3 p_1\big)\partial_2 W \\
&\quad + \big(p_3 + \mathbf{e}_1 p_2 - \mathbf{e}_2 p_1 - \mathbf{e}_3 p_0\big)\partial_3 W \\
&= \overline{D}(PW).
\end{aligned}$$

Thus, the property $\overline{D}W = 0$ is not preserved under rotations. Moreover, $W \in \ker D$ can be expanded by Taylor's formula with respect to the variables $\theta_i = x_i - \mathbf{e}_i x_0$ $(i = 1, 2, 3)$ which denote the so-called totally regular variables (see, e.g., [3, 7]), yielding

$$W(Z) = W(0) + \theta_1 a + \theta_2 b + \theta_3 c + O(|Z|^2).$$

Hereby, $\det J|_{Z=0}$ depends only on the linear part of this expansion. On the other hand, we also have the classical Taylor formula in $\mathbb{R}^4$ with respect to the real variables $x_i$:

$$W(Z) = W(0) + x_0\partial_0 W(0) + x_1\partial_1 W(0) + x_2\partial_2 W(0) + x_3\partial_3 W(0) + O(|Z|^2).$$

If $\det J|_{Z=0} = 0$, then $\partial_0 W, \ldots, \partial_3 W$ are real linearly dependent, i.e. there exist real numbers $\alpha_0, \ldots, \alpha_3$ with $\sum_{i=0}^{3} \alpha_i^2 = 1$ and

$$\alpha_0\partial_0 W(0) + \alpha_1\partial_1 W(0) + \alpha_2\partial_2 W(0) + \alpha_3\partial_3 W(0) = 0.$$

Now, let $p = (\alpha_0, \ldots, \alpha_3)^T \cong P \in \mathbb{H}$ and $Y = PZ$. We consider $\overline{D}_Z W(PZ)|_{Z=0}$. Using again $D + \overline{D} = 2\partial_{x_0}$ this results in

$$\begin{aligned}
\frac{\partial W(PZ)}{\partial x_0} &= \sum\nolimits_j \frac{\partial W}{\partial y_j}\frac{\partial y_j}{\partial x_0} \\
&= \sum\nolimits_j \frac{\partial W}{\partial y_j}\frac{\partial (PZ)_j}{\partial x_0} \\
&= \frac{\partial W}{\partial y_0}\alpha_0 + \frac{\partial W}{\partial y_1}\alpha_1 + \frac{\partial W}{\partial y_2}\alpha_2 + \frac{\partial W}{\partial y_3}\alpha_3 \\
&= 0
\end{aligned}$$

or, with other words,

$$\det\, J_f|_{Z=0} = 0 \quad \Longrightarrow \quad \exists P : |P| = 1 \wedge \overline{D}_Z W(PZ)|_{Z=0} = 0.$$

Obviously, the reverse statement is also true.

We are now ready to describe the linkage between the derivative and the corresponding Jacobian determinant:

**Theorem 3.1.** *Let $W$ be monogenic. Then*

$$\det\, J_W|_{Z=0} = 0 \quad \Longleftrightarrow \quad \exists P :\ |P| = 1 \wedge \overline{D}_Z W(PZ)|_{Z=0} = 0.$$

This theorem also implies the criterion

$$\det J_W|_{Z=0} \neq 0 \quad \Longleftrightarrow \quad \min_{P\in\mathbb{H}, |P|=1} \left|\overline{D}_Z W(PZ)|_{Z=0}\right| > 0.$$

We are interested now to describe an analogous property not only for monogenic functions but also for solutions of our Beltrami system (3).

**Theorem 3.2.** *Let $W \in C^{1,\alpha}(\Omega)$ be a solution of $DW(Z) = Q(Z)\overline{D}W(Z)$ in $\Omega$ with $Q(0) = 0$. Then*

$$\det\, J_W|_{Z=0} = 0 \quad \Longleftrightarrow \quad \exists P:\ |P| = 1 \wedge \overline{D}_Z W(PZ)|_{Z=0} = 0.$$

For the proof we need the equality $2\frac{\partial}{\partial x_0}W(Z) = (Q(Z)+1)\overline{D}W(Z)$ which comes directly from (3) by rearrangement. As above now $\det J_W|_{Z=0} \neq 0$ implies $\overline{D}W(0) \neq 0$, otherwise we get $\frac{\partial}{\partial x_0}W(0) = 0$ and $\det J_W|_{Z=0} = 0$.

Assume that $\det J_W|_{Z=0} = 0$. Then analogously to the preparation of Theorem 3.1 there exists a $P \in \mathbb{H}$ such that $|P| = 1$ and $\frac{\partial W(PZ)}{\partial x_0}|_{Z=0} = 0$. From $Q(0) = 0$ we have $DW(0) = 0$ and, consequently, $\frac{\partial W(PZ)}{\partial x_0}|_{Z=0} = 0$ implies $D_Z W(PZ)|_{Z=0} = -\overline{D}_Z W(PZ)|_{Z=0}$. Together with $D_Z W(PZ) = \overline{P}DW(Z)$ we obtain $\overline{D}_Z W(PZ)|_{Z=0} = 0$.

Now, we will deal with the property of quasiconformality of the mapping $W : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$. We discuss this property only locally for a neighbourhood of the point $Z = 0$. Suppose that $W$ is monogenic and realizes a homeomorphism. Without loss of generality (removing higher order terms from the Taylor expansion), we can consider

$$W(Z) = \theta_1 \mathbf{e}_1 + \theta_2 B + \theta_3 C$$

with $B, C \in \mathbb{H}$. We have to estimate the terms

$$\max_{|Z|=r} |W(Z) - W(0)| \qquad \text{and} \qquad \min_{|Z|=r} |W(Z) - W(0)|$$

with

$$W(Z) - W(0) = \sum_i x_i \partial_i W(0) + O(|Z|^2).$$

Due to the linear independence it is enough to consider only $|\sum x_i \partial_i W(0)|$ for the maximum and minimum on the surface of the ball $|Z| = r$. In the case of the chosen function $W(Z)$ we have for the Jacobian written in the short form of columns

$$J = \left(\tfrac{1}{2}\overline{D}W, \mathbf{e}_1, B, C\right)$$

and, therefore,

$$J^T J = \begin{pmatrix} |\frac{1}{2}\overline{D}W|^2 & \frac{1}{2}\overline{D}W \cdot \mathbf{e}_1 & \frac{1}{2}\overline{D}W \cdot B & \frac{1}{2}\overline{D}W \cdot C \\ \frac{1}{2}\overline{D}W \cdot \mathbf{e}_1 & 1 & B \cdot \mathbf{e}_1 & C \cdot \mathbf{e}_1 \\ \frac{1}{2}\overline{D}W \cdot B & B \cdot \mathbf{e}_1 & |B|^2 & B \cdot C \\ \frac{1}{2}\overline{D}W \cdot C & C \cdot \mathbf{e}_1 & B \cdot C & |C|^2 \end{pmatrix}$$

where $B \cdot C$ denotes the scalar product of $B$ and $C$ in $\mathbb{R}^4$. From this for the trace

$$\mathrm{tr} J^T J = |\tfrac{1}{2}\overline{D}W|^2 + 1 + |B|^2 + |C|^2$$

follows and we obtain $|\frac{1}{2}\overline{D}W|^2 \leq \mathrm{tr} J^T J \leq 4\sigma_{\max}$ as an estimate for the greatest eigenvalue $\sigma_{\max}$. Using the Rayleigh-quotient, we get $\sigma_{\min} = \min_{|Z|\neq 0} \frac{(J^T JZ, Z)}{(Z,Z)} \leq$

$\frac{(J^T JY,Y)}{(Y,Y)}$ $(Y \in \mathbb{R}^4)$ for the lowest eigenvalue $\sigma_{\min}$. Chosing $Y = (1,0,0,0)^T$ this results in $\sigma_{\min} \leq |\frac{1}{2}\overline{D}W|^2$. Alltogether, we obtain

$$\sigma_{\min} \leq |\tfrac{1}{2}\overline{D}W|^2 \leq 4\sigma_{\max}.$$

From the inequalities for $|\det J|$ we get

$$\frac{\sigma_{\min}}{\sigma_{\max}}|\tfrac{1}{2}\overline{D}W|^2 \leq \sqrt{|\det J|} \leq \frac{\sigma_{\max}}{\sigma_{\min}}|\tfrac{1}{2}\overline{D}W|^2$$

as well as

$$\frac{\sigma_{\min}}{\sigma_{\max}}\sqrt{|\det J|} \leq |\frac{1}{2}\overline{D}W|^2 \leq \frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{|\det J|}.$$

These inequalities show the equivalence of the Jacobian determinant and the derivative of monogenic functions.

**Remark 3.2.** It should be mentioned explicitly that all the aforementioned considerations only require $DW(0) = 0$ and not that $W(Z)$ is a monogenic function in a neighbourhood of the origin. It is enough to assume that it is a function of the class $\mathcal{C}^2$ to ensure the existence of the Taylor expansion with second order remainder term. In the next section we will apply these ideas to the problem of the existence of locally quasiconformal solutions of the Beltrami equation. The Beltrami system is usually studied in the scale of Hölder spaces. If we have solutions of (3) belonging to $C^{2,\alpha}(\Omega)$ (where $\Omega = \mathbb{R}^4$ is allowed), then the above used real Taylor expansion is ensured. The additional condition $Q(0) = 0$ implies $DW(0) = 0$ and therefore the linear part of the Taylor expansion can be written using the hypercomplex variables $\theta_i$ which was necessary for the estimation of the Jacobian determinant.

## 4. Solvability of the Beltrami-type equation

The correspondence between the quaternionic Beltrami equation (3) and the matrix equation (5) in Section 2 allows us to study the solvability of (3) by employing integral operators with entries similar to the complex operators. Indeed, taking the classical integral operator methods developed for the plane case (c.f. [28]) into account we obtain the form of the corresponding integral operator $T$ as the right inverse operator of $D$ in a straightforward manner. The crucial point is to substitute the complex Cauchy kernel $\frac{1}{2\pi}(\xi - z)$ by the generalized Cauchy kernel

$$e(\xi - Z) = \frac{1}{\omega}\frac{\overline{\xi - Z}}{|\xi - Z|^4}$$

mentioned in Section 2. In detail, with $H = (H_1, H_2)^T$ and $\xi = \xi_1 + \mathbf{e}_1\xi_2 \cong (\xi_1, \xi_2) \in \mathbb{C}^2_{(e_2)}$ we get

$$TH = -\tfrac{1}{2\pi^2}\begin{pmatrix} \int_\Omega \frac{\overline{\xi_1 - z_1}}{|\xi - Z|^4} H_1(\xi) + \frac{\overline{\xi_2 - z_2}}{|\xi - Z|^4} H_2(\xi) d\Omega_\xi \\ \int_\Omega \frac{-(\xi_2 - z_2)}{|\xi - Z|^4} H_1(\xi) + \frac{\xi_1 - z_1}{|\xi - Z|^4} H_2(\xi) d\Omega_\xi \end{pmatrix}.$$

If $\Omega$ is a bounded domain, then $T$ is a continuous operator from the space $\mathcal{W}_p^k(\Omega)$ into the space $\mathcal{W}_p^{k+1}(\Omega)$ $(1 < p < \infty, k \in \mathbb{N}_0)$. In the case of an unbounded domain $\Omega$ the operator $T$ has the continuity property for $kp \geq 4$ (see [23]). Thus, in both cases $DTH = H$. It should be noticed that $T$ also acts from $\mathcal{C}^{l,\alpha}(\Omega)$ into $\mathcal{C}^{l+1,\alpha}(\Omega)$ $(0 < \alpha \leq 1)$.

Following now the general scheme of the solution of the Beltrami equation by integral operator methods in the plane we have to transform the matrix Beltrami equation $DW = Q\overline{D}W$ into an equivalent singular integral equation. For that purpose we notice that every given solution $W \in \mathcal{W}_p^k(\Omega)$ $(1 < p < \infty, k \in \mathbb{N})$ can be represented in the form

$$W(z_1, z_2) = \Phi(z_1, z_2) + TH(z_1, z_2) \tag{6}$$

with $H \in \mathcal{W}_p^{k-1}(\Omega)$ taken as $H = DW$ and $\Phi = W - TH \in \mathcal{W}_p^k(\Omega)$. The same relations are true in the case of $W \in \mathcal{C}^{l,\alpha}(\Omega)$ with $H \in \mathcal{C}^{l-1,\alpha}(\Omega)$. Because $D\Phi = DW - DTH = DW - H = 0$, it is evident that $\Phi$ is a monogenic vector in the sense of the matrix differential operator $D$ (i.e. the corresponding quaternionic function $\Phi = \Phi(Z) \in \mathbb{H}$ is monogenic or, in other words, it is a solution of the homogeneous generalized Cauchy-Riemann equation $DW = 0$.)

Knowing now that every $\mathcal{W}_p^k$-solution of the Beltrami equation has a representation in the form (6), we use (6) as an *ansatz* for an unknown solution $W$ to obtain the aforementioned singular integral equation equivalent to (5). Indeed, let $\Phi$ be some specially chosen monogenic vector. Applying to both sides of (6) the operators $D$ and $\overline{D}$ and combining the obtained expressions according to the equation $DW = Q\overline{D}W$ we get that $H = DW$ has to satisfy the equation

$$H = Q\overline{D}\Phi + Q\Pi H \tag{7}$$

where

$$\begin{aligned} \Pi H &= \overline{D}TH \\ &= \tfrac{1}{2\pi^2}\begin{pmatrix} \int_\Omega [E_{11}(\xi, Z)H_1(\xi) + E_{12}(\xi, Z)H_2(\xi)]\, d\Omega_\xi \\ \int_\Omega [E_{21}(\xi, Z)H_1(\xi) + E_{22}(\xi, Z)H_2(\xi)]\, d\Omega_\xi \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} H_1(Z) \\ H_2(Z) \end{pmatrix} \end{aligned}$$

and the kernels $E_{ij}$ $(i,j = 1,2)$ are given by

$$\begin{aligned} E_{11}(\xi, Z) &= \frac{2(\overline{\xi}_1 - \overline{z}_1)^2 - 2(\xi_2 - z_2)(\overline{\xi}_2 - \overline{z}_2) - |\xi - Z|^2}{|\xi - Z|^6} \\ E_{12}(\xi, Z) &= \frac{2(\overline{\xi}_1 - \overline{z}_1)(\overline{\xi}_2 - \overline{z}_2) + 2(\xi_1 - z_1)(\overline{\xi}_2 - \overline{z}_2)}{|\xi - Z|^6} \\ E_{21}(\xi, Z) &= \frac{-2(\overline{\xi}_1 - \overline{z}_1)(\xi_2 - z_2) - 2(\xi_1 - z_1)(\xi_2 - z_2)}{|\xi - Z|^6} \\ E_{22}(\xi, Z) &= \frac{-2(\xi_2 - z_2)(\overline{\xi}_2 - \overline{z}_2) - |\xi - Z|^2 + 2(\xi_1 - z_1)^2}{|\xi - Z|^6}. \end{aligned}$$

This strongly singular integral operator is an $\mathbb{H}$-analogue (in the $\mathbb{C}_{(\mathbf{e}_2)}$-vector representation of quaternions) to the complex $\Pi$-operator (see [28]). Going back to the usual representation of quaternions and using the operator $\overline{D}$ in the form corresponding to (4) it is an easy task to verify that the obtained integral operator coincides with the quaternionic $\Pi$-operator investigated in detail in [5, 13].

To be able to follow the general scheme of the method we should now verify that this new matrix integral operator satisfies all necessary mapping properties between the relevant function spaces. Moreover, we also have to define relations between the corresponding norm of the $\Pi$-operator and the coefficient $Q$ in (7) to guarantee the applicability of fixed-point methods which serve for the unique determination of $H$ as solution to (7). Formula (6) shows that in view of the chosen monogenic vector $\Phi$ we obtain that way a unique solution for the Beltrami equation itself. It is evident that the choice of $\Phi$ is a degree of freedom in the determination of a solution to (5). Indeed, we will see that the specific choice of $\Phi$ allows us to obtain a homeomorphic solution of (5) in the following section.

Following the general theory of singular integral operators (cf. [16]) and the results of [5] this strongly singular integral operator acts from $\mathcal{C}^{l,\alpha}(\Omega)$ to $\mathcal{C}^{l,\alpha}(\Omega)$ $(0 < \alpha < 1)$ and it is also a bounded operator from $\mathcal{W}_p^k(\Omega)$ into $\mathcal{W}_p^k(\Omega)$ $(1 < p < \infty, k \in \mathbb{N}_0)$.

The determination of a concrete norm estimate in $\mathcal{W}_2^k(\Omega)$ has been shown in [13] by using methods of [16], particularly the relation of the norm of the quaternionic $\Pi$-operator to the absolute value of its symbol.

**Remark 4.1.** As a consequence of the aforementioned properties of the $\Pi$-operator we obtain that for $||Q|| \leq q_c < \frac{1}{||\Pi||}$ in the norm of $\mathcal{C}^{l,\alpha}(\Omega)$ $(0 < \alpha < 1)$ or $\mathcal{W}_p^k(\Omega)$ $(kp > 4)$ the singular integral equation (7) can be solved using Banach's fixed-point theorem. We notice that in the case of $kp > 4$ the space $\mathcal{W}_p^k(\Omega)$ forms a Banach algebra. Moreover, according to the well-known Sobolev embedding theorem [24] for $kp \geq 4 + lp + p\alpha$ $(l \in \mathbb{N}_0, 0 < \alpha < 1)$ our solution $H$ also belongs to the space $C^{l,\alpha}(\Omega)$. Based on this fact, in all what follows we consider the solvability of equation (7) over spaces $\mathcal{W}_p^k(\Omega)$ with $kp > 4$.

As we can see the choice of a minimal value of $p$ to guarantee the needed smoothness of the solution of (7) essentially depends from the real dimension of the space. Comparing this situation with the complex case we should notice that we cannot profit from the consequences of the theorem of M. Riesz [20] about $\|\Pi\|_{\mathcal{L}_p}^p$ being a logarithmic convex function of $p$ in $\mathbb{C}$ and the fact that for $p = 2$ the $\mathcal{L}_2$-norm is one. In the quaternionic case we are forced to a value of $p$ greater than 2 and to be able to obtain a condition for the contraction property of $||Q\Pi||$ we would have to impose stronger conditions on the module of the multiplier $Q$.

## 5. The existence of local homeomorphic solutions

As we mentioned in the previous section in the case of $Q \in \mathcal{W}_p^k(\mathbb{R}^4)$ $(kp > 4)$ with $||Q||_{\mathcal{W}_p^k} \leq q_c < \frac{1}{\|\Pi\|_{\mathcal{W}_p^k}}$ our singular integral equation (7) has a solution $H$ uniquely determined by choosing a monogenic vector $\Phi$. For the following we set

$$\Phi(Z) = \frac{1}{2} \begin{pmatrix} z_1 \\ \overline{z}_2 \end{pmatrix}$$

and obtain a solution

$$W = \tfrac{1}{2}\begin{pmatrix} z_1 \\ \overline{z}_2 \end{pmatrix} + TH$$

whereby $H$ satisfies the singular integral equation (7) with $\overline{D}\Phi = (1,0)^T$ (in quaternionic terms $\overline{D}\Phi = 1$). The special choice of $\Phi$ is motivated by the simplest form of a partial solution to the homogeneous Cauchy-Riemann equation $DW = 0$ which guaranted $Q\overline{D}W = (Q_1, Q_2)^T$ and with a Jacobian matrix closed to the identity. In the quaternions this solution has the form

$$W = \tfrac{1}{2}(z_1 + z_2\mathbf{e}_1) - \tfrac{1}{2\pi^2}\int_{\mathbb{R}^4} \frac{\overline{\xi - Z}}{|\xi - Z|^4}\big(H_1(\xi) + \mathbf{e}_1 H_2(\xi)\big) d\mathbb{R}^4_\xi.$$

Starting from this special solution we will now investigate the problem of the existence of locally quasiconformal solutions of the Beltrami equation (3).

Let us first consider the case $Q \in \mathcal{W}^2_p(\mathbb{R}^4)$ $(p > 4)$. Furthermore, suppose $||Q||_{\mathcal{W}^2_p} \leq q_c < \frac{1}{\|\Pi\|_{\mathcal{W}^2_p}}$ (c.f. Remark 4.1). Then for each neighbourhood $U_\delta(0) = \{Z : |Z| < \delta\}$ we can consider the function

$$Q^\delta(Z) = Q(Z)\varphi(Z)$$

with

$$\varphi(Z) = \begin{cases} 1 & \text{if } |Z| < \frac{1}{2}\delta \\ -192(\frac{|Z|}{\delta})^5 + 720(\frac{|Z|}{\delta})^4 - 1040(\frac{|Z|}{\delta})^3 + 720(\frac{|Z|}{\delta})^2 - 240(\frac{|Z|}{\delta}) + 32 & \text{if } \frac{1}{2}\delta \leq |Z| \leq \delta \\ 0 & \text{if } |Z| > \delta. \end{cases}$$

Hereby, we have $||Q^\delta||_{\mathcal{W}^2_p} \leq ||Q||_{\mathcal{W}^2_p}||\varphi||_{\mathcal{W}^2_p} \leq 58\pi^2\delta^4||Q||_{\mathcal{W}^2_p}$.

Let us denote by $\mathring{\mathcal{W}}^2_p(U_\delta(0))$ the space of all $\mathcal{W}^2_p(\mathbb{R}^4)$-functions with support in $U_\delta(0)$. Then we have for the operator $\Pi_\delta$ defined by $\Pi_\delta H = Q^\delta \Pi H$ the mapping property

$$\Pi_\delta : \mathring{\mathcal{W}}^2_p(U_\delta(0)) \mapsto \mathring{\mathcal{W}}^2_p(U_\delta(0)) \qquad (1 < p < \infty).$$

Moreover, from

$$||Q||_{\mathcal{W}^2_p} \leq q_c < \frac{1}{\|\Pi\|_{\mathcal{W}^2_p}} \ \ (p > 4) \qquad \text{and} \qquad ||Q^\delta||_{\mathcal{W}^2_p} \leq 58\pi^2\delta^4||Q||_{\mathcal{W}^2_p}$$

it follows that for all $\delta \leq \frac{1}{\sqrt{\sqrt{58}\pi}}$ the operator $\Pi_\delta$ is a contraction over $\mathcal{W}^2_p(U_\delta(0))$ $(p > 4)$. Based on this, we have that

$$W = \tfrac{1}{2}\begin{pmatrix} z_1 \\ \overline{z}_2 \end{pmatrix} + TH^\delta,$$

where $H^\delta$ satisfies the equation

$$H^\delta = \begin{pmatrix} Q_1^\delta \\ Q_2^\delta \end{pmatrix} + \begin{pmatrix} Q_1^\delta & -\overline{Q_2}^\delta \\ Q_2^\delta & \overline{Q_1}^\delta \end{pmatrix} \Pi H^\delta,$$

is a solution of the Beltrami equation $DW = Q^\delta \overline{D}W$ over $\mathbb{R}^4$ and it is also a solution of the Beltrami equation $DW = Q\overline{D}W$ over $U_{\frac{\delta}{2}}(0)$. Note that due to our construction of $Q^\delta$ the function $H^\delta$ vanishes outside the neighbourhood $U_\delta$ and belongs to the space $\mathcal{C}^{1,\alpha}(\mathbb{R}^4)$ $(\alpha = 1 - \frac{4}{p}$ for $p > 4)$. Furthermore, from the application of Banach's fixed-point theorem to our singular integral equation

$$\|H^\delta\|_{\mathcal{W}_p^2} \leq \frac{\|Q^\delta\|_{\mathcal{W}_p^2}}{1 - \|Q^\delta\|_{\mathcal{W}_p^2}\|\Pi\|_{\mathcal{W}_p^2}} < \frac{58\pi^2\|Q\|_{\mathcal{W}_p^2}\delta^4}{1 - 58\pi^2\|Q\|_{\mathcal{W}_p^2}\delta^4\|\Pi\|_{\mathcal{W}_p^2}}$$

follows. For $\delta < \frac{1}{\sqrt{\sqrt{116\pi}}}$ we get $\|H^\delta\|_{\mathcal{W}_p^2} < 116\pi^2\|Q\|_{\mathcal{W}_p^2}\delta^4 < \frac{116\pi^2}{\|\Pi\|_{\mathcal{W}_p^2}}\,\delta^4$. This leads to the fact that for all $\epsilon > 0$ there exists a $\delta$ such that

$$\|H^\delta\|_{\mathcal{L}_p(\mathbb{R}^4)} < \frac{\epsilon}{\|\Pi\|_{\mathcal{W}_p^2}}. \tag{8}$$

These properties allow us to solve the problem of the existence of a local homeomorphism based on Theorem 3.1. Hereby, we would like to observe that the theorem is also true if we replace the condition of monogeneity of the function $W$ by the conditions that $W$ is a $\mathcal{C}^2$-function and $DW(Z) = Q(Z)\overline{D}W(Z)$ with $Q(0) \neq -1$. This latter condition corresponds to the imposing of a direct connection of the terms $DW$ and $\overline{D}W$ given by a Beltrami-type equation. In the case $Q(0) = -1$, $W$ will never be a local homeomorphism at the point 0. (Indeed, $DW + \overline{D}W = 0$ means that $\partial_{x_0}W = 0$.) Therefore, let us assume in the following $Q(0) \neq -1$.

From Theorem 3.1 we have the condition

$$\det J_W|_{Z=0} = 0 \quad \Longleftrightarrow \quad \exists P:\ |P| = 1 \wedge \overline{D}_Z W(PZ)|_{Z=0} = 0.$$

Evaluating the terms

$$\begin{aligned}&\overline{D}_Z W(PZ)\\ &\quad = 2\left(\partial_{z_1} - \mathbf{e}_1\overline{\partial}_{z_2}\right)\\ &\qquad \times \Big(\big((p_1 z_1 - \overline{p}_2 z_2) + \mathbf{e}_1(\overline{p}_2\overline{z}_1 + p_1\overline{z}_2)\big) + TH^\delta\left(p_1 z_1 - \overline{p}_2 z_2, \overline{p}_2\overline{z}_1 + p_1\overline{z}_2\right)\Big)\end{aligned}$$

and $\overline{\overline{D}_Z W(PZ)}$ at the point zero this condition is transformed into the following problem:

Does there exist $P = p_1 + \mathbf{e}_1 p_2$ such that the system

$$\begin{pmatrix} 2 + C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & 2 + C_{22} & C_{23} & C_{24} \\ \overline{C}_{13} & \overline{C}_{14} & 2 + \overline{C}_{11} & \overline{C}_{12} \\ \overline{C}_{23} & \overline{C}_{24} & \overline{C}_{21} & 2 + \overline{C}_{22} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \overline{p}_1 \\ \overline{p}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

has non-trivial solutions? Hereby, we have for the coefficients

$$\begin{aligned}
C_{11} &= \tfrac{1}{2\pi^2}\int_{\mathbb{R}^4}\frac{2\overline{\xi}_1^2}{|\xi|^6}H_1^\delta(\xi)d\mathbb{R}^4{}_\xi + \tfrac{1}{2\pi^2}\int_{\mathbb{R}^4}\frac{2\overline{\xi}_1\overline{\xi}_2}{|\xi|^6}H_2^\delta(\xi)d\mathbb{R}^4{}_\xi \\
C_{12} &= \tfrac{1}{2\pi^2}\int_{\mathbb{R}^4}\frac{2\overline{\xi}_1\xi_1-|\xi|^2}{|\xi|^6}H_1^\delta(\xi)d\mathbb{R}^4{}_\xi + \tfrac{1}{2\pi^2}\int_{\mathbb{R}^4}\frac{2\xi_1\overline{\xi}_2}{|\xi|^6}H_2^\delta(\xi)d\mathbb{R}^4{}_\xi \\
C_{13} &= \tfrac{1}{2\pi^2}\int_{\mathbb{R}^4}\frac{2\overline{\xi}_1\xi_2}{|\xi|^6}H_1^\delta(\xi)d\mathbb{R}^4{}_\xi - \tfrac{1}{2\pi^2}\int_{\mathbb{R}^4}\frac{2\xi_1\overline{\xi}_1-|\xi|^2}{|\xi|^6}H_2^\delta(\xi)d\mathbb{R}^4{}_\xi \\
C_{14} &= -\frac{1}{\pi^2}\int_{\mathbb{R}^4}\frac{2\overline{\xi}_2\xi_2-|\xi|^2}{|\xi|^6}H_1^\delta(\xi)d\mathbb{R}^4{}_\xi + \tfrac{1}{2\pi^2}\int_{\mathbb{R}^4}\frac{2\xi_1\overline{\xi}_2}{|\xi|^6}H_2^\delta(\xi)d\mathbb{R}^4{}_\xi \\
C_{21} &= -\tfrac{1}{2\pi^2}\int_{\mathbb{R}^4}\frac{2\overline{\xi}_1\xi_2}{|\xi|^6}H_1^\delta(\xi)d\mathbb{R}^4{}_\xi - \tfrac{1}{2\pi^2}\int_{\mathbb{R}^4}\frac{2\xi_2\overline{\xi}_2-|\xi|^2}{|\xi|^6}H_2^\delta(\xi)d\mathbb{R}^4{}_\xi \\
C_{22} &= \tfrac{1}{2\pi^2}\int_{\mathbb{R}^4}\frac{2\overline{\xi}_1\xi_1-|\xi|^2}{|\xi|^6}H_1^\delta(\xi)d\mathbb{R}^4{}_\xi + \tfrac{1}{2\pi^2}\int_{\mathbb{R}^4}\frac{2\xi_1\overline{\xi}_2}{|\xi|^6}H_2^\delta(\xi)d\mathbb{R}^4{}_\xi \\
C_{23} &= -\tfrac{1}{2\pi^2}\int_{\mathbb{R}^4}\frac{2\xi_1\xi_2}{|\xi|^6}H_1^\delta(\xi)d\mathbb{R}^4{}_\xi + \tfrac{1}{2\pi^2}\int_{\mathbb{R}^4}\frac{2\xi_1\overline{\xi}_1-|\xi|^2}{|\xi|^6}H_2^\delta(\xi)d\mathbb{R}^4{}_\xi \\
C_{24} &= -\tfrac{1}{2\pi^2}\int_{\mathbb{R}^4}\frac{2\xi_2\xi_2}{|\xi|^6}H_1^\delta(\xi)d\mathbb{R}^4{}_\xi + \tfrac{1}{2\pi^2}\int_{\mathbb{R}^4}\frac{2\xi_1\xi_2}{|\xi|^6}H_2^\delta(\xi)d\mathbb{R}^4{}_\xi
\end{aligned}$$

Based on the structure of these coefficients as complex singular integral operators of Calderon-Zygmund type evaluated at the point zero, which we can also find in the representation of the Π-operator,

$$|C_{ij}| \leq \|\Pi\|_{\mathcal{L}_p}\|H^\delta\|_{\mathcal{L}_p}$$

for all indices $i, j$. Moreover, from (8) we obtain $|C_{ij}| < \epsilon$.

Based on [26] a diagonal dominant matrix $(K_{ij})$ is regular if

$$|K_{ii}| > \sum_{j=1, j\neq i}^{4} |K_{ij}|$$

for all $i$. This criterion allows to us to obtain the regularity of the linear system (9) by choosing $\epsilon = \frac{1}{3}$. Therefore, we get

$$d_i = |2 + C_{ii}| - \sum_{\substack{j=1\\ j\neq i}}^{4} |C_{ij}| \geq 2 - |C_{ii}| - \sum_{\substack{j=1\\ j\neq i}}^{4} |C_{ij}| \geq 2 - \frac{1}{3} - \frac{3}{3} = \frac{2}{3}$$

for $i = 1, 2$ as well as

$$d_i = |2 + \overline{C}_{i-2,i-2}| - \sum_{\substack{j=1\\ j\neq i-2}}^{4} |\overline{C}_{i-2,j}| \geq 2 - |\overline{C}_{i-2,i-2}| - \sum_{\substack{j=1\\ j\neq i-2}}^{4} |\overline{C}_{i-2,j}| \geq 2 - \frac{1}{3} - \frac{3}{3} = \frac{2}{3}$$

for $i = 3, 4$ and, hence, the linear system (9) is regular.

From the application of the affine transformation $\xi = Z - Z_0$ we obtain the following theorem.

**Theorem 5.1.** *Suppose that* $Q \in \mathcal{W}_p^2(\mathbb{R}^4)$ *for a certain* $p > 4$ *and* $||Q||_{\mathcal{W}_p^2} \leq q_c < \frac{1}{||\Pi||_{\mathcal{W}_p^2}}$. *Suppose also* $Q(Z) \neq -1$ *in any point* $Z \in \mathbb{R}^4$. *Then the generalized Beltrami equation*

$$DW(Z) = Q(Z)\overline{D}W(Z)$$

*has a solution, which realizes a local homeomorphism at each point* $Z_0$.

Now, let us again assume that $Q \in \mathcal{W}_p^2(\mathbb{R}^4)$ and $\|Q\|_{\mathcal{W}_p^2} \leq q_c < \frac{1}{4\|\Pi\|_{\mathcal{W}_p^2}}$ for some $p > 4$. Note that, from the application of Banach's fixed-point theorem to our singular integral equation,

$$||H||_{\mathcal{W}_p^2} < \frac{q_c}{1 - q_c||\Pi||_{\mathcal{W}_p^2}} < \frac{1}{3||\Pi||_{\mathcal{W}_p^2}}$$

and the solution

$$W = \tfrac{1}{2}\begin{pmatrix} z_1 \\ \overline{z}_2 \end{pmatrix} + TH$$

belongs to the space $\mathcal{C}^2(\mathbb{R}^4)$ due to the fact that $TH \in \mathcal{W}_p^3(\mathbb{R}^4)$ (cf. [24]). Moreover, we again have for the coefficients $C_{ij}$ of system (9) $|C_{ij}| \leq ||\Pi||_{\mathcal{L}_p}||H||_{\mathcal{L}_p}$ for all indices $i, j$ which implies $|C_{ij}| \leq \frac{1}{3}$. Therefore, we obtain once more the regularity of our linear system (9) by the above mentioned criterion.

Using the affine transformation $\xi = Z - Z_0$ and Remark 3.2 we are able to establish the following result.

**Theorem 5.2.** *If* $Q \in \mathcal{W}_p^2(\mathbb{R}^4)$, $\|Q\|_{\mathcal{W}_p^2} \leq q_c < \frac{1}{4||\Pi||_{\mathcal{W}_p^2}}$ *for some* $p > 4$ *and* $Q(Z) \neq -1$ $(Z \in \mathbb{R}^4)$, *then the function*

$$W = \frac{1}{2}\begin{pmatrix} z_1 \\ \overline{z}_2 \end{pmatrix} + TH,$$

*whereby* $H$ *satisfies the corresponding singular integral equation* (7), *is a solution of the Beltrami equation*

$$DW(Z) = Q(Z)\overline{D}W(Z) \tag{10}$$

*which realizes a local homeomorphism in each point* $Z_0$. *Furthermore, if* $Q(0) = 0$, *then the Beltrami equation* (10) *has a locally quasiconformal solution at each point* $Z_0$.

Let us finally remark that in the complex case the treatment of $Q(0) \neq 0$ is done by reduction to the previous case by means of an affine transformation. Unfortunately, we shall prove that, in general, this method is not applicable to our case. In fact, let us consider the existence of such an affine transformation, given by

$$\xi_1 = a_0 + a_1 z_1 + a_2\overline{z}_1 + a_3 z_2 + a_4\overline{z}_2$$
$$\xi_2 = b_1 + b_1 z_1 + b_2\overline{z}_1 + b_3 z_2 + a_4\overline{z}_2$$

which changes equation (3) into the new Beltrami equation

$$D_\xi W = \widetilde{Q}(\xi)\overline{D}_\xi W \tag{11}$$

possessing the desired property of $\widetilde{Q}(0) = 0$. The existence of this affine transformation leads to the correspondence

$$\begin{pmatrix} \partial_{z_1} \\ \overline{\partial}_{z_1} \\ \partial_{z_2} \\ \overline{\partial}_{z_2} \end{pmatrix} = \begin{pmatrix} a_1 & \overline{a}_2 & b_1 & \overline{b}_2 \\ a_2 & \overline{a}_1 & b_2 & \overline{b}_1 \\ a_3 & \overline{a}_4 & b_3 & \overline{b}_4 \\ a_4 & \overline{a}_3 & b_4 & \overline{b}_3 \end{pmatrix} \begin{pmatrix} \partial_{\xi_1} \\ \overline{\partial}_{\xi_1} \\ \partial_{\xi_2} \\ \overline{\partial}_{\xi_2} \end{pmatrix}$$

between the derivatives. Replacing the derivatives in the first equation of (3),

$$\overline{\partial}_{z_1} W_1 - \partial_{z_2} W_2 = Q_1(\partial_{z_1} W_1 + \partial_{z_2} W_2) - \overline{Q}_2(\overline{\partial}_{z_1} W_2 - \overline{\partial}_{z_2} W_1)$$

and rewriting the equation we get

$$\begin{aligned} (a_2 - Q_1 a_1 - \overline{Q}_2 a_4)\partial_{\xi_1} W_1 + (\overline{a}_1 - Q_1 \overline{a}_2 - \overline{Q}_2 \overline{a}_3)\overline{\partial}_{\xi_1} W_1 \\ +(b_2 - Q_1 b_1 - \overline{Q}_2 b_4)\partial_{\xi_2} W_1 + (\overline{b}_1 - Q_1 \overline{b}_2 - \overline{Q}_2 \overline{b}_3)\overline{\partial}_{\xi_2} W_1 \\ +(-a_3 - Q_1 a_3 + \overline{Q}_2 a_2)\partial_{\xi_1} W_2 + (-\overline{a}_4 - Q_1 \overline{a}_4 + \overline{Q}_2 \overline{a}_1)\overline{\partial}_{\xi_1} W_2 \\ +(-b_3 - Q_1 b_3 + \overline{Q}_2 b_2)\partial_{\xi_2} W_2 + (-\overline{b}_4 - Q_1 \overline{b}_4 + \overline{Q}_2 \overline{b}_1)\overline{\partial}_{\xi_2} W_2 = 0 \end{aligned}$$

which we can compare with the first equation of (11). This originates the system

$$\begin{aligned} \overline{a}_1 - Q_1 \overline{a}_2 - \overline{Q}_2 \overline{a}_3 &= 1 \\ b_2 - Q_1 b_1 - \overline{Q}_2 b_4 &= 0 \\ -a_3 - Q_1 a_3 + \overline{Q}_2 a_2 &= 0 \\ -\overline{b}_4 - Q_1 \overline{b}_4 + \overline{Q}_2 \overline{b}_1 &= 0 \\ a_2 - Q_1 a_1 - \overline{Q}_2 a_4 &= -\widetilde{Q}_1 \\ \overline{b}_1 - Q_1 \overline{b}_2 - \overline{Q}_2 \overline{b}_3 &= -\overline{\widetilde{Q}_2} \\ -\overline{a}_4 - Q_1 \overline{a}_4 + \overline{Q}_2 \overline{a}_1 &= \overline{\widetilde{Q}_2} \\ -b_3 - Q_1 b_3 + \overline{Q}_2 b_2 &= -(1 + \widetilde{Q}_1) \end{aligned} \tag{11}$$

valid for all $Z \in \Omega$. We should notice that in the above system the equations which determine the new $\widetilde{Q}$ function are independent of the coefficients $a_3$ and $b_4$, while in the control equations one notices the absence of the coefficients $\overline{a}_4$ and $b_3$.

Consider the case in which $Q_2(Z_0) = 0$ for some $Z_0 \in \Omega$. On this particular point we obtain from $(11)_{3-4}$ that either $Q_1(Z_0) = -1$ and, therefore, $||Q(Z_0)|| = 1$, or $a_3 = b_4 = 0$. Now, from $(11)_{3-4}$, this implies that either $a_2 = b_1 = 0$ or $Q_2 \equiv 0$ in $\Omega$. Again, this hypothesis is not possible since that would imply from $(11)_{3-4}$ that the function $Q_1$ would be constant.

It is easily seen from $(11)_{1-2}$ that $a_1 = 1$ and $b_2 = 0$, respectively. The remaining equations are now

$$\begin{aligned} \widetilde{Q}_1 &= Q_1 + \overline{Q}_2 a_4 \\ \overline{\widetilde{Q}_2} &= \overline{Q}_2 \overline{b}_3 \\ \overline{\widetilde{Q}_2} &= \overline{Q}_2 - (1 + Q_1) a_4 \\ (1 + \widetilde{Q}_1) &= (1 + Q_1) b_3. \end{aligned}$$

Again, on the point $Z_0$, we get $\widetilde{Q}_1(Z_0) = Q_1(Z_0)$ so $b_3 = 1$, which implies $\widetilde{Q}_2 \equiv Q_2$ and $a_4 = 0$. Therefore, the only affine transformation satisfying this conditions is $\xi = Z - Z_0$. Note that the above arguments are independent of the condition $\widetilde{Q}(0) = 0$.

For the case in which $Q_2(Z) \neq 0$ for all $Z \in \Omega$, the existence of an affine transformation which reduces the original Beltrami equation to the case $(11)_1$ would imply that the inverse transformation (again affine) would invert the effect, generating from $\widetilde{Q}$ the original $Q$. But the previous argument shows that $Q(Z_0) = 0$ for at least one $Z_0 \in \Omega$. Hence, no such transformation exists for $Q_2(Z) \neq 0$ for all $Z \in \Omega$.

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