Abstract. Let \( u : \Omega \to \mathbb{R}^N \) be the solution of the nonlinear elliptic system

\[
- \sum_{i=1}^{n} \partial_i F_i(x, \nabla u) = f(x) + \sum_{i=1}^{n} \partial_i f_i(x),
\]

where \( \Omega \subseteq \mathbb{R}^n \) is a bounded domain with a piecewise smooth boundary (e.g., \( \Omega \) is a polyhedron). It is assumed that a mixed boundary value condition is given. Global regularity results in Sobolev and in Nikolskii spaces are proven, in particular \([W^{s,2}(\Omega)]^N\)-regularity \((s < \frac{3}{2})\) of \( u \).

Keywords: Mixed boundary value problems, piecewise smooth boundaries, Nikolskii spaces

AMS subject classification: Primary 35J55, 35J65, secondary 35J25

0. Introduction

We treat the nonlinear elliptic system

\[
\begin{align*}
- \sum_{i=1}^{n} \partial_i F_i(x, \nabla u) & = f(x) + \sum_{i=1}^{n} \partial_i f_i(x) & \text{in } \Omega \\
u(x) & = 0 & \text{on } \Gamma_D \\
- \sum_{i=1}^{n} F_i(x, \nabla u) \nu_i & = \sum_{i=1}^{n} f_i \nu_i & \text{on } \Gamma_N
\end{align*}
\]

where \( \Omega \subseteq \mathbb{R}^n \) \((n \geq 3)\) is bounded, \( u : \Omega \to \mathbb{R}^N \) is a vector-valued function, \( \partial_i = \frac{\partial}{\partial x_i} \), \( \partial \Omega = \Gamma_D \cup \Gamma_N \) where \( \Gamma_D \) is the Dirichlet boundary and \( \Gamma_N \) is the Neumann boundary, and \( \nu \) is the outward normal of \( \partial \Omega \). We suppose that \( \partial \Omega \) is piecewise smooth (e.g., \( \Omega \) is a polyhedron or has a Lipschitz boundary).

In this paper we investigate the regularity of the solution \( u \) of (0.1). Refining the method of [8] we obtain regularity results in Nikolskii spaces and in Sobolev spaces \([W^{s,2}(\Omega)]^N\), especially \([W^{s,2}(\Omega)]^N\)-regularity \((s < \frac{3}{2})\) of \( u \) up to the boundary.
Solutions of mixed boundary value problems in non-smooth domains may have singularities on the boundary at such points where the boundary condition is changing or where \( \partial \Omega \) is not smooth.

In the case of a linear elliptic equation various authors have investigated the regularity of the solution. They have given a decomposition of the solution \( u \) into a regular and a singular part. In particular, for \( \Omega \subset \mathbb{R}^2 \) this provides an explicit description of the behaviour of \( u \) near the boundary (cf. [4, 7, 9, 11]). In the case when \( \Omega \subset \mathbb{R}^n \) \( (n \geq 3) \) there are difficulties by finding such a decomposition which describes all the singularities of \( u \) (see [2, 3, 10, 14, 17]).

In the case of nonlinear equations there are only few results. Semilinear Dirichlet problems on corner domains are treated in [12, 15] and in [5, 6], where results in weighted Sobolev spaces are given. Further, nonlinear mixed boundary value problems are investigated in [8]. Regularity results in Sobolev spaces are proven.

In this paper we generalize some results given in [8]. Let the boundary of \( \Omega \) consist of smooth \( (n - 1) \)-dimensional manifolds with piecewise smooth boundaries such that each boundary manifold is either a Dirichlet or a Neumann boundary manifold. Let us fix some point \( P \in \partial \Omega \). Then we suppose that there is a ball \( B(P) \) and a smooth mapping which maps \( \Omega \) onto a domain \( \hat{\Omega} \) such that \( B(P) \cap \hat{\Omega} \) is the intersection of \( B(P) \) and a polyhedron. In contrast to [8] we consider the case that \( B(P) \cap \partial \hat{\Omega} \) contains more than one Dirichlet boundary manifold. Further, we admit that \( B(P) \cap \hat{\Omega} \) is probably not convex. But we assume that each inner angle between a Dirichlet and a Neumann boundary manifold is not greater than \( \pi \).

We suppose that there is a function \( F(x, p) \) such that \( F^r_i(x, p) \) is the partial derivative of \( F(x, p) \) with respect to the component corresponding to \( p^r_i \) (here \( F^r_i(x, p) \) denotes the \( r \)-th component of the vector \( F_i(x, p) \)). Hence, we deal with the variational case.

The aim of this paper is to show that \( u \in [W^{s, 2}(\Omega)]^N \) for \( s < \frac{3}{2} \). This result is the best possible, for we admit that \( \hat{\Omega} \) can be a polyhedron where the inner angle between a Dirichlet and a Neumann boundary manifold is equal to \( \pi \). Otherwise, if all such angles are less than \( \pi \), we prove that \( u \in \mathcal{H}^{3, 2}(\Omega) \), where \( \mathcal{H}^{s, p}(\Omega) \) denotes a Nikolskii space. Moreover, in the case when \( N = 1 \) the solution \( u \) of equation (0.1) is Hölder continuous. Then we show that \( u \in L^p(\Omega) \) for some \( p > 3 \).

This paper is organized as follows. In Section 1 we state the assumptions on the data and the main results. Section 2 contains notations. In Section 3 the proofs of the main results are given. Finally, in Section 4 we explain the proofs with examples of tree-dimensional domains.

1. Assumptions on the data and main results

We need the following assumptions on the data.

(A1) \( \Omega \subset \mathbb{R}^n \) \( (n \geq 3) \) is a connected open domain with Lipschitz boundary.

(A2) \( \partial \Omega = \bigcup_{1 \leq i \leq M} \Gamma_i \), where \( \Gamma_i \) are open \( (n - 1) \)-dimensional manifolds, and \( \Gamma_i \cap \Gamma_j = \emptyset \) holds for \( i \neq j \).
(A3) \( \partial \Gamma_i \ (1 \leq i \leq M) \) are \((n-2)\)-dimensional Lipschitz continuous manifolds.

(A4) \( \Gamma_1, \ldots, \Gamma_\sigma \subset \Gamma_D \) and \( \Gamma_{\sigma+1}, \ldots, \Gamma_M \subset \Gamma_N \).

(A5) \( P \in \bigcap_{i \in A} \partial \Gamma_i \) implies that \(|A| \leq n\).

(A6) To each point \( P \in \partial \Omega \) there exists a mapping \( \phi \) and a ball \( B_R(\phi(P)) \) such that:

(i) \( B_R(\phi(P)) \cap \phi(\partial \Omega) \) is the intersection of \( B_R(\phi(P)) \) and a polyhedron.

(ii) \( B_R(\phi(P)) \cap \phi(\partial \Omega) \) is simply connected.

(iii) \( \phi, \phi^{-1} \in W^{2,\infty}_{loc}(\mathbb{R}^n) \) and the Jacobian of \( \phi \) is positive definite.

(iv) If \( \Gamma_i \in \Gamma_D, \Gamma_j \in \Gamma_N, \) and \( \partial \Gamma_i \cap \partial \Gamma_j \neq \emptyset \), then \( \angle(\phi(\Gamma_i), \phi(\Gamma_j)) \leq \pi \).

(v) At most one pair of boundary manifolds \( \Gamma_i, \Gamma_j \ (i \neq j, \partial \Gamma_i \cap \partial \Gamma_j \neq \emptyset) \) satisfies \( \angle(\phi(\Gamma_i), \phi(\Gamma_j)) = \pi \).

Remark.

(i) By \( \angle(\phi(\Gamma_i), \phi(\Gamma_j)) \) we denote the inner angle between \( \phi(\Gamma_i) \cap B_R(\phi(P)) \) and \( \phi(\Gamma_j) \cap B_R(\phi(P)) \) where it is assumed that \( \phi(\Gamma_i) \cap B_R(\phi(P)) \neq \emptyset \) and \( \phi(\Gamma_j) \cap B_R(\phi(P)) \neq \emptyset \).

(ii) We assume that the inner angle between a boundary manifold of \( \phi(\Gamma_D) \) and another one of \( \phi(\Gamma_N) \) is not greater than \( \pi \) (cf. assumption (A6)/(ii)). But it is admitted that the inner angle between two boundary manifolds is greater than \( \pi \) if there is no change of the boundary value condition.

(iii) It is also possible to treat domains with a slit. Then instead of assumption (A6)/(v) we need the assumption that at most one pair of boundary manifolds \( \Gamma_i, \Gamma_j \ (i \neq j, \partial \Gamma_i \cap \partial \Gamma_j \neq \emptyset) \) satisfies \( \angle(\phi(\Gamma_i), \phi(\Gamma_j)) = \pi \).

Let \( x \in \overline{\Omega} \) and \( p \in \mathbb{R}^{nN} \) with components \( x_i \ (1 \leq i \leq n) \) and \( p_r \ (1 \leq r \leq N) \), respectively. We suppose that there is a \( C^2 \)-function \( F(x,p) : \Omega \times \mathbb{R}^{nN} \to \mathbb{R} \) such that

\[
\frac{\partial}{\partial x_i} F(x,p) = F_i(x,p), \quad F_i(x,p) = \frac{\partial}{\partial x_k} F_i(x,p), \quad F_{i,k}(x,p) = \frac{\partial}{\partial p_k} F_{i,k}(x,p)
\]

for \( 1 \leq i, k \leq n \) and \( 1 \leq r, s \leq N \). Furthermore, we suppose that there are functions \( g_0, g_x, g_i, \) and \( g_{i,z_k} \ (1 \leq i, k \leq n) \) such that:

(H1) \( c_0 + c_0 |p|^2 \leq F(x,p) \leq g_0(x) + c|p|^2 \) for \( g_0 \in L^\infty(\Omega) \) and \( c_0 > 0 \).

(H2) \( |F_{i,z}(x,z)| \leq g_{i,z}(x) + c|p|^2 \) for \( g_{i,z} \in L^1(\Omega) \).

(H3) \( |F_i(x,p)| \leq g_i(x) + c|p| \) for \( g_i \in L^2(\Omega) \).

(H4) \( |F_{i,k}(x,z)| \leq g_{i,z_k}(x) + c|p| \) for \( g_{i,z_k} \in L^2(\Omega) \).

(H5) \( |F_{i,k}^z(x,z)| \leq c \).

(H6) There is a constant \( k_0 > 0 \) independent of \( x \) and \( p \) such that for all \( \xi \in \mathbb{R}^{nN} \)

\[
k_0 |\xi|^2 \leq \sum_{r,s=1}^n \sum_{i,k=1}^n F_{i,k}^r(x,p)\xi_i^r \xi_k^s.
\]

(H7) \( F_i(x) \in L^2(\Omega) \) and \( F_{i,k}^r(x) \in W^{1,2}(\Omega) \cap L^\infty(\Omega) \) for \( 1 \leq i \leq n \) and \( 1 \leq r \leq N \).
Remark. Hypothesis (H6) can be replaced by the weaker condition

\[(H6') \quad \text{There are constants } k_0 > 0 \text{ and } k_1 \text{ independent of } x \text{ and } p \text{ such that for all } \xi \in [H^1(\Omega)]^N \]

\[ k_0 \int_{\Omega} |\nabla \xi|^2 dx - k_1 \int_{\Omega} |\xi|^2 dx \leq \int_{\Omega} \sum_{r,s=1}^{N} \sum_{i,k=1}^{n} F_{r,k}^*(x, \nabla u) \partial_i \xi^r \partial_k \xi^s dx. \]

Let us note that the changes to be made in the proofs are obvious.

Under the above hypotheses there exists a unique weak solution \( u \in [W^{1,2}(\Omega)]^N \) of problem (0.1) (see [16]).

We use the usual Sobolev spaces \( W^{s,p}(\Omega) \) and the Nikolskii spaces \( \mathcal{H}^{s,p}(\Omega) \) (cf. [1]). In detail, let \( s \) be no integer, let \( z \in \mathbb{R}^n, s = m + \sigma \) where \( 0 < \sigma < 1 \) and \( m \) is an integer, \( \Omega_\eta = \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \eta \} \), and \( 1 \leq p < \infty \). The spaces \( W^{s,p}(\Omega) \) and \( \mathcal{H}^{s,p}(\Omega) \) consist of all functions \( u \) for which the norms

\[ ||u||_{W^{s,p}(\Omega)} = \left( \int_{\Omega} \int_{\Omega} \frac{|\partial^\sigma u(x) - \partial^\sigma u(y)|^p}{|x - y|^{n + \sigma}} \, dx \, dy \right)^{\frac{1}{p}} \]

and

\[ ||u||_{\mathcal{H}^{s,p}(\Omega)} = \left( \int_{\Omega} \sup_{\sigma > 0} \int_{\Omega_\eta} \frac{|\partial^\sigma u(x + z) - \partial^\sigma u(x)|^p}{|z|^{p\sigma}} \, dx \right)^{\frac{1}{p}} \]

are finite.

We will prove the following results:

**Theorem 1.1.**

a) The solution \( u \) of equation (0.1) satisfies

\[ u \in [W^{s,2}(\Omega)]^N, \quad \text{for all } s < \frac{3}{2}. \]  \hspace{1cm} (1.1)

b) If \( \angle(\Gamma_i, \Gamma_j) \neq \pi \) for each pair of boundary manifolds \( \Gamma_i, \Gamma_j \) (\( i \neq j \), \( \partial \Gamma_i \cap \partial \Gamma_j \neq \emptyset \)), then

\[ u \in [\mathcal{H}^{\frac{3}{2},2}(\Omega)]^N \]  \hspace{1cm} (1.2)

holds.

**Remark.**

(i) By assumption we consider the case when \( n \geq 3 \). But our proofs of (1.1) and (1.2) also hold when \( n = 2 \).

(ii) \( \angle(\Gamma_i, \Gamma_j) \neq \pi \) implies that \( \angle(\phi(\Gamma_i), \phi(\Gamma_j)) \neq \pi \), for \( \phi \) is smooth.

Using the Sobolev imbedding theorem and (1.1) we get \( u \in [W^{1,s}(\Omega)]^N \) for \( s < \frac{2n}{n-1} \).

Let us note that \( s < 3 \) for \( n > 3 \). The next theorem improves this result in the case when \( N = 1 \).
Theorem 1.2. Let $N = 1$ and let the functions $g_{x_i}$, $g_i$, $g_{i,x_k}$, $f$ and $f_k$ given in hypotheses (H1) - (H7) satisfy

$$g_i \in L^{3\alpha/2}(\Omega), \quad g_{x_i}, g_i, g_{i,x_k}, f, \partial_i f_k \in L^{3\alpha/2}(\Omega)$$

for $1 \leq i, k \leq n$ and some $\delta > 0$. Then there exists a constant $\varepsilon_0 > 0$ independent of $n$ such that the solution $u$ of equation (0.1) satisfies

$$\nabla u \in L^s(\Omega) \quad \text{for} \quad s = 3 + \varepsilon_0.$$  

Remark. The results of Theorem 1.1 and Theorem 1.2 also hold for solutions $u(x,t)$ of parabolic systems. Let $u(x,0) \in [W^{1,2}(\Omega)]^N$. Then we get the results (1.1), (1.2), and (1.4) in the spaces $[L^2(0,T; W^{s,p}(\Omega))]^N$ and $[L^2(0,T; H^{s,p}(\Omega))]^N$.

2. Notations

Let $B_R(x) = \{y \in \mathbb{R}^n : |x - y| < R\}$. The boundary of $\Omega$ is piecewise smooth. By assumption to each point $P \in \partial \Omega$ there is a constant $R_0 > 0$ and a $W^{2,\infty}$-mapping

$$\phi^* : x \to \hat{x}$$

such that $B_{R_0}(\hat{P}) \cap \hat{\Omega}$ is the intersection of $B_{R_0}(\hat{P})$ and a polyhedron. (We use the denotations $\hat{P} = \phi^*(P)$, $\hat{\Omega} = \phi^*(\Omega)$ etc. and we will write $B_R$ instead of $B_R(\hat{P})$.)

In the sequel we suppose that $\hat{P}$ and $R_0 \in (0,1]$ are fixed such that $\hat{P}$ is the only vertex of $B_{R_0}(\hat{P}) \cap \partial \hat{\Omega}$ or that there is no vertex of $\partial \hat{\Omega}$ in $B_{R_0}(\hat{P})$. Further, let $\hat{P} \in \partial \hat{\Gamma}_k$ for some $k \in \{1, \ldots, M\}$.

We need appropriate basis vectors $\{\zeta^1, \ldots, \zeta^n\}$ in $B_{R_0}(\hat{P})$. Let $\Lambda_1$, $\Lambda_2$, and $\Lambda_3$ be disjoint index sets (some of them possibly empty) such that $\bigcup_{i=1}^n \Lambda_i = \{1, \ldots, n\}$. Let $\alpha^* > 0$, $|\zeta^i| = 1$ for $1 \leq i \leq n$, and $\text{angle}(\zeta^i, \zeta^j) \geq \alpha^*$ for $1 \leq i < j \leq n$. We assume the following:

1) $y + s\zeta^i \in (\hat{\Omega} \cup \partial \hat{\Omega})$ for $y \in (\partial \hat{\Omega} \cap B_{R_0})$, $0 < s < R_0$, and $1 \leq i \leq n$.

2) If $\hat{\Gamma}_D \cap B_{R_0} \neq \emptyset$, then $\zeta^i$ ($i \in \Lambda_1$) is parallel to $\hat{\Gamma}_D \cap B_{R_0}$.

3) If $\hat{\Gamma}_D \cap B_{R_0} = \emptyset$, then $\Lambda_1 = \{1, \ldots, n\}$.

4) If $i \in \Lambda_1$, $y \in (\hat{\Gamma}_D \cap B_{R_0})$, $s > 0$, and $y + s\zeta^i \in B_{R_0}$, then $y + s\zeta^i \in \hat{\Gamma}_D$.

5) If $\hat{\Gamma}_N \cap B_{R_0} \neq \emptyset$, then $\zeta^i$ ($i \in \Lambda_2$) is parallel to $\hat{\Gamma}_N \cap B_{R_0}$.

6) If $\hat{\Gamma}_N \cap B_{R_0} = \emptyset$, then $\Lambda_2 = \{1, \ldots, n\}$.

7) $\zeta^i$ ($i \in \Lambda_2$) satisfies
   
   i) $\text{angle}(\zeta^i, \hat{\Gamma}_D \cap B_{R_0}) \geq \alpha^*$
   
   ii) $y - s\zeta^i \notin (\hat{\Omega} \cup \partial \hat{\Omega})$ for $y \in (\hat{\Gamma}_D \cap B_{R_0})$, and $0 < s < R_0$.

8) If $\text{angle}(\hat{\Gamma}_i, \hat{\Gamma}_j) = \pi$ ($i \neq j, \hat{\Gamma}_i \cap \hat{\Gamma}_j \cap B_{R_0} \neq \emptyset$), then $\Lambda_3 = \{n\}$, otherwise $\Lambda_3 = \emptyset$.

9) $\zeta^n$ ($n \in \Lambda_3$) satisfies $\text{angle}(\zeta^n, \hat{\Gamma}_i \cup \hat{\Gamma}_j) \cap B_{R_0}) \geq \alpha^*$ where $i, j$ are given in Assumption 8).
Remark.

i) Let us note that there is such a basis. Some examples how to choose the basis vectors are given in Section 4.

ii) We can find a constant $\alpha^*$ depending only on $n$ and on the geometry of $\partial \Omega$.

In the sequel let $h > 0$. We define $E^\sigma_t y = y + \sigma \zeta^i$, $E^\sigma_t f(y) = f(y + \sigma \zeta^i)$,

$$D^h_t f(y) = \frac{E^h_t f(y) - f(y)}{h} \quad \text{and} \quad D^{-h}_t f(y) = \frac{f(y) - E^{-h}_t f(y)}{h}$$

and we will write $E^\sigma_t f(y)g(y)$ instead of $(E^\sigma_t f(y))g(y)$.

We set $R = \frac{R_0}{8}$, $B = B_R \cap \hat{\Omega}$, $B' = B_{4R} \cap \hat{\Omega}$, and

$$\hat{\Omega}^h_i = \left\{ y \in B_{R_0} : y \neq x + h \zeta^i, x \in B_{R_0} \right\}$$

$$\hat{\Omega}^{-h}_i = \left\{ y \in B_{R_0} \setminus \hat{\Omega} : y = x - h \zeta^i, x \in B_{R_0} \setminus \hat{\Omega} \right\}.$$  

Let $\tau_0$ be a cut-off function with $\tau_0 \equiv 1$ in $B$, supp $\tau_0 = B_{4R}$, and $|\nabla \tau_0| \leq c$, where $c$ depends only on $R_0$. By $\tau$ we denote the restriction of $\tau_0$ onto $\hat{\Omega} \cup \partial \hat{\Omega}$.

Moreover, we need appropriate extensions of functions into $\hat{\Omega}^{-h}_i$ for $i \in \Lambda_2$. Let the function $g(y)$ be defined on $\hat{\Omega}$. Let $z_0 \in \partial \hat{\Omega} \cap B_{R_0}$ and $z_0 - \lambda \zeta^i \in \hat{\Omega}^{-h}_i$ for $0 < \lambda \leq h$. Then we set

$$g(z_0 - \lambda \zeta^i) = g(z_0 + \lambda \zeta^i). \quad (2.1)$$

This is an $W^{1,2}$-extension if $g \in W^{1,2}(\hat{\Omega})$. In particular, it holds that $\|g\|_{W^{1,2}(\hat{\Omega}^{-h}_i)} \leq \|g\|_{W^{1,2}(\hat{\Omega})}$, where the constant $c$ depends only on the data, for $\alpha^*$ depends only on $n$ and on the geometry of $\partial \Omega$.

Next, we define an appropriate extension of $v = u \circ (\phi^*)^{-1}$ into $\hat{\Omega}^{-h}_i$ for $i \in \Lambda_2$. Let $\gamma \in \partial \hat{\Omega} \cap \partial \hat{\Omega}^{-h}_i$, $0 < \lambda \leq h$, and $y - \lambda \zeta^i \in \hat{\Omega}^{-h}_i$. We set

$$v(y - \lambda \zeta^i) = 0. \quad (2.2)$$

This provides an $W^{1,2}$-extension of $v$, for $i \in \Lambda_2$ implies that $(\partial \hat{\Omega} \cap \partial \hat{\Omega}^{-h}_i) \subset \gamma_D$. In particular, it holds for $1 \leq r \leq N$ that

$$\|v^r\|_{\mathcal{H}^{\frac{3}{2},2}(\hat{\Omega}^{-h}_i)} \leq c\|v^r\|_{\mathcal{H}^{\frac{3}{2},2}(\hat{\Omega})},$$

where $c$ and $c'$ depend only on the data and $v^r$ is the $r$-th component of $v$. Thus, extension (2.2) is an $\mathcal{H}^{\frac{3}{2},2}$-extension (cf. [8]).

In what follows we will write $\sum_{i,k, l}$ and $\sum_{s, r}$ instead of $\sum_{i,k, l=1}^n$ and $\sum_{r, s=1}^N$, respectively. Further, $\nabla v$ is an $\mathbb{R}^{nN}$-vector and $|\nabla v|^2 = \sum_r \sum_i |\partial_i v^r|^2$. The point $\cdot$ denotes the Euclidean scalar product and $c$ denotes a constant which will be allowed to vary from equation to equation.
3. The regularity of the solution

In this section we prove Theorem 1.1 and Theorem 1.2.

Let $A$ be the matrix whose elements are defined by $a_{ik} = \frac{\partial}{\partial x_i}(\Phi^k)$, where $\Phi^k$ denotes the $k$-th component of $\Phi(x)$. Let $y = \tilde{z}$. In the sequel we only deal with functions defined onto $\tilde{\Omega}$. For simplicity we will write $f(y)$ instead of $f((\Phi^*)^{-1}(y))$ etc.

The function $v = u \circ (\Phi^*)^{-1}$ is the weak solution of

$$- \sum_i \tilde{\alpha}_i F_i(y, \tilde{\nabla} v) = f(y) + \sum_i \tilde{\alpha}_i f_i(y)$$

(3.1)

where $\tilde{\alpha}_i v(y) = \sum_k a_{ik}(y) \partial_k v(y)$.

In detail, $A$ is positive definite, the smallest eigenvalue $\lambda_0 > 0$ depends only on the geometry of $\partial \Omega$, and

$$a_{ik}(y) \in W^{1,\infty}(\tilde{\Omega})$$

(3.2)

holds. Further, let us note that $v(y) \in [W^{1,2}(\tilde{\Omega})]^N$.

We need several propositions.

**Proposition 3.1.** It holds that

$$\sup_{0 < h < 4 \rho} \int_B \tau h |D_i^h \tilde{\nabla} v|^2 dy \leq c \quad \text{for } i \in \Lambda_1$$

(3.3)

where the constant $c$ depends only on $\rho_0$ and the data.

**Proof.** Let $0 < h < 4 \rho$. First, we suppose that $1 \in \Lambda_1$ and we prove (3.3) for $i = 1$. The Taylor expansion of $F(y, p)$ ($p \in \mathbb{R}^N$) entails

$$\sum_r \sum_i (p' - p)_i F_i^r(y, p) = F(y, p') - F(y, p)$$

$$- \sum_{r,s} \sum_{i,k} (p' - p)^r_i (p' - p)_k \int_0^1 (1 - t) F_{i,k}^{rs}(y, tp' + (1 - t)p) dt.$$  

(3.4)

Let

$$m_{ik}^r(h) = \int_0^1 (1 - t) F_{i,k}^{rs}(y, tE_i^h \tilde{\nabla} v + (1 - t) \tilde{\nabla} v) dt$$

for $1 \leq i, k \leq n$ and $1 \leq r, s \leq N$. We set $p = \tilde{\nabla} v$ and $p' = E_i^h \tilde{\nabla} v$. Thus, $(p' - p)_i = hD_i^h \tilde{\nabla} v^r \equiv \sum_l hD_i^h (a_{il} \partial_l v^r)$ and

$$\sum_r \sum_i F_i^r(y, \tilde{\nabla} v) D_i^h(a_{il} \partial_l v^r) = \frac{F(y, E_i^h \tilde{\nabla} v) - F(y, \tilde{\nabla} v)}{h}$$

$$- \sum_{r,s} \sum_{i,k} h (\sum_l D_i^h(a_{il} \partial_l v^r)) (\sum_l D_i^h(a_{kl} \partial_l v^s)) m_{ik}^r(h).$$

(3.5)
The function \( \varphi = \tau D^h_1 v \) is an admissible test function. Multiplying (3.1) by \( \varphi \) yields

\[
\sum_{i,l} \int_{B'} F_i(y, \tilde{v} v) \cdot \partial_l (a_{il} \tau) D^h_1 v + \sum_{i,l} \int_{B'} F_i(y, \tilde{v} v) \cdot (a_{il} \tau) \partial_l D^h_1 v
\]

\[
= \int_{B'} \tau f \cdot D^h_1 v - \sum_{i,l} \int_{B'} f_i \cdot \partial_l (a_{il} \tau D^h_1 v)
\]

where the point \( \cdot \) denotes the Euclidean scalar product in \( \mathbb{R}^N \). Applying (3.5) we obtain

\[
(I) = \int_{B'} \tau \sum_{r,s} \sum_{i,k} h \left( \sum_l D^h_1 (a_{il} \partial_l v^r) \right) \left( \sum_l D^h_1 (a_{kl} \partial_l v^s) \right) m_{ik}^r(h)
\]

\[
= \int_{B'} \frac{F(y, E^h_1 \tilde{v} v) - F(y, \tilde{v} v)}{h} - \sum_{i,l} \int_{B'} \tau F_i(y, \tilde{v} v) \cdot D^h_1 a_{il} \partial_l E^h_1 v
\]

\[
+ \sum_{i,l} \int_{B'} f_i(y, \tilde{v} v) \cdot \partial_l (a_{il} \tau) D^h_1 v - \int_{B'} \tau f \cdot D^h_1 v + \sum_{i,l} \int_{B'} f_i \cdot \partial_l (a_{il} \tau D^h_1 v)
\]

\[
= (II) + \ldots + (VI).
\]

The identity \( D^h_1 (g \tilde{g}) = D^h_1 g E^h_1 \tilde{g} + g D^h_1 \tilde{g} \) yields

\[
(I) = \int_{B'} \tau \sum_{r,s} \sum_{i,k} h \left( \sum_l (D^h_1 a_{il} \partial_l E^h_1 v^r + a_{il} D^h_1 \partial_l v^r) \right)
\]

\[
\times \left( \sum_l (D^h_1 a_{kl} \partial_l E^h_1 v^s + a_{kl} D^h_1 \partial_l v^s) \right) m_{ik}^r(h).
\]

By (3.2) and hypothesis (H5) it follows that

\[
\left| \int_{B'} \tau \sum_{r,s} \sum_{i,k} h \left( \sum_l D^h_1 a_{il} \partial_l E^h_1 v^r \right) \left( \sum_l D^h_1 a_{kl} \partial_l E^h_1 v^s \right) m_{ik}^r(h) \right|
\]

\[
\leq c h \| \nabla E^h_1 v \|_{L^2(B')}^2
\]

and

\[
\left| \int_{B'} \tau \sum_{r,s} \sum_{i,k} h \left( \sum_l D^h_1 a_{il} \partial_l E^h_1 v^r \right) \left( \sum_l a_{kl} D^h_1 \partial_l v^s \right) m_{ik}^r(h) \right|
\]

\[
\leq \frac{c h}{\eta} \| \nabla E^h_1 v \|_{L^2(B')}^2 + \eta h \int_{B'} \tau |D^h_1 \nabla v|^2
\]
for $\eta > 0$. Hypothesis (H6) entails

$$\int_{B_{r}} \tau \sum_{i,k} \sum_{t,s} h \left( \sum_{i} a_{ii} D_{i}^{h} \partial_{i} v^{r} \right) \left( \sum_{i} a_{kk} D_{k}^{h} \partial_{k} v^{s} \right) m_{ik}^{t}(h)$$

$$\geq \frac{k_{0}}{2} \int_{B_{r}} \tau \sum_{i,k} h \left( \sum_{i} a_{ii} D_{i}^{h} \partial_{i} v^{r} \right)^{2}$$

$$= \frac{k_{0}}{2} \int_{B_{r}} \tau h D_{i}^{h} \nabla v^{r} \cdot (A^{T} A) D_{i}^{h} \nabla v^{r}$$

$$\geq \frac{k_{0} \lambda_{0}^{2}}{2} \int_{B_{r}} \tau h \left| D_{i}^{h} \nabla v \right|^{2}.$$

Altogether we obtain

$$(I) \geq c \int_{B_{r}} \tau h \left| D_{i}^{h} \nabla v \right|^{2} - ch$$

for a sufficiently small $\eta > 0$. Further, using Taylor expansion and summation by parts we get

$$(II) = \int_{B_{r}} \tau \sum_{i,k} \sum_{t,s} \zeta_{1k}^{t} F(t, E_{1}^{h} \tilde{v}) - F(t E_{1}^{h} y, E_{1}^{h} \tilde{v}) + \int_{B_{r}} \tau D_{i}^{h} F(t, \tilde{v})$$

$$= \int_{B_{r}} \tau \sum_{k} \zeta_{1k}^{t} F_{t+k} \left( t y + (1-t) E_{1}^{h} y, E_{1}^{h} \tilde{v} \right) d t d y$$

$$+ \int_{B_{r}} \tau D_{i}^{h} \left( t F(t, \tilde{v}) \right) - \int_{B_{r}} D_{i}^{h} \tau F(t E_{1}^{h} y, E_{1}^{h} \tilde{v})$$

$$= (II)_{1} + (II)_{2} + (II)_{3}$$

where $\zeta_{1k}$ denotes the $k$-th component of the basis vector $\zeta^{1}$. Hypotheses (H2) and (H1) entail

$$| (II)_{1} | \leq c \left( \sum \sup_{t \leq t \leq 1} \left\| g_{t} (y + t \zeta^{1}) \right\|_{L^{1}(B_{r})} + \left\| E_{1}^{h} \tilde{v} \right\|_{L^{2}(B_{r})}^{2} \right) \leq c$$

$$(II)_{2} = - h^{-1} \int_{\mathcal{W}_{1}(B_{r})} \tau F(t, \tilde{v})$$

$$(II)_{3} \leq c \int_{B_{r}} \left( \left| E_{1}^{h} g_{0} \right| + \left| E_{1}^{h} \tilde{v} \right| \right)^{2} \leq c.$$
Next, summation by parts yields

\[(VI) = \sum_{i,l} \int_{B'} f_i \cdot \partial_l (\tau a_{il}) D^h_{i} v - \sum_{i,l} \int_{B'} D^h_{i} (\tau a_{il} f_i) \cdot \partial_l E^h_{i} v + \sum_{i,l} \int_{B'} D^h_{i} (\tau a_{il} f_i) \cdot \partial_l v\]

\[= (VI)_1 + (VI)_2 + (VI)_3.\]

Due to (3.2) and hypothesis (H7) we obtain

\[|(VI)_1| \leq c \left( \sum_i \|f_i\|_{L^2(B')}^2 + \|D^h_{i} v\|_{L^2(B')}^2 \right) \leq c\]

\[|(VI)_2| \leq c \left( \sum_i \|f_i\|_{L^2(B')}^2 + \sum_i \|D^h_{i} f_i\|_{L^2(B')}^2 + \|\nabla E^h_{i} v\|_{L^2(B')}^2 \right) \leq c.\]

Applying hypothesis (H1) we get for \(\eta > 0\)

\[|(VI)_3| = \frac{1}{h} \sum_{i,l} \int_{\hat{\Omega}_1^h} \tau a_{il} f_i \cdot \partial_l v \]

\[\leq \frac{c}{\eta h} \left\|\hat{\Omega}_1^h \right\| L^\infty(\hat{\Omega}_1^h) + \frac{\eta}{h} \int_{\hat{\Omega}_1^h} \tau |\nabla v|^2 \]

\[\leq c + \frac{\eta}{c_0^h} \int_{\hat{\Omega}_1^h} \tau F(y, \nabla v).\]  

(3.7)

Let \(\eta = \frac{c_0^h}{2}\). Then (3.6), (3.7), and hypothesis (H1) yield

\[(II)_2 + |(VI)_3| \leq c \frac{1}{2h} \int_{\hat{\Omega}_1^h} \tau F(y, \nabla v) \leq c - \frac{c_0}{2h} \left|\hat{\Omega}_1^h\right| \leq c.\]

Altogether we obtain assertion (3.3) for \(i = 1\). Finally, let us note that the proof of (3.3) for arbitrary \(i \in \Lambda_1\) follows in the same way.

**Proposition 3.2.** There exists a constant \(c\) depending only on \(R_0\) and the data such that

\[\sup_{0 < h < \eta} \int_{B'} \tau h \|D^{-h}_i \nabla v\|^2 \, dy \leq c \quad \text{for } i \in \Lambda_2.\]  

(3.8)

**Proof.** Let \(0 < h < 4R\). We give the proof of (3.8) for some fixed number \(i \in \Lambda_2\), say \(i = 1\).

First, we extend \(v\) into \(\hat{\Omega}_1^{-h}\) by using (2.2), and the functions \(F(\cdot, p), g_0, \tau, a_{ik} (1 \leq i, k \leq n)\) by using (2.1). Now, let us verify that \(\varphi = -\tau D_i^{-h} v\) is an admissible test function. The conditions on \(\zeta^i (i \in \Lambda_2)\) imply that \(y - h\zeta \not\in \hat{\Omega} \cup \partial \hat{\Omega}\) for \(y \in \hat{\Gamma}_D \cap B'\). Hence, the extension (2.2) yield

\[v(y - h\zeta^i) = 0 \quad \text{for } y \in \hat{\Gamma}_D \cap B',\]
thus
\[ \varphi(y) = \tau h^{-1}(v(y - h \zeta^l) - v(y)) = 0 \quad \text{for } y \in \hat{\Gamma}_D \cap B'. \]

Multiplying (3.1) by \( \varphi \) and integrating over \( \hat{\Omega} \) we get
\[
- \int_{B'} \tau f \cdot D_1^{-h} v + \sum_{i,l} \int_{B'} f_i \cdot \partial_l (a_{il} \tau D_1^{-h} v) + \sum_{i,l} \int_{B'} F_i (y, \tilde{v}) \cdot \partial_l (a_{il} \tau) D_1^{-h} v \\
= - \sum_{i,l} \int_{B'} F_i (y, \tilde{v}) \cdot (\tau a_{il}) \partial_l D_1^{-h} v \\
= \sum_{i,l} \int_{B'} \tau F_i (y, \tilde{v}) \cdot [- D_1^{-h} (a_{il} \partial_l v) + D_1^{-h} a_{il} E_1^{-h} \partial_l v] \tag{3.9}
\]

where we have used the identity \( D_1^{-h} (g \tilde{g}) = D_1^{-h} g E_1^{-h} \tilde{g} + g D_1^{-h} \tilde{g} \). The Taylor expansion of \( F(y, \cdot) \) yields
\[
\sum_r \sum_{i,k} (p' - p)_i F_i^r (y, p) \\
= F(y, p') - F(y, p) \\
- \sum_r \sum_{i,k} (p' - p)_i (p' - p)_k \int_0^1 (1 - t) F_{i,k}^r (y, tp' + (1 - t)p) \, dt.
\]

We set
\[
m_{i,k}^r (-h) = \int_0^1 (1 - t) F_{i,k}^r (y, tE_1^{-h} \tilde{v} + (1 - t) \tilde{v}) \, dt
\]

for \( 1 \leq i, k \leq n \) and \( 1 \leq r, s \leq N \). Let us put \( p = \tilde{v} \) and \( p' = E_1^{-h} \tilde{v} \). Then we obtain
\[
- \sum_r \sum_{i,l} F_i^r (y, \tilde{v}) D_1^{-h} (a_{il} \partial_l v') \\
= \frac{1}{h} (F(y, E_1^{-h} \tilde{v}) - F(y, \tilde{v})) \\
- \sum_r \sum_{i,k} h \left( \sum_l D_1^{-h} (a_{il} \partial_l v') \right) \left( \sum_l D_1^{-h} (a_{kl} \partial_l v^*) \right) m_{i,k}^r (-h).
\]

Thus, (3.9) yields
\[
(I) = \int_{B'} \tau h \sum_{r,s} \sum_{i,k} \left( \sum_l D_1^{-h} (a_{il} \partial_l v') \right) \left( \sum_l D_1^{-h} (a_{kl} \partial_l v^*) \right) m_{i,k}^r (-h)
\]
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\[= \int_{B'} \tau h^{-1}(F(y, E_1^{-h}\tilde{\nabla}v) - F(y, \tilde{\nabla}v)) + \sum_{i=1}^{n} \int_{B'} \tau F_i(y, \tilde{\nabla}v) \cdot D_1^{-h} a_{ij} \partial_i E_1^{-h} v \]

\[= \int_{B'} \tau F(y, \tilde{\nabla}v) \cdot \partial_i(\tau h^{-1} D_1^{-h} E_1^{-h} v) + \sum_{i=1}^{n} \int_{B'} \tau F_i(y, \tilde{\nabla}v) \cdot \partial_i(\tau h^{-1} D_1^{-h} E_1^{-h} v) \]

\[= \int_{B'} \tau f \cdot D_1^{-h} v \]

\[= \sum_{i=1}^{n} \int_{B'} \tau f_i \cdot \partial_i(\tau h^{-1} D_1^{-h} E_1^{-h} v) \]

\[= (III) + \ldots + (VI). \]

Hypothesis (H6) entails

(I) \[\geq \frac{k_0}{2} \int_{B'} \tau h D_1^{-h} \tilde{\nabla} v \cdot D_1^{-h} \tilde{\nabla} v = \frac{k_0}{2} \int_{B'} \tau h \sum_r |D_1^{-h}(A \nabla v^r)|^2. \]

We use

\[
\int_{B'} \tau h AD_1^{-h} \nabla v^r \cdot AD_1^{-h} \nabla v^r \geq \lambda_0^2 \int_{B'} \tau h |D_1^{-h} \nabla v^r|^2
\]

\[
\int_{B'} \tau h (D_1^{-h} A) \nabla E_1^{-h} v^r \cdot (D_1^{-h} A) \nabla E_1^{-h} v^r \leq c \int_{B'} \tau h |\nabla E_1^{-h} v^r|^2 \leq c
\]

\[2 \int_{B'} \tau h (D_1^{-h} A) \nabla E_1^{-h} v^r \cdot AD_1^{-h} \nabla v^r \leq \frac{c}{\eta} \int_{B'} \tau h |\nabla E_1^{-h} v^r|^2 + \eta \int_{B'} \tau h |D_1^{-h} \nabla v^r|^2
\]

for \(\eta > 0\). Putting \(\eta = \frac{k_0 \lambda_0^2}{4}\) it follows that

(I) \[\geq \frac{k_0 \lambda_0^2}{4} \int_{B'} \tau h |D_1^{-h} \nabla v|^2 - c. \]

Next,

(II) \[= - \int_{B'} \tau D_1^{-h} F(y, \tilde{\nabla}v) + \int_{B'} \tau h^{-1} (F(y, E_1^{-h} \tilde{\nabla}v) - F(E_1^{-h} y, E_1^{-h} \tilde{\nabla}v)) \]

\[= (II)_1 + (II)_2. \]

Summation by parts entails

(II)_1 \[= - \int_{B' \cup B''} \tau D_1^{-h} F(y, \tilde{\nabla}v) \]

\[= - \int_{B' \cup B''} D_1^{-h} (\tau F(y, \tilde{\nabla}v)) + \int_{B' \cup B''} D_1^{-h} \tau F(E_1^{-h} y, E_1^{-h} \tilde{\nabla}v) \]

\[= (II)_{11} + (II)_{12}. \]
Mixed Boundary Value Problems

where

\[ B'' = \{ y \in B_{R_0} \setminus B' : y = x + h\zeta, x \in B' \}. \]

The extensions (2.1) and (2.2) entail

\[ |(II)_{11}| = \frac{1}{h} \int_{\Omega_{\frac{1}{2}}} \tau F(y, \nabla v) \leq \frac{1}{h} \int_{\Omega_{\frac{1}{2}}} \| g_0 \|_{L^\infty(\Omega_{\frac{1}{2}})} \frac{1}{h} |\nabla \zeta|^2 \leq c. \]

Further, using hypothesis (H1) we obtain

\[ |(II)_{12}| \leq c \int_{B'} |F(E_{-h}y, E_{-h}^2 v)| \leq c \int_{B'} (|E_{-h}^2 g_0| + |E_{-h}^2 \nabla v|^2) \leq c. \]

Let \( \zeta^{1k} \) be the \( k \)-th component of the basis vector \( \zeta^1 \). Hypothesis (H2) and the Taylor expansion entail

\[ |(II)_{22}| \leq c \left( \sum \sup_{0 \leq l \leq 1} \| g_{x_k}(y - th\zeta^1) \|_{L^1(B')} + \| E_{-h}^2 \nabla v \|_{L^2(B')}^2 \right) \]

\[ \leq c. \]

By (3.2) and Hypotheses (H3) and (H7) we get

\[ |(III)| \leq c \left( \sum \| g_i \|_{L^2(B')}^2 + \| \nabla \|_{L^2(B')}^2 + \| \nabla E_{-h}^2 v \|_{L^2(B')}^2 \right) \]

\[ \leq c. \]

\[ |(IV)| \leq c \left( \sum \| g_i \|_{L^2(B')}^2 + \| \nabla \|_{L^2(B')}^2 + \| D_{-h}^2 v \|_{L^2(B')}^2 \right) \]

\[ \leq c. \]

Next,

\[ (VI) = -\sum_{i,l} \int_{B'} f_i \cdot \partial_j(a_{il} \tau) D_{-h}^2 v - \sum_{i,l} \int_{B'} \tau a_{il} f_i \cdot D_{-h}^2 \partial_j v \]

\[ = (VI)_1 + (VI)_2. \]

Due to (3.2) and Hypothesis (H1)

\[ |(VI)_1| \leq c \left( \sum \| f_i \|_{L^2(B')}^2 + \| D_{-h}^2 v \|_{L^2(B')}^2 \right) \leq c \]

follows. Using summation by parts we obtain

\[ (VI)_2 = -\sum_{i,l} \int_{B' \cup B''} \tau a_{il} f_i \cdot D_{-h}^2 \partial_j v \]

\[ = \sum_{i,l} \int_{B' \cup B''} D_{-h}^2 (\tau a_{il} f_i) \partial_j E_{-h}^2 v - \sum_{i,l} \int_{B' \cup B''} D_{-h}^2 (\tau a_{il} f_i \partial_j v) \]

\[ = (VI)_3 + (VI)_4. \]
In view of hypothesis (H7) we get
\[ |(VI)_3| = \sum_{i,t} \int_{B'} (D_1^{-h}(\tau a_{it})f_i + E_1^{-h}(\tau a_{it})D_1^{-h}f_i) \partial_t E_1^{-h}v \]
\[ \leq c \left( \sum_i \| f_i \|_{L^2(B')}^2 + \sum_i \| D_1^{-h}f_i \|_{L^2(B')}^2 + \| \nabla E_1^{-h}v \|_{L^2(B')}^2 \right) \]
\[ \leq c. \]

The extension (2.2) yields \( \partial_v = 0 \) in \( \tilde{\Omega}_1^{-h} \). This implies that
\[ (VI)_4 = \frac{1}{h} \sum_{i,t} \int_{\tilde{\Omega}_1^{-h}} \tau a_{it} f_i \partial_v = 0. \]

Thus, the assertion follows.

**Proposition 3.3.** Let \( \Lambda_3 = \{ n \} \) and \( 0 < \delta < \frac{1}{2} \). Then there exists a constant \( c \) depending only on \( R_0, \delta, \) and the data such that
\[ \sup_{0<\delta<\frac{1}{2}} \int_{B} h^{1+6} |D_1^n \nabla v|^2 dy \leq c. \]  
(3.10)

The proof of this proposition follows as in [8] using (3.1), (3.3), (3.8), and Fourier series.

Now, we are able to prove the main results.

**Proof of Theorem 1.1.** a) Recall that \( \Omega_{\eta} = \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \eta \} \) and note that the basis vectors \( \zeta^i \) fulfil \( \angle(\zeta^i, \zeta^j) \geq \alpha^* \) for \( 1 \leq i < j \leq n \), where the constant \( \alpha^* \) depends only on the geometry of \( \partial \Omega \). It holds that \( \tau \equiv 1 \) in \( B \). Thus, (3.3), (3.8), and (3.10) yield for all \( \delta \in (0, \frac{1}{2}) \)
\[ \sup_{\delta<\delta<\frac{1}{2}} \int_{((\phi^*)^{-1}(B))_n} \frac{|\nabla u(x + z) - \nabla u(x)|^2}{|z|^{1-\delta}} dx \leq c \]  
(3.11)

where the constant \( c \) depends only on the data, \( \delta, \) and on \( R_0 \). Further, let us note that \( R_0 \) depends only on the shape of \( \partial \Omega \).

Next, there are a finite set of points \( \{ \hat{P}_1, \ldots, \hat{P}_k \} \) and a set of balls \( B_{R_i}(\hat{P}_i) \) such that
\[ \partial \Omega \subset \bigcup_{i=1}^{k} (B^i \cap \partial \Omega), \quad \text{where} \quad B^i = (\phi^*)^{-1}(B_{R_i}(\hat{P}_i)), \]
and \( \hat{P}_i \) is the only vertex of \( \partial \tilde{\Omega} \) in \( B_{R_i}(\hat{P}_i) \) or \( B_{R_i}(\hat{P}_i) \cap \partial \tilde{\Omega} \) contains no vertex of \( \partial \tilde{\Omega} \). Further, the radii \( R_i \) \( (1 \leq i \leq k) \) depend only on the data, for they are determined by the geometry of \( \Omega \). Thus,
\[ u \in \mathcal{H}^{3-\frac{3}{2};2}(\Omega)^N \quad \text{for} \quad \delta \in (0, \frac{1}{2}) \]
follows. The imbedding theorem of Nikolskii spaces into Sobolev spaces (cf. [1])
\[ \mathcal{H}^{s,p}(\Omega) \rightarrow W^{3-\epsilon, p}(\Omega) \quad \text{for} \quad \epsilon > 0 \]
entails \( u \in \mathcal{W}^{s,2}(\Omega)^N \) for all \( s < \frac{3}{2} \). This yields assertion (1.1).

b) Using (3.3) and (3.8) we get (3.11) for \( \delta = 0 \). Proceeding as above we obtain
\[ u \in \mathcal{H}^{3,2}(\Omega)^N \]
Proof of Theorem 1.2. We only sketch the proof. Assumption (1.3) yields \( f \in L^q(\Omega) \) and \( f \in L^{2q}(\Omega) \) for some \( q > \frac{\alpha}{2} \). Now, \( N = 1 \) holds. Following [13] we see that \( u \in C^{0,\alpha}(\overline{\Omega}) \) for some \( \alpha > 0 \). Thus, we can proceed as in [8]. The Hölder continuity and the equation yield 
\[
\int_{B_r(y_0) \cap \Omega} \frac{|\nabla u(y)|^2}{|y - y_0|^{n-2+2\varepsilon}} \, dy \leq c
\]
for some \( \varepsilon > 0 \). Replacing the test functions \( \varphi \) by \( r^{-\varepsilon} \varphi \) in Propositions 3.1 and 3.2 and recalling the proof of Proposition 3.3 we get 
\[
\int_{B_r(y_0) \cap \Omega} r^{3-\varepsilon-n} |h^{\frac{1+\varepsilon}{2}} D_i^h \nabla v|^2 \leq c
\]
for \( 1 \leq i \leq n, 0 < r \leq R_0 \) and \( 0 < \delta < \frac{1}{2} \). Applying an imbedding theorem of Morrey-Nikolskii type we obtain the assertion \( \blacksquare \).

4. Examples

In this section we give some explicit examples of the index sets \( \Lambda_1, \Lambda_2, \Lambda_3 \), and the basis vectors \( \zeta^1, \ldots, \zeta^n \).

Let \( \Omega \subset \mathbb{R}^3 \) be a polyhedron. We consider three typical situations: an edge of \( \partial \Omega \) (Example 1), the case when \( \text{angle}(\Gamma_D, \Gamma_N) = \pi \) (Example 2), and a corner point (Example 3).

Let \( P = (0,0,0)^T, B_{R_0} = \{ y : |y| < \frac{1}{2} \} \), and let \( e_k \) \( (1 \leq k \leq 3) \) be the \( k \)-th unit vector in \( \mathbb{R}^3 \).

**Example 1.** Let
\[
\begin{align*}
\Gamma_1 &= \{ y \in B_{R_0} : y_1 = 0, y_3 > 0 \} \\
\Gamma_2 &= \{ y \in B_{R_0} : y_3 = 0, y_1 > 0 \}
\end{align*}
\]
and 
\[\Omega \cap B_1 = \{ y \in B_1 : y_1 > 0, y_3 > 0 \}.\]

**Case 1:** \( \Gamma_D \cap B_{R_0} = \overline{\Gamma_1} \) and \( \Gamma_N \cap B_{R_0} = \overline{\Gamma_2} \). Let us put \( \zeta^1 = e_2 \) and \( \zeta^2 = e_3 \). Then \( \zeta^1 \) and \( \zeta^2 \) are parallel to \( \Gamma_D \cap B_{R_0} \), thus, \( \Lambda_1 = \{1,2\} \). Next, we put \( \Lambda_2 = \{3\} \). We must choose \( \zeta^3 \) such that \( \zeta^3 \) is parallel to \( \Gamma_N \cap B_{R_0} \) and \( \text{angle}(\zeta^3, \Gamma_D \cap B_{R_0}) \geq \alpha^* \) for some suitable large constant \( \alpha^* > 0 \) (i.e., \( \alpha^* \sim \text{angle}(\Gamma_1, \Gamma_2) \)). Thus, let \( \zeta^3 = e_3 \).

**Case 2:** \( \Gamma_D \cap B_{R_0} = \emptyset \) and \( \Gamma_N \cap B_{R_0} = \overline{\Gamma_1} \cup \overline{\Gamma_2} \). It holds that \( \Lambda_1 = \{1,2,3\} \). We must choose \( \zeta^i \) \( (1 \leq i \leq 3) \) such that
i) \( y + s\zeta^i \in \overline{\Omega} \) for \( y \in \partial \Omega \cap B_{R_0} \) and \( 0 < s < R_0 \)
ii) \( \text{angle}(\zeta^i, \zeta^j) \geq \alpha^* \) for \( 1 \leq i < j \leq 3 \) and some suitable constant \( \alpha^* > 0 \).
Thus, let \( \zeta^i = e_i \) for \( 1 \leq i \leq 3 \).

**Case 3:** \( \Gamma_D \cap B_{R_0} = \overline{\Gamma_1} \cup \overline{\Gamma_2} \) and \( \Gamma_N \cap B_{R_0} = \emptyset \). Now, it holds that \( \Lambda_2 = \{1,2,3\} \). The basis vectors \( \zeta^i \) \( (1 \leq i \leq 3) \) must fulfil
i) \( y + s\zeta^i \in \Omega \) for \( y \in \partial \Omega \cap B_{R_0} \) and \( 0 < s < R_0 \)

ii) \( \text{angle}(\zeta^i, \zeta^j) \geq \alpha^* \) for \( 1 \leq i < j \leq 3 \) and \( \alpha^* > 0 \)

iii) \( \text{angle}(\zeta^i, \Gamma_D \cap B_{R_0}) \geq \alpha^* \)

where \( \alpha^* > 0 \) is suitable. Thus, let \( \zeta^1 = \frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_3, \zeta^2 = \frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_3, \) and \( \zeta^2 = \frac{1}{3}e_2 + \frac{2}{3}(e_1 + e_3) \).

**Example 2.** Let

\[
\Omega \cap B_{R_0} = \{ y \in B_{R_0} : y_3 > 0 \}
\]

and

\[
\Gamma_D \cap B_{R_0} = \{ y \in B_{R_0} : y_3 = 0, y_1 \geq 0 \}
\]

\[
\Gamma_N \cap B_{R_0} = \{ y \in B_{R_0} : y_3 = 0, y_1 < 0 \}.
\]

We choose \( \zeta^1 = e_1 \) and \( \zeta^2 = e_2 \). Then \( y + s\zeta^i \in \Gamma_D \cap B_{R_0} \) holds for \( y \in \Gamma_D \cap B_{R_0}, s > 0 \), and \( y + s\zeta^i \in B_{R_0} \). Thus, \( \Lambda_1 = \{1, 2\} \). Further, \( \Lambda_2 = \emptyset \) and \( \Lambda_3 = \{3\} \). Let us put \( \zeta^3 = e_3 \).

**Example 3.** Let \( \Omega = [0, 1]^3 \).

**Case 1:** \( \Gamma_D = \{ y \in \partial \Omega : y_3 = 0 \} \) and \( \Gamma_N = \partial \Omega \setminus \Gamma_D \). The two vectors \( e_1 \) and \( e_2 \) are parallel to \( \Gamma_D \cap B_{R_0} \) and \( e_3 \) is parallel to \( \Gamma_N \cap B_{R_0} \). Thus, let \( \Lambda_1 = \{1, 2\}, \zeta^1 = e_1, \zeta^2 = e_2, \Lambda_2 = \{3\}, \) and \( \zeta^3 = e_3 \).

**Case 2:** \( \Gamma_D = \{ y \in \partial \Omega : y_2 = 0 \vee y_3 = 0 \} \) and \( \Gamma_N = \partial \Omega \setminus \Gamma_D \). Now, \( e_1 \) is parallel to \( \Gamma_D \cap B_{R_0}, \) thus, \( \Lambda_1 = \{1\} \) and \( \zeta^1 = e_1 \). Further, the two vectors \( e_2 \) and \( e_3 \) are parallel to \( \Gamma_N \cap B_{R_0}, \) thus, \( \Lambda_2 = \{2, 3\} \). We must choose \( \zeta^i (i = 2, 3) \) such that

i) \( \text{angle}(\zeta^i, \Gamma_D \cap B_{R_0}) \geq \alpha^* \)

ii) \( \text{angle}(\zeta^2, \zeta^3) \geq \alpha^* \)

for some suitable constant \( \alpha^* > 0 \). Thus, let \( \zeta^2 = \frac{\sqrt{3}}{2}e_2 + \frac{1}{2}e_3 \) and \( \zeta^3 = \frac{1}{2}e_2 + \frac{\sqrt{3}}{2}e_3 \).

**Case 3:** \( \Gamma_D = \emptyset \) and \( \Gamma_N = \partial \Omega \). It holds that \( \Lambda_1 = \{1, 2, 3\} \). Let \( \zeta^i = e_i \) for \( 1 \leq i \leq 3 \).

**Case 4:** \( \Gamma_D = \partial \Omega \) and \( \Gamma_N = \emptyset \). Now, it holds that \( \Lambda_2 = \{1, 2, 3\} \). We choose \( \zeta^i (1 \leq i \leq 3) \) such that

i) \( \text{angle}(\zeta^i, \Gamma_D \cap B_{R_0}) \geq \alpha^* \)

ii) \( \text{angle}(\zeta^i, \zeta^j) \geq \alpha^* \) for \( 1 \leq i < j \leq 3 \) and \( \alpha^* > 0 \)

iii) \( y + s\zeta^i \in \Omega \) for \( y \in \partial \Omega \cap B_{R_0} \) and \( 0 < s < R_0 \)

where \( \alpha^* > 0 \) is suitable.
References


Received 08.07.1998; in revised form 22.03.1999