On the Existence of $C^1$ Functions with Perfect Level Sets

E. D’Aniello and U. B. Darji

Abstract. Given a closed set $M \subset [0,1]$ of Lebesgue measure zero, we construct a $C^1$ function $f$ with the property that $f^{-1}(\{y\})$ is a perfect set for every $y$ in $M$.

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1. Introduction

Bruckner and Garg in [1] give a full description of the level set structure of a typical continuous function. In [2] the description of the level set structure of a typical $C^1$ function is given. It follows from [2: Theorem 2] that a typical $C^1$ function is either strictly monotone or has the property that all of its level sets are countable.

In this article we investigate the behaviour of $C^1$ functions in the other direction. It is easy to show that given a $C^1$ function $f : [0,1] \to [0,1]$, the set of points where the level sets of $f$ are uncountable has Lebesgue measure zero [3: p. 226/Lemma 6.3]. We show that given a closed set $M \subset [0,1]$ of Lebesgue measure zero, there exists a $C^1$ function $f : [0,1] \to [0,1]$ with the property that $f^{-1}(\{y\})$ is a perfect set for every $y$ in $M$ as well as finite for every $y$ in $[0,1]\setminus M$.

2. Preliminaries

In this section a few definitions, notations and lemmas are stated that are used throughout the article.

Definition 2.1. A subset $B \subset \mathbb{R}^2$ is a box if $B = I \times J$ for some compact intervals $I$ and $J$ of the real line $\mathbb{R}$. A signed box is an ordered pair $(B, \ast)$ where $B$ is a box and $\ast \in \{+, -\}$. We will generally use $B^\ast$ to denote the signed box $(B, \ast)$.

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Throughout $\pi_1$ and $\pi_2$ will be the coordinate projections. We shall say that two intervals of the real line are disjoint if their intersection is empty or coincides with a single point.

**Definition 2.2.** Let $B = [a, b] \times [c, d]$ be a box. A $C^1$ function $f : [a, b] \rightarrow [c, d]$ is diagonal to $B^+$, denoted by $f \uparrow B^+$, if

1. $f$ is increasing on $[a, b]$ with $f(a) = c$ and $f(b) = d$
2. $f'(a) = f'(b) = 0$

where by the derivatives at the end points we mean the right and the left derivatives, respectively. Further, we say that $f$ is diagonal to $B^-$, denoted by $f \downarrow B^-$, if condition (2) above is satisfied and the condition

3. $f$ is decreasing on $[a, b]$ with $f(a) = d$ and $f(b) = c$

is satisfied. Finally, we say that $f$ is diagonal to $B$, denoted by $f \uparrow B$ or $f \downarrow B$. If $f \uparrow B^*$, we say that $*$ is the sign induced on $B$ by $f$.

**Definition 2.3.** Let $B = I \times J$ be a box. Then the slope of $B$, denoted by $\text{sl}(B)$, is $\frac{\lambda(J)}{\lambda(I)}$.

The proof of the following lemma is a very easy exercise.

**Lemma 2.4.** Let $B^*$ be a signed box and $\varepsilon > 0$. Then there exists a $C^1$ function $f$ such that $f \uparrow B^*$ and $|f'(x)| < \text{sl}(B) + \varepsilon$ for all $x \in \pi_1(B)$.

**Definition 2.5.** Let $B = I \times J$ be a box. We say:

(i) $f : I \rightarrow J$ is jagged inside $B^+$ if $f|_{B_L} \uparrow B^+_L$, $f|_{B_M} \uparrow B^+_M$ and $f|_{B_R} \uparrow B^+_R$, where $B_i = I_i \times J$ and $\{I_L, I_M, I_R\}$ is the partition of $I$ into three equal pieces ordered from the left to the right.

(ii) $f$ is jagged inside $B^-$ if $f|_{B_L} \downarrow B^-_L$, $f|_{B_M} \downarrow B^-_M$ and $f|_{B_R} \downarrow B^-_R$, where the $B_i$’s are the same as above.

(iii) $f$ is jagged inside $B$ if $f$ is jagged inside $B^+$ or $B^-$. We call $\{B_L, B_M, B_R\}$ vertical splitting of $B$ into three equal pieces.

**Lemma 2.6.** Let $B^*$ be a signed box and $\varepsilon > 0$. Then there is a $C^1$ function $g$ jagged inside $B^*$ such that $|g'(x)| < 3\text{sl}(B) + \varepsilon$ for all $x \in \pi_1(B)$.

**Proof.** If $* = +$, we apply Lemma 2.4 to $B^+_L$, $B^-_M$ and $B^+_R$. If $* = -$, we apply Lemma 2.4 to $B^-_L$, $B^+_M$ and $B^-_R$.
3. Proof of the main result

In this section we first prove a lemma essential to our main result.

Lemma 3.1. Suppose $B$ is a box, $*$ is a sign (i.e. $* \in \{+, -\}$), $M \subseteq \pi_2(B)$ is a closed set with $\lambda(M) = 0$, $N > 0$ is an integer and $\delta > 0$. Then there exist a $C^1$ function $h$ and a sequence of pairwise disjoint boxes $B_1, \ldots, B_n$ contained in $B$ such that:

1. $h \not\subseteq B^*$.
2. $M \subseteq \pi_2(\bigcup_{i=1}^n B_i)$, $\pi_2(B_i) \cap M \neq \emptyset$ and $\lambda(\bigcup_{i=1}^n \pi_2(B_i)) < \delta$.
3. $\{\pi_2(B_i)\}$ is a pairwise disjoint finite sequence, and so is $\{\pi_1(B_i)\}$.
4. $\text{sl}(B_i) = \frac{1}{N}$ (1 $\leq i \leq n$).
5. $h|_{B_i} \not\subseteq B_i^\dagger$ (1 $\leq i \leq n$).
6. $|h'(x)| < \text{sl}(B) + \delta$ for all $x \notin \bigcup_{i=1}^n \pi_1(B_i)$.
7. $|h'(x)| < \frac{1}{N} + \frac{\delta}{2}$ for all $x \in \bigcup_{i=1}^n \pi_1(B_i)$.

Proof. Let $B = I \times J$, where $I = [a, b]$ and $J = [c, d]$. Without loss of generality let us assume that $* = +$. Since $M$ is a closed subset of a compact set and $\lambda(M) = 0$, there exist a finite number of disjoint closed intervals $J_1, \ldots, J_n$ such that $M \subseteq \bigcup_{i=1}^n J_i$, $J_i \cap M \neq \emptyset$, $\lambda(\bigcup_{i=1}^n J_i) < \delta$ and

$$0 < \frac{\lambda(J) - \lambda(\bigcup_{i=1}^n J_i)}{\lambda(I) - N\lambda(\bigcup_{i=1}^n J_i)} < \frac{\delta}{2}.$$ 

We shall find a finite sequence of pairwise disjoint intervals $I_1, \ldots, I_n$ contained in $I$ and $h \in C^1(I)$ such that

(i) $\lambda(I_i) = N\lambda(J_i)$ (1 $\leq i \leq n$)
(ii) $h$ is diagonal to $B^+$
(iii) $h|_{B_i} \not\subseteq B_i^\dagger$ (1 $\leq i \leq n$) where $B_i = I_i \times J_i$
(iv) $0 \leq h'(x) < \frac{\lambda(J) - \lambda(\bigcup_{i=1}^n J_i)}{\lambda(I) - N\lambda(\bigcup_{i=1}^n J_i)} + \frac{\delta}{2}$ (x $\in I \setminus \bigcup_{i=1}^n I_i$).

Without loss of generality we may assume that $z \geq y$ for all $z \in J_{i+1}$ and all $y \in J_i$ (1 $\leq i \leq n$). Let

$$m = \frac{\lambda(J) - \lambda(\bigcup_{i=1}^n J_i)}{\lambda(I) - N\lambda(\bigcup_{i=1}^n J_i)}$$

and set

$$d_1 = d(J_1, c), I_1 = \left[ a + \frac{d_1}{m}, a + \frac{d_1}{m} + N\lambda(J_1) \right]$$
$$d_2 = d(J_1, J_2), I_2 = \left[ a + \frac{d_1}{m} + \frac{d_2}{m} + N\lambda(J_1), a + \frac{d_1}{m} + \frac{d_2}{m} + N(\lambda(J_1) + \lambda(J_2)) \right]$$
$$\vdots$$
$$d_n = d(J_{n-1}, J_n), I_n = \left[ a + \sum_{i=1}^n \frac{d_i}{m} + N \sum_{i=1}^{n-1} \lambda(J_i), a + \sum_{i=1}^n \frac{d_i}{m} + N \sum_{i=1}^n \lambda(J_i) \right].$$
Apply Lemma 2.4 to $B_i^+$ and $\delta$, and also to $[a, a + \frac{d_n}{m}] \times [c, c + d_1]$ and $\frac{\delta}{2}$, to

$$
\left[a + \sum_{i=1}^{n} \frac{d_i}{m} + N \sum_{i=1}^{n} \lambda(J_i), b \right] \times \left[\max J_n, d \right]
$$

and $\frac{\delta}{2}$, and to

$$
\tilde{B}_t^+ = \left[a + \sum_{i=1}^{t} \frac{d_i}{m} + N \sum_{i=1}^{t} \lambda(J_i), a + \sum_{i=1}^{t+1} \frac{d_i}{m} + N \sum_{i=1}^{t} \lambda(J_i) \right] \times \left[\max J_t, \min J_{t+1} \right]
$$

and $\frac{\delta}{2}$, for $t = \{1, \ldots, n - 1\}$. We paste together the resulting functions and call this function $h$. Then $h$ is a $C^1$ function which satisfies the desired properties (1) - (7). 

**Theorem 3.2.** Let $M \subseteq [0, 1]$ be a closed set with $\lambda(M) = 0$. Then there exists a $C^1$ function $f : [0, 1] \to [0, 1]$, onto, such that $f^{-1}(\{y\})$ is a nowhere dense perfect set for all $y \in M$ as well as finite for all $y \in [0, 1] \setminus M$.

**Proof.** We shall construct the desired function $f$ as $C^1$ limit of a sequence of piecewise monotone $C^1$ functions. At stage $n$ we shall have a piecewise monotone function $f_n$ with $f_n(0) = 0$, $f_n(1) = 1$ and a finite collection of boxes. Then $f_{n+1}$ will be an appropriate modification of $f_n$ inside these boxes. This sequence $\{f_n\}$ will be constructed inductively.

Let us first construct $f_1$. We apply Lemma 3.1 with $B = [0, 1] \times [0, 1]$, $\ast = +$, $M = M$, $N = 3 \cdot 2$ and $\delta = \frac{1}{2}$ and obtain a function $h$ and a finite collection of boxes $B_1, \ldots, B_n$ which satisfy the conclusions of Lemma 3.1. Using Lemma 2.6 with $\varepsilon = \frac{1}{2}$, for each $i$, $1 \leq i \leq n$, we obtain a $C^1$ function $g_i$ which is jagged inside $B_i^+$. Let

$$
f_1(x) = \begin{cases} 
h(x) & \text{if } x \in [0, 1] \setminus \cup_{i=1}^{n} \pi_1(B_i) \\
g_i(x) & \text{if } x \in \pi_1(B_i) \text{ for some } i.
\end{cases}
$$

Now, for each $i$, let $B_{i,L}$, $B_{i,M}$, $B_{i,R}$ be the vertical splitting of $B_i$ into three equal pieces and let $\mathcal{G}_1 = \{B_{i,L}, B_{i,M}, B_{i,R} : 1 \leq i \leq n\}$. At the end of stage 1, the following properties hold:

1. $f_1$ is a piecewise monotone $C^1$ function with $f_1(0) = 0$ and $f_1(1) = 1$.
2. $f_1|_{B \square B}$ for all $B \in \mathcal{G}_1$.
3. $|f'_1(x)| < \text{sl}(B) + \delta = 1 + \frac{1}{2}$ for all $x \in [0, 1] \setminus \cup_{i=1}^{n} \pi_1(B_i)$.
4. $|f'_1(x)| < 3 \cdot \frac{1}{N} + \delta = 3 \cdot \frac{1}{3 \cdot 2} + \frac{1}{2} = 1$ for all $x \in \cup_{i=1}^{n} \pi_1(B_i)$.

Now let us assume that we are at stage $k > 1$, $f_k$ and $\mathcal{G}_k$ have been constructed so that the following properties are satisfied:

1. $M \subseteq \pi_2(\cup \mathcal{G}_k)$, $\pi_2(B) \cap M \neq \emptyset$ for all $B \in \mathcal{G}_k$, and $\lambda(\pi_2(\cup \mathcal{G}_k)) < \frac{1}{(k+1)!}$.
2. $\text{sl}(B) = \frac{1}{2^n}$ for all $B \in \mathcal{G}_k$.
3. Suppose that $y \in M$ and $B \in \mathcal{G}_{k-1}$ are such that $y \in \pi_2(B)$. Then there exist disjoint boxes $B_1$ and $B_2$ in $\mathcal{G}_k$ contained in $B$ such that $y \in \pi_2(B_1) \cap \pi_2(B_2)$.
(iv) $f_k$ is a piecewise monotone $C^1$ function with $f_k(0) = 0$ and $f_k(1) = 1$.
(v) $f_k(x) = f_{k-1}(x)$ if $x \in [0, 1] \setminus \pi_1(\cup \mathcal{G}_{k-1})$.
(vi) $f_k|B \subseteq B$ for all $B \in \mathcal{G}_k$.
(vii) $|f_k'(x)| < \frac{1}{2k} + \frac{1}{2k}$ for all $x \in \cup_{B \in \mathcal{G}_k} \pi_1(B)$.
(viii) $|f_k'(x)| < \frac{1}{2k} + \frac{1}{2k}$ for all $x \in \pi_1(\cup \mathcal{G}_{k-1}) \setminus \pi_1(\cup \mathcal{G}_k)$.
(ix) If $(x, f_k(x))$ is such that $f_k(x) \in M$, then $(x, f_k(x)) \in \cup \mathcal{G}_k$.

Now we proceed to construct $f_{k+1}$. Fix a box $B \in \mathcal{G}_k$ and let $\ast$ be the sign induced on $B$ by $f_k|B$. Now we apply Lemma 3.1 to $B^\ast$, $M \cap \pi_2(B)$, $N = 3 \cdot 2^{k+1}$ and $\delta = \frac{1}{(k+2)!} \lambda(\pi_2(B))$ and obtain a function $h : \pi_1(B) \rightarrow \pi_1(B)$, onto, and a finite collection of pairwise disjoint boxes $B_1, \ldots, B_n$ which satisfy the conclusions of Lemma 3.1.

Using Lemma 2.6 with $\varepsilon = \frac{1}{2k+1}$, for each $i$, $1 \leq i \leq n$, we obtain a $C^1$ function $g_i$ which is jagged inside $B_i^\ast$. Now let $h_B : \pi_1(B) \rightarrow \pi_2(B)$ be defined as

$$h_B(x) = \begin{cases} h(x) & \text{if } x \in \pi_1(B) \setminus \cup_{i=1}^n \pi_1(B_i) \\ g_i(x) & \text{if } x \in \pi_1(B_i) \text{ for some } i. \end{cases}$$

We observe the following:

1. $h_B$ is a piecewise monotone $C^1$ function.
2. $|h_B'(x)| < \text{sl}(B) + \delta < \frac{1}{2k} + \frac{1}{2k} \pi_1(B_i)$ for all $x \in \pi_1(B) \setminus \cup_{i=1}^n \pi_1(B_i)$.
3. $|h_B'(x)| < 3 \cdot \frac{1}{2k+1} + \frac{1}{2k} \pi_1(B)$ for all $x \in \cup_{i=1}^n \pi_1(B_i)$.
4. $h_B$ and $h_B'$ agree with $f_k$ and $f_k'$, respectively, at the end points of $\pi_1(B)$.

Let

$$\mathcal{G}_{k,B} = \{B_{i,L}, B_{i,M}, B_{i,R} : 1 \leq i \leq n\}$$

and observe that, for every $C \in \mathcal{G}_{k,B}$, $\text{sl}(C) = \frac{1}{2k+1}$. Now for each $B$ obtain such function $h_B$ and a collection of boxes $\mathcal{G}_{k,B}$. Define $f_{k+1} : [0, 1] \rightarrow [0, 1]$ by

$$f_{k+1}(x) = \begin{cases} f_k(x) & \text{if } x \in [0, 1] \setminus \pi_1(\cup \mathcal{G}_k) \\ h_B(x) & \text{if } x \in \pi_1(B) \text{ for some } B \in \mathcal{G}_k. \end{cases}$$

Let $\mathcal{G}_{k+1} = \cup_{B \in \mathcal{G}_k} \mathcal{G}_{k,B}$. It is easy to verify that $f_{k+1}$ satisfies the induction hypotheses (i) - (ix). Fix $k$. Then

$$|f_k'(x) - f_{k+1}'(x)| \leq \sup_{x \in \cup \mathcal{G}_k} |f_k'(x)| + \sup_{x \in \cup \mathcal{G}_k} |f_{k+1}'(x)| \leq \frac{7}{2} \cdot \frac{1}{2k}$$

for all $x \in [0, 1]$. Since $f_k(0) = 0$, $\{f_k\}$ converges to some $C^1$ function $f$.

Next let us show that $f^{-1}(y)$ is perfect for every $y \in M$. Clearly, $f^{-1}(y) \neq \emptyset$, since $f : [0, 1] \rightarrow [0, 1]$ is onto. Let $x \in [0, 1]$ be such that $f(x) = y$ and let $\varepsilon > 0$. We shall show that there exists $x'$ such that $|x - x'| < \varepsilon$ and $f(x') = y$. Let $k$ be large enough so that $\frac{2^k}{(k+1)!} < \varepsilon$. Notice that, for each $B \in \mathcal{G}_k$, $\lambda(\pi_1(B)) < \frac{2^k}{(k+1)!}$. By induction hypotheses (v) and (ix), $(x, f_k(x)) \in B_0$ for some $B_0 \in \mathcal{G}_k$. Notice that $\pi_1(B_0) \subseteq (x-\varepsilon, x+\varepsilon)$. By hypothesis (iii) there exist disjoint boxes $B_1$ and $B_2$ in $\mathcal{G}_{k+1}$. 


contained in $B_0$ such that $y \in \pi_2(B_1) \cap \pi_2(B_2)$. For each $l > k$, $f_l(\pi_1(B_1)) = \pi_2(B_1)$ and $f_l(\pi_1(B_2)) = \pi_2(B_2)$. Since $\{f_k\}$ converges to $f$ in the $C^1$ norm, we have $f(\pi_1(B_1)) = \pi_2(B_1)$ and $f(\pi_1(B_2)) = \pi_2(B_2)$. There is at least one point $x' \neq x$ in $(x - \varepsilon, x + \varepsilon)$ such that $f(x') = y$. Therefore $f^{-1}(\{y\})$ is perfect. From the same argument it follows that $f^{-1}(\{y\})$ is nowhere dense.

Now we want to show that, for each $y \in [0, 1] \setminus M$, $f^{-1}(\{y\})$ is finite. Let $k$ be large enough so that $y \not\in \pi_2(\bigcup G_k)$. Then, for all $l > k$, $f^{-1}(\{y\}) = f_l^{-1}(\{y\})$. Since $f_l$ is piecewise monotone, $f_l^{-1}(\{y\})$ is finite.

References


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