Conditional Stability of a Real Inverse Formula for the Laplace Transform

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Abstract. We establish a conditional stability estimate of a real inverse formula for the Laplace transform of functions under the assumption that the Bergman-Selberg norms of the Laplace transform of those functions are uniformly bounded. The rate of the stability estimate is shown to be of logarithmic order.

Keywords: Laplace transform, real inversion formulas, conditional stability, Bergman-Selberg space, error estimates, Mellin transform, Gauss formula, convolution, reproducing kernels

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1. Introduction and main results

We are concerned with the Laplace transform

\[(\mathcal{L}F)(x) = \int_0^\infty F(t)e^{-xt}dt \quad (x > 0).\]

Our main purpose is to get some estimates of \(F(t)\) (\(t > 0\)) by means of \(\sup_{x \geq 0}|(\mathcal{L}F)(x)|\). In particular, we are interested in estimates of \(F\) that are small when \(\sup_{x \geq 0}|(\mathcal{L}F)(x)|\) is small. This kind of estimates is called *stability estimate* for the inverse Laplace transform and, in general, we cannot expect such stability estimates because the Laplace transform \(\mathcal{L}\) advances the regularity of \(F\) very much. For example, consider \(F_n(t) = \sin nt\) (\(n \in \mathbb{N}\)). Then \((\mathcal{L}F_n)(x) = \frac{n}{x^2 + n^2} \quad (x > 0)\) and \(\sup_{x > 0}|(\mathcal{L}F_n)(x)| = \frac{1}{n} \to 0\) as \(n \to \infty\), but \(\lim_{n \to \infty} \|F_n\|_{L^\infty(0, \infty)} \neq 0\).

The lack of stability implies the ill-posedness in taking the inverse of the Laplace transform if we choose \(L^\infty\)-norms for functions under consideration. However, it is possible to obtain some stability estimates provided that we restrict ourselves to some

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reasonable set of functions. They are called conditional stability estimates, and there are many such estimates depending on the choice of norms and "reasonable" functions classes. In this paper, we establish such a conditional stability estimate in $L^\infty$-norm for a subclass of Hölder continuous functions. The image of this space under the Laplace transform turns out to be a Bergman-Selberg space.

For $q > 0$, we can define a norm equivalent to the Bergman-Selberg norm $\| \cdot \|_{L^q_+(\mathbb{R}^+)}$ by

$$\|f\|_{L^q_+(\mathbb{R}^+)}^2 = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n + 2q + 1)} \int_0^{\infty} |\partial_x^n(xf'(x))|^2 x^{2n + 2q - 1} \, dx.$$  \hfill (1.1)

It is known (see, e.g., Saitoh [4: Chapter 5]) that

$$\|F\|_{L^q_+(\mathbb{R}^+)} = \left( \int_0^{\infty} |F(t)|^2 t^{1 - 2q} \, dt \right)^{\frac{1}{2}} = \|L F\|_{L^q_+(\mathbb{R}^+)}.$$  \hfill (1.2)

Equality (1.2) means that the Laplace transform is an isometry between the norms $\| \cdot \|_{L^q_+(\mathbb{R}^+)}$ and $\| \cdot \|_{H^q_+(\mathbb{R}^+)}$ for any fixed $q > 0$. The norm $\| \cdot \|_{H^q_+(\mathbb{R}^+)}$ specifies our choice of an admissible set.

We state our main results.

**Theorem 1.** Let $\frac{1}{4} < q < 1$, $M > 0$, and

$$\max\left\{\frac{1}{2}, 2q - 1\right\} < \alpha < \min\{1, 2q\}.$$  \hfill (1.3)

Set

$$U = \left\{ f : \|f\|_{H^q_+(\mathbb{R}^+)} \leq M \text{ and } \|x^\alpha f(\cdot)\|_{H^{q - \frac{1}{2}}_+(\mathbb{R}^+)} \leq M \right\}.$$  \hfill (1.4)

Then for $0 < t_0 < t_1 < \infty$ and $0 < \gamma < \frac{2\alpha - 1}{4}$ there exists a constant $C = C(U, t_0, t_1, \gamma) > 0$ such that

$$\|F\|_{L^\infty(t_0, t_1)} \leq C \left( \frac{-1}{\log \|LF\|_{L^\infty(0, \infty)}} \right)^{\gamma}$$  \hfill (1.5)

if $LF \in U$.

The right-hand side of (1.5) tends to 0 as $\|LF\|_{L^\infty(0, \infty)} \to 0$, but with the logarithmic rate. So the conditional stability estimate is worse than any Hölder continuity.

The subset $U$ is defined on the set of images of the Laplace transform with the Bergman-Selberg norm. Sometimes it is more desirable to have a characterization based on the original functions.

**Theorem 2.** Let $\alpha, \gamma, q, t_0, t_1, M, C$ be defined as in Theorem 1 and set

$$V = \left\{ F \in C^1[0, \infty) : F(0) = 0, \|F\|_{L^q_+} \leq M, \|F'\|_{L^2_{\frac{3}{2} + q - 1}} \leq \frac{M \Gamma\left(\frac{3}{2} - \alpha\right)}{\sqrt{\pi}} \right\}.$$  \hfill (1.6)

Then estimate (1.5) holds for all $F \in V$.

In the next section Preliminaries we shall show that the condition $\alpha < 1$ in (1.3) is sharp. That is, this assumption is needed essentially in the paper [2], which is the base of Theorems 1 and 2.
2. Preliminaries and sharp condition for $\alpha$

The keys to the proofs of Theorems 1 and 2 are the real inversion formula of the Laplace transform (Byun and Saitoh [3], Saitoh [4]) and the error estimate of this real inversion formula (Amano, Saitoh and Yamamoto [2]):

**Proposition 1** (see [3, 4]). Let $q > 0$ be fixed, $\|F\|_{L^2_q} < \infty$ and $f = \mathcal{L}F$. Then the inversion formula

$$F(t) = s - \lim_{N \to \infty} \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)\,dx \quad (t > 0)$$

is valid where the limit is taken in the space $L^2_q$ and the polynomials $P_{N,q}$ are given by the formulas

$$P_{N,q}(\xi) = \sum_{0 \leq \nu \leq n \leq N} (-1)^{\nu+1}\frac{\Gamma(2n+2q)}{\nu!(n-\nu)!\Gamma(n+2q+1)\Gamma(n+\nu+2q)} \xi^{n+\nu+2q-1}$$

$$\times \left\{ \frac{2(n+q)}{n+\nu+2q} \xi^2 - \left( \frac{2(n+q)}{n+\nu+2q} + 3n+2q \right) \xi + n(n+\nu+2q) \right\}.$$ 

Moreover, the series

$$\sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n (xf'(x))|^2 x^{2n+2q-1} dx$$

converges and the inequality

$$\left\| F(t) - \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)\,dx \right\|^2_{L^2_q}$$

$$\leq \sum_{n=N+1}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n (xf'(x))|^2 x^{2n+2q-1} dx$$

holds.

**Proposition 2** (see [2]). Let (1.3) hold. Then for $f \in H_q(\mathbb{R}^+)$ there exists a constant $M_1 = M_1(q, \alpha) > 0$ such that

$$\left| F(t) - \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)\,dx \right| \leq M_1 \|x^\alpha f(\cdot)\|_{H_q^{-\frac{\alpha}{2}}(\mathbb{R}^+)} \frac{t^{q-1+\frac{\alpha}{2}}}{N^{2\alpha-\frac{1}{4}}}.$$  \hspace{1cm} (2.1)

Theorem 1 in [2] only asserts that for $N \to \infty$

$$\left| F(t) - \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)\,dx \right| \leq t^{q-1+\frac{\alpha}{2}} \left( \frac{1}{N^{2\alpha-\frac{1}{4}}} \right).$$  \hspace{1cm} (2.2)

However, from the proof given in [2] we easily specify the dependency of the coefficient at the right-hand side of (2.2) to obtain inequality (2.1).

In order to see the necessity of the restriction $\alpha < 1$ in condition (1.3), recall
Proposition 3 (see [2: Lemma]). If \( f \in C^\infty(0, \infty) \) and
\[
I_{q,\alpha}(f) := \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n + 2q + 1)} \int_0^\infty \left| \partial_x^n(xf'(x)) \right|^2 x^{2n+2q-1+\alpha} dx < \infty \tag{2.3}
\]
for a fixed \( \alpha > \max(\frac{1}{2}, 2q - 1) \), then
\[
\left| \sum_{n=N+1}^{\infty} \frac{1}{n! \Gamma(n + 2q + 1)} \int_0^\infty \partial_x^n(xf'(x)) \partial_x^n(x\partial_x(e^{-tx})) x^{2n+2q-1} dx \right| = t^{\alpha-2q} o(N^{\frac{1-2\alpha}{4}})
\]
as \( N \to \infty \).

Proposition 2 was proved with the use of Proposition 3. We will analyze the relation of the restriction \( \alpha < 1 \) with condition (2.3). Set
\[
F_N(t) = \int_0^\infty f(x)e^{-xt} P_{N,q}(xt) dx
\]
for any \( q > 0 \) and \( F \in L^2_q \). Then, as shown in [3, 4],
\[
F_N(t) = \sum_{n=0}^{N} \frac{t^{2q-1}}{n! \Gamma(n + 2q + 1)} \int_0^\infty \partial_x^n(xf'(x)) \partial_x^n(x\partial_x(e^{-tx})) x^{2n+2q-1} dx
\]
and, by Proposition 1, \( s - \lim_{N \to \infty} F_N = F \).

We examine now properties of functions \( f \) satisfying (2.3). For the Mellin transform \((Mf)(s) = \int_0^\infty f(x)x^{s-1}dx\) of \( f \) recall the identity
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|(Mf)(q-\mathrm{i}t)|^2}{|\Gamma(q-\mathrm{i}t)|^2} dt = \|f\|_{H_q(\mathbb{R}^+)}^2
\]
and notice that
\[
\int_{-\infty}^{\infty} |(Mf)(q-\mathrm{i}t)|^2(q^2 + t^2)^2 \{(q + 1)^2 + t^2\} \cdots \{(q + n - 1)^2 + t^2\} dt = 2\pi \int_0^\infty \left| \partial_x^n(xf'(x)) \right|^2 x^{2n+2q-1} dx
\]
(see [4: Page 207/Formula (28)]). Hence,
\[
2\pi \int_0^\infty \left| \partial_x^n(xf'(x)) \right|^2 x^{2n+2q+\alpha-1} dx
\]
\[
= \int_{-\infty}^{\infty} \left|Mf\right| \left(q + \frac{\alpha}{2} - \mathrm{i}t\right)^2 \left\{ \left(q + \frac{\alpha}{2}\right)^2 + t^2 \right\}^2 \times \left\{ \left(q + \frac{\alpha}{2} + 1\right)^2 + t^2 \right\} \cdots \left\{ \left(q + \frac{\alpha}{2} + n - 1\right)^2 + t^2 \right\} dt
\]
and so

\[
I_{q,\alpha}(f) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n + 2q + 1)} \times \int_{-\infty}^{\infty} |(Mf)(q + \frac{\alpha}{2} - it)|^2 \left\{ \left( q + \frac{\alpha}{2} + t^2 \right) \right\} \times \left\{ \left( q + \frac{\alpha}{2} + n - 1 \right)^2 + t^2 \right\} dt
\]

\[
= \frac{1}{2\pi \Gamma(2q+1)} \int_{-\infty}^{\infty} |(Mf)(q + \frac{\alpha}{2} - it)|^2 \left\{ \left( q + \frac{\alpha}{2} + t^2 \right) \right\} \times \sum_{n=0}^{\infty} \frac{(q + \frac{\alpha}{2} + it)_n (q + \frac{\alpha}{2} - it)_n}{(2q + 1)_n n!} dt
\]

where \((a)_n = a(a + 1) \cdots (a + n - 1) = \frac{\Gamma(a+n)}{\Gamma(a)}\). Applying the famous Gauss summation formula (see [1: Page 556/Formulas (15.1.20) and (15.1.1)])

\[
\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{(Re}(c-a-b) > 0, c \notin \mathbb{N}_0) \]

and using the property \(\Gamma(z) = \Gamma(z)\) we obtain

\[
\sum_{n=0}^{\infty} \frac{(q + \frac{\alpha}{2} + it)_n (q + \frac{\alpha}{2} - it)_n}{(2q + 1)_n n!} = \frac{\Gamma(2q+1)\Gamma(1-\alpha)}{|\Gamma(q + 1 - \frac{\alpha}{2} + it)|^2}
\]

when \(\alpha < 1\). Hence

\[
I_{q,\alpha}(f) = \frac{\Gamma(1-\alpha)}{2\pi} \int_{-\infty}^{\infty} |(Mf)(q + \frac{\alpha}{2} - it)|^2 \left\{ \left( q + \frac{\alpha}{2} + t^2 \right) \right\} \times \frac{(q + \frac{\alpha}{2})^2 + t^2}{|\Gamma(q + 1 - \frac{\alpha}{2} + it)|^2} dt
\]

\[
= \frac{\Gamma(1-\alpha)}{2\pi} \int_{-\infty}^{\infty} \frac{|(Mf)(q + \frac{\alpha}{2} - it)|^2}{|\Gamma(q - \frac{\alpha}{2} + it)|^2} \left\{ \left( q + \frac{\alpha}{2} + t^2 \right) \right\} \times \frac{(q + \frac{\alpha}{2})^2 + t^2}{(q - \frac{\alpha}{2})^2 + t^2} dt
\]

\[
\leq C \int_{-\infty}^{\infty} \frac{|(Mf)(q + \frac{\alpha}{2} - it)|^2}{|\Gamma(q - \frac{\alpha}{2} - it)|^2} dt.
\]

Note that

\[(Mf)(q + \frac{\alpha}{2} - it) = \int_{0}^{\infty} f(x)x^q+\frac{\alpha}{2}-it-1dx = M(f(x)x^\alpha)(q - \frac{\alpha}{2} - it)\]

Hence,

\[
I_{q,\alpha}(f) \leq C \int_{-\infty}^{\infty} \frac{|(M(x^\alpha f(x)))(q - \frac{\alpha}{2} - it)|^2}{|\Gamma(q - \frac{\alpha}{2} - it)|^2} dt
\]

\[
= C\|x^\alpha f(x)\|^2_{H_{q-\frac{\alpha}{2}}(\mathbb{R}^+)}.
\]
We see that if \( x^\alpha f(x) \in H_{q-\frac{\alpha}{2}}(\mathbb{R}^+) \), then the function \( f \) satisfies condition (2.3). Thus we get Proposition 2 under the condition \( \alpha < 1 \).

The condition \( \alpha < 1 \) is sharp. In the case \( \alpha \geq 1 \) we shall show that (2.3), that is (2.4) does not converge for \( f \neq 0 \). Indeed, from (2.4)

\[
I_{q,\alpha}(f) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n+2q+1)} \times \int_{-\infty}^{\infty} \left|(Mf)\left(q+\frac{\alpha}{2} - it\right)\right|^2 \left\{ \left(q+\frac{\alpha}{2}\right)^2 + t^2 \right\} \cdot \left\{ \left(q+\frac{\alpha}{2} + n - 1\right)^2 + t^2 \right\} dt \\
\geq \frac{1}{2\pi \Gamma(2q+1)} \int_{-\infty}^{\infty} \left|(Mf)\left(q+\frac{\alpha}{2} - it\right)\right|^2 dt \\
\times \left(q+\frac{\alpha}{2}\right)^2 \sum_{n=0}^{\infty} \frac{(q+\frac{\alpha}{2})_n(q+\frac{\alpha}{2})_n}{(2q+1)_n n!}.
\]

Since \((2q+1) - (q+\frac{\alpha}{2}) - (q+\frac{\alpha}{2}) \leq 0\), the last series is divergent, and \( I_{q,\alpha}(f) \) is finite only if

\[
\int_{-\infty}^{\infty} \left|(Mf)\left(q+\frac{\alpha}{2} - it\right)\right|^2 dt = 0,
\]

that is, if \( f \equiv 0 \).

### 3. Proof of Theorem 1

We divide the proof into two steps.

**First Step.** We set \( f = LF \) and

\[
F_N(t) = \int_{0}^{\infty} f(x)e^{-xt} P_{N,q}(xt) \, dx \quad (t > 0).
\]

In this step, we will estimate \( |F_N(t)| \) \( (t \in [t_0, t_1]) \). We have

\[
|F_N(t)| \leq \|f\|_{L^\infty(0,\infty)} \int_{0}^{\infty} |e^{-xt} P_{N,q}(xt)| \, dx \\
= \frac{1}{t} \|f\|_{L^\infty(0,\infty)} \int_{0}^{\infty} |e^{-\xi} P_{N,q}(\xi)| \, d\xi \quad (3.1) \\
\leq \frac{1}{t} \|f\|_{L^\infty(0,\infty)} S(N,q).
\]
Here we set
\[ S(N, q) = \sum_{0 \leq \nu \leq n \leq N} \frac{\Gamma(2n + 2q)}{\nu! (n - \nu)! \Gamma(n + 2q + 1) \Gamma(n + \nu + 2q)} \times \int_0^\infty \left\{ \frac{2(n + q)}{n + \nu + 2q} \xi^{n+\nu+2q+1} e^{-\xi} \right\} \left\{ \frac{2(n + q)}{n + \nu + 2q} + 3n + 2q \right\} \xi^{n+\nu+2q} e^{-\xi} + n(n + \nu + 2q) \xi^{n+\nu+2q-1} e^{-\xi} \right\} d\xi. \]

It is sufficient to estimate \( S(N, q) \). Noting that
\[ \Gamma(n + \nu + 2q + 2) = (n + \nu + 2q + 1)(n + \nu + 2q)\Gamma(n + \nu + 2q) \]
and
\[ \Gamma(n + \nu + 2q + 1) = (n + \nu + 2q)\Gamma(n + \nu + 2q) \]
we obtain
\[ S(N, q) = \sum_{0 \leq \nu \leq n \leq N} \frac{\Gamma(2n + 2q)(4(n + q) + (6n + 4q)(n + \nu + 2q))}{\nu! (n - \nu)! \Gamma(n + 2q + 1)} \leq \sum_{n=0}^N \left( \sum_{\nu=0}^n \frac{n!}{\nu! (n - \nu)!} \right) \frac{1}{n! \Gamma(n + 2q + 1)} \frac{\Gamma(2n + 2q)}{(4(n + q) + (6n + 4q)(2n + 2q))} \]
\[ \leq \sum_{n=0}^N 2^n(4n + 4q)(3n + 2q + 1) \frac{\Gamma(2n + 2q)}{n! \Gamma(n + 2q + 1)} \]
\[ \leq C4^N \sum_{n=0}^N \frac{\Gamma(2n + 2q)}{n! \Gamma(n + 2q + 1)}. \]

Here and henceforth \( C > 0 \) denotes a generic constant dependent only on \( M, q, \alpha, \beta, t_0, t_1 \) and we note that \( (4n + 4q)(3n + 2q + 1) \leq C2^N \) for \( 0 \leq n \leq N \).

Moreover, we have
\[ \Gamma(2n + 2q) = \frac{1}{\sqrt{2\pi}} 2^{2n+2q-\frac{3}{2}} \Gamma(n + q) \Gamma\left(n + q + \frac{1}{2}\right) \]
(see, e.g., Abramowitz and Stegun [1: p. 256]), and so
\[ \sum_{n=0}^N \frac{\Gamma(2n + 2q)}{n! \Gamma(n + 2q + 1)} = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^N 2^{2n+2q-\frac{3}{2}} \frac{\Gamma(n + q) \Gamma(n + q + \frac{1}{2})}{n! \Gamma(n + 2q + 1)} \]
\[ \leq C4^N \sum_{n=0}^N \frac{\Gamma(n + q) \Gamma(n + q + \frac{1}{2})}{n! \Gamma(n + 2q + 1)} \]
\[ \leq C4^N \sum_{n=0}^\infty \frac{\Gamma(n + q) \Gamma(n + q + \frac{1}{2})}{n! \Gamma(n + 2q + 1)} \]
\[ = C4^N \frac{\Gamma(2q + 1) \Gamma\left(\frac{1}{2}\right)}{\Gamma(q + 1) \Gamma\left(\frac{1}{2}\right)}. \]
At the last equality, we use the Gauss summation formula (2.5). Therefore we obtain

\[ S(N, q) \leq C 16^N, \]

so that inequality (3.1) yields

\[ |F_N(t)| \leq \frac{C}{t} 16^N \| f \|_{L^\infty((0, \infty))} \quad (t > 0). \]

**Second Step.** It is sufficient to prove (1.5) for sufficiently small \( \| f \|_{L^\infty((0, \infty))} \). Let \( 0 < t_0 \leq t \leq t_1 \). We have

\[
|F(t)| \leq |F_N(t)| + |F(t) - F_N(t)| \\
\leq \frac{C}{t} 16^N \| f \|_{L^\infty((0, \infty))} + M_1 \| x^\alpha f(\cdot) \|_{H^{q-\frac{1}{2}}(\mathbb{R}^+)} t^{q-1+\frac{\alpha}{2}} \frac{1}{N^{2a-\frac{1}{2}}} \\
\leq C \left( 16^N \| f \|_{L^\infty((0, \infty))} + \frac{1}{N^{2a-\frac{1}{2}}} \right) 
\]  

for all \( N \in \mathbb{N} \). Here we note that \( f \in \mathcal{U} \) implies \( \| x^\alpha f(\cdot) \|_{H^{q-\frac{1}{2}}(\mathbb{R}^+)} \leq M \). We set \( \eta = \| f \|_{L^\infty((0, \infty))} \) for simplicity. Let \( 0 < \gamma < \frac{2a-1}{\alpha} \) be chosen arbitrarily. We fix \( N \in \mathbb{N} \) such that

\[
\left( \log \frac{1}{\eta} \right)^{\frac{4\gamma}{2a-1}} \leq N < 1 + \left( \log \frac{1}{\eta} \right)^{\frac{4\gamma}{2a-1}}. 
\]

Then we can see that

\[
\frac{1}{N^{\frac{2a-1}{\alpha}}} \leq \left( \log \frac{1}{\eta} \right)^{-\gamma}. \tag{3.3}
\]

Moreover, we have

\[
16^N \| f \|_{L^\infty((0, \infty))} = \eta \exp((\log 16)N) \leq \eta \exp \left( (\log 16) + (\log 16) \left( \log \frac{1}{\eta} \right)^{\frac{4\gamma}{2a-1}} \right). 
\]

Since \( \frac{4\gamma}{2a-1} < 1 \), we can easily verify

\[
\lim_{\eta \to 0} \eta \exp \left( (\log 16) \left( \log \frac{1}{\eta} \right)^{\frac{4\gamma}{2a-1}} \right) \left( \log \frac{1}{\eta} \right)^{\gamma} = 0. 
\]

Consequently, we see that

\[
16^N \| f \|_{L^\infty((0, \infty))} \leq \frac{C}{\left( \log \frac{1}{\eta} \right)^{\gamma}}. \tag{3.4}
\]

Application of (3.3) and (3.4) in (3.2) yields conclusion (1.5). Thus the proof of Theorem 1 is complete.
4. Proof of Theorem 2

It is sufficient to show that \( \mathcal{L} \mathcal{V} \subset \mathcal{U} \). Let

\[
(I_0^\alpha F)(t) = \int_0^t \frac{(t - \tau)^{\alpha - 1}}{\Gamma(\alpha)} F(\tau) \, d\tau \quad (\alpha > 0)
\]

be the fractional integral of order \( \alpha \). Recall the Hardy inequality for the fractional integral (see [5: Formula (5.46')] )

\[
\int_0^\infty t^{-2\alpha - 2\gamma} |I_0^{\alpha} t^\gamma F(t)|^2 \, dt \leq \frac{\pi}{\Gamma^2(\alpha + \frac{1}{2})} \int_0^\infty |F(t)|^2 \, dt.
\]

By replacing
\[
\alpha \quad \text{by} \quad 1 - \alpha
\]
\[
\gamma \quad \text{by} \quad \frac{\alpha}{2} + q - \frac{3}{2}
\]
\[
F(t) \quad \text{by} \quad t^{\frac{3}{2} - \frac{q}{2}} F'(t)
\]

we obtain

\[
\int_0^\infty t^{1 - 2(q - \frac{\alpha}{2})} |I_0^{1 - \alpha} F'(t)|^2 \, dt \leq \frac{\pi}{\Gamma^2(\frac{q}{2} + \alpha)} \int_0^\infty x^{1 - (\alpha + 2q - \frac{2}{2})} |F'(x)|^2 \, dx
\]

\[
= \frac{\pi}{\Gamma^2(\frac{q}{2} + \alpha)} \|F'\|_{L^2_{\frac{q}{2} + q - 1}}^2.
\]

Hence, if \( F \in \mathcal{V} \), then \( F' \in L^2_{\frac{q}{2} + q - 1} \) and

\[
\|I_0^{1 - \alpha} F'\|_{L^2_{\frac{q}{2} + q - 1}} \leq \sqrt{\frac{\pi}{\Gamma^2(\frac{q}{2} + \alpha)}} \|F'\|_{L^2_{\frac{q}{2} + q - 1}} \leq M,
\]

so the corresponding Bergman-Selberg norm of its Laplace transform is also bounded by \( M \),

\[
\| \mathcal{L} I_0^{1 - \alpha} F' \|_{H^2_{q - \frac{2}{2}}(\mathbb{R}^+)} \leq M.
\]

We have (see [5: Formula (7.14)])

\[
(\mathcal{L} I_0^{1 - \alpha} F')(x) = x^{\alpha - 1}(\mathcal{L} F')(x).
\]

Since \( F(0) = 0 \), it is clear that

\[
(\mathcal{L} F')(x) = x(\mathcal{L} F)(x).
\]

Hence,

\[
(\mathcal{L} I_0^{1 - \alpha} F')(x) = x^\alpha (\mathcal{L} F)(x).
\]

Thus

\[
\| x^\alpha (\mathcal{L} F)(x) \|_{H^2_{q - \frac{2}{2}}(\mathbb{R}^+)} \leq M.
\]

As \( F \in \mathcal{V} \), we also have \( \| \mathcal{L} F \|_{H^2_{q}(\mathbb{R}^+)} = \| F \|_{L^2_q} \leq M \). Consequently, \( \mathcal{L} \mathcal{V} \subset \mathcal{U} \).
References


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