Estimates for Quasiconformal Mappings
onto
Canonical Domains

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Abstract. In this paper we establish estimates for $K$-quasiconformal mappings $z = g(w)$ of a domain bounded by two circles $|w| = 1, |w| = q$ and $n$ continua situated in $q < |w| < 1$ onto a circular ring $Q(g) < |z| < 1$ that has been slit along $n$ arcs on the circles $|z| = R_j(g)$ $(j = 1, \ldots, n)$ such that $|z| = 1$ and $|z| = Q$ correspond to $|w| = 1$ and $|w| = q$, respectively. The bounds in the estimates for $Q, R_i$ and $|g(w)|$ are explicitly given, most of them are optimal. They are deduced mainly from [17].

Keywords: $K$-quasiconformal mappings, Riemann moduli of a multiply-connected domain, monotony of the modulus of a doubly-connected domain

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1. Introduction and notations

The generalization of Carleman's (see [1: p. 212], [2: p. 177], [12: p. 15]) area inequality for doubly-connected domains to multiply-connected domains in [15] improves many Grötzsch's [4, 6, 8] and Rengel's [13] significant circular slits theorems for conformal mappings. In [16] we establish further area inequalities for $K$-quasiconformal mappings (see the definition in [10: p. 16]). Combining this with Grötzsch's [4, 5, 7] inequalities yields in [17] sharp or asymptotic sharp estimates for $K$-quasiconformal mappings of the circular ring $Q < |z| < 1$ with concentric circular slits onto domains lying in $q < |w| < 1$. In this paper we establish estimates for the inverse mappings of those studied in [17]. Here the consideration is partly similar to the case of conformal mappings ($K = 1$, see [14: pp. 121 - 124]) using two auxiliary functions introduced in Section 4.

Let now $B$ be any domain given in the $w$-plane, bounded by two circles $|w| = 1, |w| = q$ and $m (p, n \in \mathbb{N})$ boundary components $\sigma_1, \ldots, \sigma_m$ lying in $(0 <) q < |w| < 1$, and transformed into itself by the rotation $t = e^{i\frac{2\pi}{p}}w$. We shall write $B = B_0$ when all $\sigma_j$ are circular arcs concentric with the circular ring. Let $G$ be the family of all $K$-quasiconformal mappings $z = g(w)$ each of which maps $B$ onto a circular ring $Q(g) < |z| < 1$ that has been slit along $m$ circular arcs $L_1(g), \ldots, L_m(g)$ concentric with the

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circular ring such that \(|z| = 1, |z| = Q\) and \(L_j\) correspond to \(|w| = 1, |w| = q\) and 
\(\sigma_j\) \((j = 1, \ldots, pn)\), respectively, \(g(1) = 1\) and satisfies the \(p\) times rotation symmetry 
\[ g(e^{i2\pi \frac{w}{r}} w) = e^{i2\pi \frac{w}{r}} g(w) \quad (w \in B). \tag{1.1} \]

It is clear that the symmetry condition is trivial for \(p = 1\). Therefore we see that the 
inverse mapping \(f\) of \(g, g \in G\), belongs to the family \(F\) studied in \([17]\), because from 
(1.1) for all \(z \in A, A = g(B)\), and \(f = g^{-1}, g \in G\), the relation 
\[ e^{i2\pi \frac{w}{r}} f(z) = f(e^{i2\pi \frac{w}{r}} z) \]
follows. Put 
\[ R_j(g) = |z| \text{ with } z \in L_j(g) \text{ and } g \in G \]
\[ c_j = \min \{|w| \text{ and } d_j = \max \{|w| \text{ with } w \in \sigma_j \} \} \text{ (j = 1, \ldots, pn)} \]
\[ c = \min \{c_1, \ldots, c_{pn}\} \text{ and } d = \max \{d_1, \ldots, d_{pn}\} \]
\[ \mu = \sup \prod_{j=1}^{pn} \frac{r_j}{r_j'}, \text{ where } r_j < |w| < r_j' \text{ are pairwise disjoint sets in } B. \]

Furthermore, denote 
by \(S\) the inner area of \(B\) 
by \(s_j\) the external area of the compact set bounded by \(\sigma_j\).

Clearly, 
\[ \frac{c}{qd} \leq \mu \leq \frac{1}{q} \tag{1.2} \]
and 
\[ S + s = \pi(1 - q^2) \quad \text{with } s = \sum_{j=1}^{pn} s_j. \tag{1.3} \]

Our task is to estimate the radii \(Q(g)\) and \(R_j(g)\) \((g \in G; j = 1, \ldots, pn)\) that are nothing 
but the Riemann moduli of the domain \(B\) in the case \(K = 1\) (see \([11: p. 334]\)); as well 
as \(|g(w)|\) \((g \in G, w \in B)\), by at most the quantities \(K, p, q, c_j, d_j, s_j, \mu, S\) and \(|w|\). 
The bounds in the estimates will be explicitly calculated by simple functions or the auxiliary 
function \(R(p, t, s)\) introduced in Section 4. Most of them are the best among the bounds 
that depend on the same quantities.

2. Estimates of \(Q\)

The estimate of \(Q\) plays an important role in establishing estimates for other quantities. 
Therefore we begin with this estimate.

**Theorem 1.** Under the above hypothesis and notations we have, for every \(g \in G\), 
the estimate 
\[ \left(1 + \frac{S}{\pi q^2}\right)^{-\frac{k}{2}} \leq Q(g) \leq \mu^{-\frac{k}{2}}, \tag{2.1} \]
where the equality on the left-hand side holds if and only if \(B = B_0\) and \(g(w) = w|w|^{k-1}\) \((w \in B)\), and the equality on the right-hand side holds if and only if \(B = B_0\) 
and \(g(w) = w|w|^{k-1}\) \((w \in B)\).
Proof. Applying [17: Theorem 3.1] to the mapping $f = g^{-1}$, $g \in G$, we have

$$S \geq \pi q^2(Q - \frac{1}{K} - 1),$$

hence the lower bound of $Q$ in (2.1) follows. Here the equality holds if and only if $f(z) = z|z|^{1/K - 1}$ ($z \in A$), i.e., $B = B_0$ and $g(w) = w|w|^{K-1}$ ($w \in B$). On the other hand, applying [16: Theorem 1] to the mapping $g \in G$, we obtain

$$\pi \geq \pi Q^2 \mu_k^*,$$

hence the upper bound of $Q$ in (2.1) follows. Here with the help of [17: Formula (2.5)] we have the assertion on the occurrence of the equality.

Corollary 1. By (1.3), the lower bound of $Q$ in (2.1) may be written in the form

$$Q(g) \geq q^K \left(1 - \frac{s}{\pi}\right)^{-\frac{1}{K}} \quad (g \in G) \quad (2.2)$$

hence

$$Q(g) \geq q^K \quad (g \in G). \quad (2.3)$$

Equality in (2.2) and (2.3) can only occur if $B = B_0$ and $g(w) = w|w|^{K-1}$ ($w \in B$).

Corollary 2. From (2.1) and (1.2) we obtain the estimate

$$Q(g) \leq \left(\frac{d}{c}\right)^{\frac{1}{K}} \quad (< 1) \quad (g \in G)$$

where the equality can only occur if $c = d$ and $g(w) = w|w|^{K-1}$ ($w \in B$).

3. Lower bounds of $R_j$

Since $R_j(g) > Q(g)$ ($g \in G; j = 1, \ldots, pn$), with the help of (2.2) or (2.3) we can get lower bounds of $R_j$. However, we want to establish other relations that, in certain situations, may give sharper estimates.

Theorem 2. Under the hypothesis and notations given in Section 1, for every $g \in G$ with $Q(g) = Q > q^K$ we have the estimates

$$R_j(g) > Q \left[\frac{ps_j}{\pi(Q K - q^2)}\right]^{\frac{1}{K}} \quad (j = 1, \ldots, pn) \quad (3.1)$$

and

$$\max_{1 \leq j \leq pn} R_j(g) > Q \left[\frac{s}{\pi(Q K - q^2)}\right]^{\frac{1}{K}}. \quad (3.2)$$

Proof. First, we notice that by (1.1), each circle $|z| = R_j$ contains at least $p$ slits belonging to the boundary of $A = g(B), g \in G$. By [17: Formula (3.1)], we therefore have

$$ps_j \leq \pi R_j^2 (1 - q^2 Q - \frac{1}{K}),$$

hence for every $g \in G$ with $q < Q^{1/K}$, i.e., $Q(g) > q^K$, estimate (3.1) follows. Similarly we obtain, with the help of [17: formula (3.2)], estimate (3.2).
4. The auxiliary functions $R(p, t, s)$ and $T(p, r, s)$

In order to establish other estimates of $R_j$, $Q$ and $|g(w)|$ we will introduce the following two real functions.

**Definition.** The real functions

\[
\begin{align*}
  r &= R(p, t, s) \quad (0 \leq s < t < 1) \\
  t &= T(p, r, s) \quad (0 \leq s < r < 1)
\end{align*}
\]

are defined in such a way that the circular ring $s < |w| < 1$ with $p$ radial slits

\[
P_j = \left\{ w \mid s \leq |w| \leq t \text{ and } \arg w = \frac{2\pi}{p} \right\} \quad (j = 0, \ldots, p - 1)
\]

and the circular ring $r < |z| < 1$ can be schlicht conformal mapped onto each other.

Because of the monotony of the modulus of a doubly-connected domain (see [2: p. 176]) we have the following monotonies of the auxiliary function $R(p, t, s)$ with $p \in \mathbb{N}$:

\[
\begin{align*}
  s &< R(p, t, s) < t \quad (0 \leq s < t < 1) \quad (4.1) \\
  R(p, t_1, s) &< R(p, t_2, s) \quad (0 \leq s < t_1 < t_2 < 1) \quad (4.2) \\
  R(p, t, s_1) &< R(p, t, s_2) \quad (0 \leq s_1 < s_2 < t < 1) \quad (4.3) \\
  R(p, t, s) &> R(1, t, s) \quad (0 \leq s < t < 1; p \geq 2)
\end{align*}
\]

With the help of Hersch's [9: p. 316] and Nehari's [11: p. 295] formulae, I myself found in [14: pp. 101 - 104] the following expression for $R(p, t, s)$:

\[
R(p, t, s) = \exp \left\{ -\pi \frac{K'(tp)}{2pK(tp)} \right\} \quad (0 < t < 1; p \in \mathbb{N}) \quad (4.4)
\]

with

\[
K(k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}
\]

\[
K'(k) = K(\sqrt{1 - k^2})
\]

and for $0 < s < t < 1$

\[
R(p, t, s) = \exp \left\{ -\pi \frac{K'(u)}{2pK(u)} \right\}
\]

with

\[
u = 1 + h - \sqrt{h(2 + h)}
\]

where

\[
h = \frac{(1 - k)(1 - ak)}{k(1 + a)}, \quad k = 4s^p \prod_{n=1}^{\infty} \left[ \frac{1 + s^{4pn}}{1 + s^{2p(2n-1)}} \right]^4
\]

\[
a = sn \left( b + \frac{i2pb}{\pi} \log \frac{t}{s}, k \right), \quad b = K(k).
\]
Here $sn(z, k)$ means the Jacobian elliptic sinus with the parameter $k$. Another expression for $R(p, t, s)$ was shown by Graeser in [3: pp. 77 - 78].

In view of [14: Formula (1.21)] we have the estimate

$$4^{-1} t < R(p, t, s) < t \quad (0 \leq s < t < 1; p \in \mathbb{N}),$$

hence $R(p, t, s) \approx t$ when $p \to \infty$.

The evaluation of $K(t_p)$ and $K'(t_p)$ (see [18: p. 177]) yields the asymptotic behaviour of $R(p, t, 0)$:

$$R(p, t, 0) \approx 4^{-1} t \quad \text{when } t \to 0$$

and

$$1 - R(p, t, 0) \approx \frac{\pi^2}{2p \log \frac{8}{p(1-t)}} \quad \text{when } t \to 1.$$

Successive approximations for $R(1, t, 0)$ are given by Lehto [10: p. 64]. The expression for $T(p, r, s)$, that is not needed here, was shown by Thao ([14: pp. 102 - 105] or [17: p. 61]).

5. Other estimates of $R_j, Q$ and $|g(w)|$

Using the auxiliary functions studied in Section 4, other estimates for $R_j, Q$ and $|g(w)|$ will be given. In particular, when $s_j = 0$ or $s = 0$, they may be sharper than ones of (3.1) and (2.2).

**Theorem 3.** Under the hypothesis and notations given in Sections 1 and 4, for every $g \in G$, $w \in B$ and $j = 1, \ldots, pn$, we have the estimates

$$R^K(p, d, q) < R_j(g) < Q(g) R^{-K} \left(p, \frac{q}{c_j}, q\right)$$

$$Q(g) > R^K(p, d, q) R^K \left(p, \frac{q}{c_j}, q\right)$$

$$R^K(p, |w|, q) < |g(w)| < Q(g) R^{-K} \left(p, \frac{q}{|w|}, q\right).$$

**Proof.** Considering the mapping $f = g^{-1}, g \in G$, with the help of [17: Theorem 6.1] we obtain

$$d_j < T(p, R^+_j, q) = t \quad \text{and} \quad \frac{q}{c_j} < T \left[p, \left(\frac{Q}{R_j}\right)^{\frac{1}{K}}, q\right] = t'.$$

Hence the definition of the auxiliary functions and the monotony (4.2) yield the relations

$$R^+_j = R(p, t, q) > R(p, d, q) \quad \text{and} \quad \left(\frac{Q}{R_j}\right)^{\frac{1}{K}} = R(p, t', q) > R\left(p, \frac{q}{c_j}, q\right).$$

Thus we have estimate (5.1). Estimate (5.2) is just a consequence of (5.1). Using [17: Formula (6.10)], we obtain similar estimate (5.3).
Corollary 3. From Theorem 3 and (4.5), for every $g \in G$, $w \in B$ and $j = 1, \ldots, pn$ we obtain the simple estimates

\[ 4^{-\frac{k}{q}} d_j^K < R_j(g) < 4^K Q(g) \left( \frac{c_j}{q} \right)^K \]  
(5.4)

\[ Q(g) > 4^{-2\frac{k}{q}} \left( \frac{qd_j}{c_j} \right)^K \]  
(5.5)

\[ 4^{-\frac{k}{q}} |w|^K < |g(w)| < 4^K Q(g) \left( \frac{|w|}{q} \right)^K \]  
(5.6)

In view of (4.3) and (4.6) we see that the coefficients in (5.4) - (5.6) are the best possible.

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References


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