A Nonlinear Boundary Value Problem for a Nonlinear Ordinary Differential Operator in Weighted Sobolev Spaces

N. T. Long, B. T. Dung and T. M. Thuyet

Abstract. We use the Galerkin and compactness method in appropriate weighted Sobolev spaces to prove the existence of a unique weak solution of the nonlinear boundary valued problem

\[-\frac{1}{x^\gamma} \frac{d}{dx} M(x, u'(x)) + f(x, u(x)) = F(x) \quad (0 < x < 1)\]

\[\lim_{x \to 0^+} x^{\gamma/p} u'(x) < +\infty\]

\[M(1, u'(1)) + h(u(1)) = 0\]

where \(\gamma > 0, p \geq 2\) are given constants and \(f, F, h, M\) are given functions.

Keywords: Boundary value problems, ordinary differential operators, weak solutions, existence and uniqueness, Galerkin method, weighted Sobolev spaces

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1. Introduction

We consider the nonlinear boundary value problem

\[-\frac{1}{x^\gamma} \frac{d}{dx} M(x, u'(x)) + f(x, u(x)) = F(x) \quad (0 < x < 1)\]

\[\lim_{x \to 0^+} x^{\gamma/p} u'(x) < +\infty\]

\[M(1, u'(1)) + h(u(1)) = 0\]

where

\[\gamma > 0\] and \(p \geq 2\) are given constants

\(f, F, h\) are given functions

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$M : (0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and is monotonically increasing with respect to the second variable.

In the case of $\gamma = 0$ the problem

$$
\begin{align*}
\frac{1}{x^\gamma} \frac{d}{dx} M(x, u'(x)) + f(x, u(x)) &= F(x) \quad (0 < x < 1) \\
u(0) &= 0 \\
M(1, u'(1)) + \gamma_1 G(1) \sin u(1) &= 0
\end{align*}
$$

(1.2)

is related to the buckling of a nonlinear elastic bar with specific weight $\gamma_0$ immersed in a fluid with specific weight $\gamma_1$ that Tucsnak [1] has constructed in the case of

$$f(x, u) - F(x) = [-\lambda + (\gamma_0 - \gamma_1)g(x) - G'(1)] \sin u$$

where $\lambda > 0$ is a constant, $g$ and $G$ are given functions with some mechanical meaning, and $u(x)$ is the angle between the tangent of the bar in the buckled state of a point with curvilinear abscissa $x$ and vertical axis $Oy$. Then, in the case of $g = \text{const}$ and $M(x, u') = M(u')$ being monotonically increasing and sufficiently smooth Tucsnak has studied the bifurcation of integral equations equivalent to problems (1.1) and (1.2) depending on a parameter $\lambda$.

We note that problem (1.1) with $\gamma = 0$ and $u'M(x, u') \geq C_1 |u'|^p$ ($p > 1, C_1 > 0$) independent of $x$ had been considered in [2]. In [6] problem (1.1) with $p = 2, M(x, u') = x^\gamma u'$ with $\gamma > 0$ and the boundary condition $u'(1) + h_1 u(1) = h_2$ with given constants $h_1 > 0$ and $h_2$ has been studied. At least, in [3, 4] the nonlinear Bessel differential equation

$$
-\frac{1}{x} \frac{d}{dx} M(x, u'(x)) + u^2 - u = 0 \quad (x > 0)
$$

(1.3)

has been studied.

In this paper we use the Galerkin and compactness method in appropriate weighted Sobolev spaces to prove the existence of a unique weak solution of problem (1.1). The results obtained here generalize those of [1 - 4, 6].

2. Preliminary results, notations, function spaces

Put $\Omega = (0, 1)$ and $p' = \frac{p}{p-1}$. We omit the definitions of the usual function spaces $C^m(\overline{\Omega}), L^p(\Omega), H^m(\Omega)$ and $W^{m,p}(\Omega)$. We denote by $L^p(\Omega) \equiv L^p$ the class of all measurable functions $u$ defined on $\Omega$ for which

$$
\|u\|_{p, \gamma} < \infty \quad (1 \leq p \leq \infty)
$$

(2.1)

where

$$
\|u\|_{p, \gamma} = \left( \int_0^1 x^\gamma |u(x)|^p dx \right)^{\frac{1}{p}} \quad (1 \leq p < \infty)
$$

$$
\|u\|_{\infty, \gamma} = \text{ess sup} \sup_{0 < x < 1} |u(x)|
$$
and were we identify functions that are equal almost everywhere on \( \Omega \). The elements of \( L_0^p \) are thus actually equivalence classes of measurable functions satisfying (2.1), two functions being equivalent if they are equal a.e. in \( \Omega \). Then \( L_0^p \) is also a Banach space with respect to the norm \( \| \cdot \|_{p,\gamma} \). In particular, \( L_0^2 \) is a Hilbert space with usual scalar product \( (u, v) = \int_0^1 x^\gamma u(x)v(x) \, dx \) and norm \( \| u \|_{2,\gamma} = \sqrt{(u, u)} \). We denote by

\[
W_1^1, p(\Omega) \equiv W_1^1, p = \{ u \in L_0^p : u' \in L_0^p \} \quad (1 \leq p \leq \infty)
\]

the real Banach space with respect to the norm

\[
\| u \|_{1, p, \gamma} = (\| u \|^p_{p, \gamma} + \| u' \|^p_{p, \gamma})^{\frac{1}{p}} \quad (1 \leq p < \infty)
\]

\[
\| u \|_{1, \infty, \gamma} = \max \{ \| u \|_{\infty, \gamma}, \| u' \|_{\infty, \gamma} \}
\]

with derivatives in the sense of distributions [8]. In defining the function space \( W_1^1, p(\Omega) \) with weight \( x^\gamma \), we can also define \( W_1^1, p(\Omega) \) as the completion of the space

\[
S_1 = \{ u \in C^1(\Omega) : \| u \|_{1, p, \gamma} < \infty \}
\]

with respect to the norm \( \| \cdot \|_{1, p, \gamma} \) (see Adams [8]).

The following imbedding inequality will be used in the sequence.

**Lemma 2.1.** For every \( u \in C^1(\Omega), \gamma > 0 \) and \( p > 1 \) we have

\[
\begin{align*}
\| u \|^p_{p, \gamma} & \leq |u(1)|^p + K_1 \| u' \|^p_{p, \gamma} \\
|u(1)| & \leq K_2 \| u \|_{1, p, \gamma} \\
x^{\frac{\gamma}{2}} |u(x)| & \leq K_3 \| u \|_{1, p, \gamma} \\
\| u \|^p_{1, \gamma} & \leq K_4 \| u \|_{1, p, \gamma} (|u(1)|^p + \| u' \|^p_{p, \gamma})^{\frac{1}{p}} \quad (p \geq 2 - \frac{1}{\gamma})
\end{align*}
\]

where

\[
K_1 = \left( \frac{p-1}{\gamma} \right)^{p-1} \\
K_2 = (\gamma + p)^{\frac{1}{p}} \\
K_3 = \max \left\{ 2^{\frac{1}{p}}, (\gamma + 2p - 1)^{\frac{1}{\gamma}} \right\} \\
K_4 = K_3 \left( \frac{2^{p-1}}{1 + (p-1)^{\frac{1}{\gamma}}} \right)^{\frac{1}{p}}.
\]

**Proof.** (i) Integrating by parts in the following integral, we get

\[
\| u \|^p_{p, \gamma} = \frac{|u(1)|^p}{1 + \gamma} - \frac{p}{1 + \gamma} \int_0^1 x^{1+\gamma}|u(x)|^{p-2} u(x) u'(x) \, dx
\]

\[
= \frac{|u(1)|^p}{1 + \gamma} - \frac{p}{1 + \gamma} I
\]

where by using the Hölder inequality

\[
|I| = \left| \int_0^1 x^{\frac{\gamma}{2}} u'(x) x^{1+\frac{\gamma}{p}} |u(x)|^{p-2} u(x) \, dx \right| \leq \| u' \|_{p, \gamma} \| u \|_{p, \gamma}^{p-1}.
\]
It follows that
\[(1 + \gamma)\|u\|_{p, \gamma}^p \leq |u(1)|^p + p\|u'\|_{p, \gamma}\|u\|_{p, \gamma}^{p-1}.
\]
Using the Hölder inequality
\[ab \leq \frac{1}{p} \varepsilon^{-p}a^p + \frac{1}{p'} \varepsilon^{p'}b^{p'} \quad (\varepsilon > 0, a \geq 0, b \geq 0)
\]
it follows that
\[(1 + \gamma)\|u\|_{p, \gamma}^p \leq |u(1)|^p + \varepsilon^{-p}\|u'\|_{p, \gamma}^p + (p - 1)\varepsilon^{p'}\|u\|_{p, \gamma}^p
\]
where \((p - 1)\varepsilon^{p'} = \gamma\). Hence (2.2) is deduced.

(ii) Similarly, it follows from (2.3), (2.4) and the Hölder inequality with \(\varepsilon = 1\) that
\[|u(1)|^p = (1 + \gamma)\|u\|_{p, \gamma}^p + p I \leq (p + \gamma)\|u\|_{p, \gamma}^p + \|u'\|_{p, \gamma}^p
\]
where \((p - 1)\varepsilon^{p'} = \gamma\). Hence (2.2) is deduced.

(iii) We have for all \(x \in [0, 1]\)
\[x^\gamma |u(x)|^p = |u(1)|^p - \int_x^1 \frac{d}{dy}(y^\gamma |u(y)|^p) dy
\]
\[= |u(1)|^p - \gamma \int_x^1 y^{\gamma-1} |u(y)|^p dy - p \int_x^1 y^\gamma |u(y)|^{p-2} u(y) u'(y) dy
\]
where by using the Hölder inequality the later integral in the right-hand side is estimated as
\[\left|\int_x^1 y^\gamma |u(y)|^{p-2} u(y) u'(y) dy\right| \leq \|u\|_{p, \gamma}^{p-1} \|u'\|_{p, \gamma}^p.
\]
Taking together we deduce that
\[x^\gamma |u(x)|^p \leq |u(1)|^p + p\|u\|_{p, \gamma}^{p-1} \|u'\|_{p, \gamma}^p.
\]
We again use the Hölder inequality with \(\varepsilon = 1\) to get from (2.5) that
\[x^\gamma |u(x)|^p \leq (p + \gamma)\|u\|_{p, \gamma}^p + \|u'\|_{p, \gamma}^p + (p - 1)\|u\|_{p, \gamma}^p + \|u'\|_{p, \gamma}^p.
\]
Hence (2.2) is proved.

(iv) Let \(p \geq 2 - \frac{1}{\gamma}\) and \(p > 1\). We have from (2.2) that
\[\|u\|_{2, \gamma}^2 = \int_0^1 x^\gamma \int_0^1 x^\gamma |u(x)| dx \leq K_3 \|u\|_{1, \gamma} \int_0^1 x^\gamma |u(x)| dx.
\]
On the other hand, using the Hölder inequality we obtain the inequalities
\[|u(x)|^p \leq 2^{p-1} \left[|u(1)|^p + \left(\int_x^1 |u'(y)|^p dy\right)^{p}\right]
\]
\[\leq 2^{p-1} \left[|u(1)|^p + (1 - x)^{p-1} \int_x^1 |u'(y)|^p dy\right].
\]
and
\[ \left( \int_0^1 x^{-\gamma} |u'(x)| \, dx \right)^p \leq \int_0^1 x^{(p-1)\gamma} |u(x)|^p \, dx. \]

Hence, taking together we deduce that
\[ \left( \int_0^1 x^{-\gamma} |u(x)| \, dx \right)^p \leq \frac{2^{p-1} |u(1)|^p}{1 + (p-1)\gamma} + 2^{p-1} \int_0^1 x^{(p-1)\gamma} (1 - x)^{p-1} \, dx \int_x^1 |u'(y)|^p \, dy. \] (2.7)

Inverting the variables of integration \( x \) and \( y \) in the last integral we estimate that integral as
\[
\int_0^1 x^{(p-1)\gamma} (1 - x)^{p-1} \, dx \int_x^1 |u'(y)|^p \, dy
= \int_0^1 |u'(y)|^p \, dy \int_0^y x^{(p-1)\gamma} (1 - x)^{p-1} \, dx
\leq \int_0^1 |u'(y)|^p \, dy \int_0^y x^{(p-1)\gamma} \, dx
\leq \frac{1}{1 + (p-1)\gamma} \int_0^y y^{1+(p-1)\gamma} |u'(y)|^p \, dy
\] (2.8)

and note that \( y^{1+(p-1)\gamma} \leq y^\gamma \) for all \( y \in [0, 1] \) and \( p \geq 2 - \frac{1}{\gamma} \). Then (2.2) is deduced from (2.6) - (2.8)

**Remark 1.** The results (2.2) proves that \( (|u(1)|^p + |u'|_{p,\gamma}^p)^{\frac{1}{p}} \) and \( \|u\|_{1,p,\gamma} \) are two equivalent norms on \( W^{1,p}_\gamma(\Omega) \) and
\[ \frac{1}{1 + K_1} \|u\|_{p,\gamma}^p \leq |u(1)|^p + \|u'|_{p,\gamma}^p \leq (1 + K_1^p) \|u\|_{p,\gamma}^p \] (2.9)
for all \( u \in W^{1,p}_\gamma(\Omega) \).

**Lemma 2.2.** The imbedding \( W^{1,p}_\gamma(\Omega) \hookrightarrow L^2_\gamma(\Omega) \) \( (p > 1) \) is continuous if \( p \geq 2 - \frac{1}{\gamma} \), and compact if \( p \geq 2 \).

**Proof.** For \( p \geq 2 - \frac{1}{\gamma} \) the continuity of the imbedding \( W^{1,p}_\gamma(\Omega) \hookrightarrow L^2_\gamma(\Omega) \) is deduced from (2.2) and (2.9). For \( p \geq 2 \) we have \( W^{1,p}_\gamma(\Omega) \hookrightarrow W^{1,2}_\gamma(\Omega) \hookrightarrow L^2_\gamma(\Omega) \) and on the other hand the imbedding \( W^{1,2}_\gamma(\Omega) \hookrightarrow L^2_\gamma(\Omega) \) is compact (see [5]). Hence, \( W^{1,p}_\gamma(\Omega) \hookrightarrow L^2_\gamma(\Omega) \) is also compact.

**Remark 2.** We also note that
\[ \lim_{x \to 0^+} x^{-\gamma} u(x) = 0 \quad (u \in W^{1,p}_\gamma(\Omega)) \] (2.10)
(see [7: p. 128/Lemma 5.40]). On the other hand, by \( W^{1,p}(\varepsilon,1) \hookrightarrow C^0([\varepsilon,1]) \) \( (0 < \varepsilon < 1) \) and
\[ \varepsilon^{-\gamma} \|u\|_{1,p,\gamma} \leq \|u\|_{1,p,\gamma} \quad (u \in W^{1,p}_\gamma, 0 < \varepsilon < 1) \] (2.11)
From (2.10) and (2.12) we deduce that

$$x^\beta u \in C^0(\Omega) \quad (u \in W^{1,p}_\gamma(\Omega)).$$

Put $H = L^2_\gamma(\Omega)$ and $V = W^{1,p}_\gamma(\Omega)$ with $p > 1$ and $p \geq 2 - \frac{1}{\gamma}$. From the result of Lemma 2.2 with $p \geq 2 - \frac{1}{\gamma}$, $V$ is continuously embedded into $H$. Furthermore, $V$ is dense in $H$ since $C^1(\overline{\Omega})$ is dense in $H$; identifying $H$ with $H'$ (the dual of $H$), we have $V \hookrightarrow H \hookrightarrow V'$. On the other hand, the notation $\langle \cdot, \cdot \rangle$ is used for the pairing between $V$ and $V'$.

### 3. Theorem on existence and uniqueness

We assume that $p \geq 2$ and formulate the hypotheses

\begin{itemize}
  \item[(M1)] $M : (0,1] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory condition, i.e. $M(\cdot, y)$ is measurable on $(0,1]$ for every $y \in \mathbb{R}$ and $M(x, \cdot)$ is continuous on $\mathbb{R}$ for a.e. $x \in (0,1]$.
  \item[(M2)] There exist a constant $C_1 > 0$ and a function $q_1 \in L^1(\Omega)$ such that $yM(x,y) \geq C_1x^\gamma|y|^p - |q_1(x)|$.
  \item[(M3)] There exist a constant $C_2 > 0$ and a function $q_2$ with $x^{-\frac{2}{\gamma}}q_2 \in L^p(\Omega)$ and $\lim_{x \to 0^+} x^{-\frac{2}{\gamma}}|q_2(x)| < \infty$ such that $|M(x,y)| \leq C_2x^\gamma|y|^{p-1} + |q_2(x)|$.
  \item[(M4)] $M$ is monotonically increasing with respect to the second variable, i.e. $(M(x,y) - M(x,\hat{y}))(y - \hat{y}) \geq 0$ for all $y, \hat{y} \in \mathbb{R}$ and a.e. $x \in \Omega$.
\end{itemize}

Furthermore, we formulate the hypotheses

\begin{itemize}
  \item[(F1)] $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory condition.
  \item[(F2)] There exist constants $C_3 > 0$ and $1 < r < p$ and a function $q_3 \in L^1(\Omega)$ such that $yf(x,y) + C_3|y|^r \geq -|q_3(x)|$ for all $y \in \mathbb{R}$ and a.e. $x \in \Omega$.
  \item[(F3)] There exist a constant $C_4 > 0$ and a function $q_4 \in L^p(\Omega)$ such that $|f(x,y)| \leq C_4|y|^{p-1} + |q_4(x)|$ for all $y \in \mathbb{R}$ and a.e. $x \in \Omega$.
\end{itemize}

Finally, we formulate the hypothesis

\begin{itemize}
  \item[(H1)] For $h \in C^0(R; R)$ there exist two constants $C_5, C'_5 > 0$ with $uh(u) \geq C_5|u|^p - C'_5$ for all $u \in \mathbb{R}$.
\end{itemize}

Suppose that

$$F \in V'.$$  

**Remark 3.** In hypothesis (F2), $r = p$ still holds if $C_3 > 0$ is sufficiently small (see Remark 6).

The weak solution of problem (1.1) is formed from the following variational
Problem. Find $u \in V$ such that

$$
\int_0^1 M(x, u'(x))u'(x)\,dx + h(u(1))u(1) + (f(x, u(x)), v) = (F, v) \quad (3.2)
$$

for all $v \in V$.

Remark 4. By (2.13), the terms $u(1)$ and $v(1)$ appearing in (3.2) are defined for every $u, v \in V$. We obtain (3.2) by formally multiplying both sides of (1.1) by $x^\gamma v \in V$ and then integrating by parts when taking conditions (1.1)$_2$, (2.10) and hypothesis (M$_3$).

Then we have the following

Theorem 1. Let $F \in V'$ and let hypotheses (M$_1$) - (M$_4$), (F$_1$) - (F$_3$) and (H$_1$) hold. Then the variational problem (3.2) has a solution. Furthermore, if $M(x, \cdot), f(x, \cdot), h$ are non-decreasing, i.e.

$$
\begin{align*}
(M(x, y) - M(x, \bar{y}))(y - \bar{y}) & \geq 0 \\
(f(x, y) - f(x, \bar{y}))(y - \bar{y}) & \geq 0 \\
(h(y) - h(\bar{y}))(y - \bar{y}) & \geq 0
\end{align*}
$$

for all $y, \bar{y} \in \mathbb{R}$ and a.e. $x \in \Omega$ where two of the three inequalities above are strict in the case $y \neq \bar{y}$, then the solution is unique.

On the other hand, uniqueness of the solution also holds if condition (3.3) is replaced by the hypothesis

(A$_1$) There exist constants $C_6, C_7, C_8 > 0$ with $0 < C_1 < \min \{C_8, \frac{C_6}{K_1}\}$ such that

(i) $(M(x, y) - M(x, \bar{y}))(y - \bar{y}) \geq C_6 x^\gamma |y - \bar{y}|^p$

(ii) $(f(x, y) - f(x, \bar{y}))(y - \bar{y}) \geq -C_7 |y - \bar{y}|^p$

(iii) $(h(y) - h(\bar{y}))(y - \bar{y}) \geq C_8 |y - \bar{y}|^p$

for all $y, \bar{y} \in \mathbb{R}$ and a.e. $x \in \Omega$.

Proof. Since $V$ is separable there exists a sequence of linear independent elements \{w$_j$\} which is total in $V$. We find $u_m$ under the form

$$
u_m = \sum_{j=1}^m c_{mj}w_j \quad (3.4)
$$

where $c_{mj}$ satisfy the nonlinear equation system

$$
\int_0^1 M(x, u_m'(x))w_j'(x)\,dx + h(u_m(1))w_j(1) + (f(x, u_m(x)), w_j) = (F, w_j). \quad (3.5)
$$

By the Brouwer lemma (see [8: p. 53/Lemma 4.3]) it follows from hypotheses (M$_1$) - (M$_3$), (F$_1$) - (F$_3$) and (H$_1$) that system (3.4) - (3.5) has a solution $u_m$. Multiplying the
By means of (3.10), (3.11) and Lemma 2.2 the sequence \( \{u_m\} \) has a subsequence still denoted by \( \{u_m\} \) such that

\[
\begin{align*}
& u_m \rightarrow u \quad \text{in } V \text{ weakly} \\
& u_m \rightarrow u \quad \text{in } H \text{ strongly and a.e. in } \Omega \\
& x^{-\frac{r}{p}} M(x, u_m') \rightarrow \chi \quad \text{in } L^{p'} \text{ weakly}
\end{align*}
\]
Note that because the embedding \( W^{1,p}(\varepsilon, 1) \hookrightarrow C^0([\varepsilon, 1]) \quad (0 < \varepsilon < 1) \) is compact, by (2.11) and (3.10) \( \{u_m\} \) has a subsequence still denoted \( \{u_m\} \) such that \( u_m|_{[\varepsilon, 1]} \rightharpoonup u|_{[\varepsilon, 1]} \) in \( C^0([\varepsilon, 1]) \). Hence

\[
\begin{align*}
& u_m(1) \rightharpoonup u(1) \\
& h(u_m(1)) \rightharpoonup h(u(1))
\end{align*}
\]

(3.14)

On the other hand, it follows from hypothesis \( (F_1) \) and (3.13)\(_2\) that

\[
x^{\frac{2}{p'}} f(x, u_m) \rightharpoonup x^{\frac{2}{p'}} f(x, u) \quad \text{a.e. } x \in \Omega.
\]

(3.15)

We shall need the following lemma, the proof of which can be found in [9].

**Lemma 3.1.** Let \( Q \) be an open bounded set of \( \mathbb{R}^N \) and \( G, G_m \in L^q(Q) \quad (1 < q < \infty) \) such that \( G_m \rightharpoonup G \) a.e. in \( Q \) and \( \|G_m\|_{L^q(Q)} \leq C \), with \( C \) being a constant independent of \( m \). Then \( G_m \rightharpoonup G \) weakly in \( L^q(Q) \).

Applying Lemma 3.1 with \( N = 1, q = p', Q = \Omega, G_m = x^{\frac{2}{p'}} f(x, u_m) \) and \( G = x^{\frac{2}{p'}} f(x, u) \) we deduce from (3.12) and (3.15) that

\[
x^{\frac{2}{p'}} f(x, u_m) \rightharpoonup x^{\frac{2}{p'}} f(x, u) \quad \text{weakly in } L^{p'}.
\]

(3.16)

If we pass to the limit in equation (3.5) we find without difficulty from (3.13)\(_3, (3.14)\)_2 and (3.16) that \( u \) satisfies the equation

\[
\int_0^1 x^{\frac{2}{p'}} \chi u'(x) \, dx + h(u(1))u(1) + \langle f(x, u), v \rangle = \langle F, v \rangle
\]

(3.17)

for all \( u \in V \). So we shall prove the existence of the solution of the variational problem (3.2) if we show that \( \chi = x^{\frac{2}{p'}} M(x, u') \). From (3.4) and (3.5) we can deduce

\[
\int_0^1 M(x, u_m^')(x)u_m'(x) \, dx = -h(u_m(1))u_m(1) - \langle f(x, u_m(x)), u_m \rangle + \langle F, u_m \rangle.
\]

(3.18)

By using (3.13)\(_1,2, (3.14), (3.16)\) and (3.17) and passing to the limit in (3.18) as \( m \to +\infty \) we have

\[
\lim_{m \to +\infty} \int_0^1 M(x, u_m'(x))u_m'(x) \, dx = \int_0^1 x^{\frac{2}{p'}} \chi(x)u'(x) \, dx.
\]

(3.19)

We deduce from (3.13)\(_1,3\) and (3.19) that

\[
\lim_{m \to +\infty} \int_0^1 (M(x, u_m'(x)) - M(x, \theta(x)))(u_m'(x) - \theta(x)) \, dx = \int_0^1 (x^{\frac{2}{p'}} \chi(x) - M(x, \theta(x)))(u'(x) - \theta(x)) \, dx
\]
for all \( \theta \in L^p_\gamma \). Using the monotonicity property of \( M \), we obtain

\[
\int_0^1 \left( x^{-\frac{\gamma}{2}} \chi(x) - M(x, \theta(x)) \right) (u'(x) - \theta(x)) \, dx \geq 0
\]

for all \( \theta \in L^p_\gamma \). If we choose here \( \theta = u' - \lambda w \) with \( \lambda > 0 \) and \( w \in L^p_\gamma \) and let \( \lambda \to 0^+ \), we easily deduce that \( \chi = x^{-\frac{\gamma}{2}} M(x, u') \) and the existence proof is completed.

To prove uniqueness let \( u \) and \( v \) be two solutions of the variational problem (3.2). Then \( w = u - v \) satisfies the equality

\[
\int_0^1 \left( M(x, u'(x)) - M(x, v'(x)) \right) w'(x) \, dx + (h(u(1)) - h(v(1))) w(1) + (f(x, u) - f(x, v), w) = 0.
\]

(3.20)

If (3.3) holds, then evidently \( u = v \). If hypothesis (A1) holds, by (3.20) and (2.7) we have

\[
C_6 \|w'\|_{p, \gamma}^p + C_8 |w(1)|^p \leq C_7 \|w\|_{p, \gamma}^p
\]

and

\[
C_6 \|w'\|_{p, \gamma}^p + C_8 |w(1)|^p \geq \min \left\{ C_8, \frac{C_6}{K_1} \right\} (K_1 \|w'\|_{p, \gamma}^p + |w(1)|^p)
\]

\[
\geq \min \left\{ C_8, \frac{C_6}{K_1} \right\} \|w\|_{p, \gamma}^p,
\]

respectively, and since \( 0 < C_7 < \min \{C_8, \frac{C_6}{K_1} \} \) we deduce that \( w = 0 \). Theorem 1 is proved completely \( \blacksquare \)

**Remark 5.** In [3], corresponding to \( p = 2 \) and \( \gamma = 1 \), we have proved that the nonlinear Bessel differential equation (1.4) associated with the boundary conditions \( u(0) = 1 \) and \( u(\infty) = 0 \) has at least one solution. Wherein, the nonlinear term \( u^2 - u \) is non-monotonic. One of the solutions above is constructed from the boundary value problem (1.4) in the interval \( a \leq x \leq b \) associated with the boundary condition \( u(a) = 1 \) and \( u(b) = 0 \) wherein \( x_i < a < b < x_{i+1} \) and \( x_i, x_{i+1} \) are two consecutive zeros of the first order Bessel function \( J_0 \). Formation of a counterexample for the function \( f(x, u) \) not satisfying the assumption to be monotonically increasing with respect to \( u \) so that the solution of (3.2) is not unique is an open problem.

**Remark 6.** Theorem 1 still holds if hypothesis (F2) is replaced by the hypothesis (F'2)

There exist a constant \( C_3 \) with \( 0 < C_3 < \min \{C_5, \frac{C_4}{K_1} \} \) and a function \( q_3 \in L^1_\gamma \) such that \( yf(x, y) + C_3 |y|^p \geq -|q_3(x)| \) for all \( y \in \mathbb{R} \) and a.e. \( x \in \Omega \).

In fact, from hypotheses (M2), (F'2'), (H1) and (3.1), (3.6) we can obtain the following inequality similar to (3.7)

\[
C_1 \|u'_m\|_{p, \gamma}^p + C_5 |u_m(1)|^p
\]

\[
\leq C_3 \|u_m\|_{p, \gamma}^p + \|f\|_{L^1_\gamma} \|u'_m\|_{1, p, \gamma} + \|q_1\|_{L^1(\Omega)} + \|q_3\|_{1, \gamma} + C_5'.
\]
Choosing $C^*_3$ such that $0 < C_3 < C^*_3 < \min \{C_5, \frac{C_4}{K^*_1} \}$ it follows from (2.2) that

$$\left(1 - \frac{C^*_3}{C_3}\right) \frac{\min\{C_1, C_3\}}{1 + K_1} \|u_m\|_{1, p, \gamma}^p \leq \|F\|_{V'} \|u_m\|_{1, p, \gamma} + \|q_1\|_{L^1(\Omega)} + \|q_3\|_{1, \gamma} + C^*_5.$$

Hence, we obtain (3.10).

**Remark 7.** In Theorem 1 hypotheses (M2), (M3), (F1), (H1) are implied by hypothesis (A1). Indeed, it follows from (A1) that

\begin{align*}
(M_2) & \quad yM(x, y) \geq \tilde{C}_1 x^\gamma |y|^p - |\tilde{q}_1(x)| \\
(F_2) & \quad yf(x, y) + \tilde{C}_3 |y|^p \geq -|\tilde{q}_3(x)| \\
(H_1) & \quad yh(y) \geq \tilde{C}_5 |y|^p - \tilde{C}'_5
\end{align*}

where

\begin{align*}
\tilde{C}_1 & = C_6 - \frac{\epsilon^p}{p} > 0 \\
\tilde{C}_3 & = C_7 + \frac{\epsilon^p}{p} > 0 \\
\tilde{C}_5 & = C_8 - \frac{\epsilon^p}{p} > 0 \\
\tilde{C}'_5 & = \frac{\epsilon^{-p'}}{p'} |h(0)|^{p'}
\end{align*}

and

\begin{align*}
\tilde{q}_1(x) & = \frac{\epsilon^{-p'}}{p'} x^{-\frac{2p'}{p'}} |q_2(x)|^{p'} \in L^1 \\
\tilde{q}_3(x) & = \frac{\epsilon^{-p'}}{p'} |q_4(x)|^{p'} \in L^1
\end{align*}

From the condition $0 < C_7 < \min \{C_8, \frac{C_6}{K^*_1} \}$ we obtain the condition $0 < \tilde{C}_3 < \min \{\tilde{C}_5, \frac{\tilde{C}_4}{K^*_1} \}$ with $\epsilon > 0$ sufficiently small. We then have the following

**Theorem 2.** Let $F \in V'$ and let hypotheses (M1), (M3), (F1), (F3), (A1) hold. Then problem (3.2) has a unique solution.

**Remark 8.** Theorem 2 still holds if hypothesis (A1) is implied by the following hypothesis

(A2) There exist constants $C_6, C_7, C_8$ with $0 < C_8 < \frac{1}{K^*_1} \min \{C_6, C_7\}$ such that, for all $y, \tilde{y} \in \mathbb{R}$ and a.e. $x \in \Omega$.

(i) $(M(x, y) - M(x, \tilde{y}))(y - \tilde{y}) \geq C_6 x^\gamma |y - \tilde{y}|^p$

(ii) $(f(x, y) - f(x, \tilde{y}))(y - \tilde{y}) \geq C_7 |y - \tilde{y}|^p$

(iii) $(h(y) - h(\tilde{y}))(y - \tilde{y}) \geq -C_8 |y - \tilde{y}|^p$.

In fact, from (3.1), (3.6) and hypotheses (A2), (M1), (M3), (F1), (F3) we obtain

$$\min\{\tilde{C}_1, \tilde{C}_3\} \|u_m\|_{1, p, \gamma}^p \leq \left(\tilde{C}_5 K^*_1 + \frac{\epsilon^p}{p}\right) \|u_m\|_{1, p, \gamma}^p + \frac{\epsilon^{-p'}}{p'} \|F\|_{V'}^p + \|\tilde{q}_1\|_{L^1(\Omega)} + \|\tilde{q}_3\|_{1, \gamma} + \tilde{C}'_5.$$
for all $\varepsilon > 0$ where

$$
\begin{align*}
\tilde{C}_1 &= C_6 - \frac{\varepsilon^p}{p} \\
\tilde{C}_3 &= C_7 - \frac{\varepsilon^p}{p} \\
\tilde{C}_5 &= C_8 + \frac{\varepsilon^p}{p} \\
\tilde{C}_5' &= \frac{\varepsilon^{-p'}}{p'} |h(0)|^{p'}
\end{align*}
$$

and

$$
\begin{align*}
\tilde{q}_1(x) &= \frac{\varepsilon^{-p'}}{p'} x^{-\frac{2\varepsilon^p}{p} |q_2(x)|^{p'}} \\
\tilde{q}_3(x) &= \frac{\varepsilon^{-p'}}{p'} |q_4(x)|^{p'}
\end{align*}
$$

It follows from the condition $0 < C_8 < \frac{1}{k^2} \min\{C_6, C_7\}$ that there exists $\varepsilon > 0$ such that $\min\{\tilde{C}_1, \tilde{C}_3\} > \tilde{C}_5 R^2 + \frac{\varepsilon^p}{p}$. Hence we obtain that $\|u_m\|_{1,p,\gamma} \leq C$ where $C$ is a constant independent of $m$. We then have the following

**Theorem 3.** Let (3.1) and let hypotheses $(A_2)$, $(M_1)$, $(M_3)$, $(F_1)$, $(F_3)$ hold. Then problem (3.2) has a unique solution.

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**References**


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