A New Minimal Point Theorem in Product Spaces

A. Göpfert, Chr. Tammer and C. Zălinescu

Abstract. We derive a minimal point theorem for a subset $A$ in a cone in product spaces under a weak assumption concerning the boundedness of the considered set $A$. Using this result we improve two vectorial variants of Ekeland’s variational principle. Finally, a new characterization of well-based cones is given.

Keywords: Minimal point theorems, Ekeland’s variational principle, well-based cones

AMS subject classification: 49J40, 90C29, 46A40, 90C26

Assume that $(X, d)$ is a complete metric space, $Y$ is a separated locally convex space, $Y^*$ is its topological dual, $K \subset Y$ is a convex cone, i.e. $K + K \subset K$ and $[0, \infty) \cdot K \subset K$,

$$K^+ = \{ y^* \in Y^* : (y, y^*) \geq 0 \text{ for all } y \in K \}$$

is the dual cone of $K$ and

$$K^\# = \{ y^* \in Y^* : (y, y^*) > 0 \text{ for all } y \in K \setminus \{0\} \}.$$

In this note we suppose that $K$ is pointed, i.e. $K \cap (-K) = \{0\}$. The cone $K$ determines an order relation on $Y$, denoted in the sequel by $\leq_K$; so, for $y_1, y_2 \in Y$, $y_1 \leq_K y_2$ if $y_2 - y_1 \in K$. It is well known that "$\leq_K$" is reflexive, transitive and antisymmetric. Let $k^0 \in K \setminus \{0\}$; using the element $k^0$ we introduce an order relation on $X \times Y$, denoted by "$\leq_{k^0}$", in the following manner:

$$(x_1, y_1) \leq_{k^0} (x_2, y_2) \text{ iff } y_1 + k^0 d(x_1, x_2) \leq_K y_2.$$

Note that "$\leq_{k^0}$" is reflexive, transitive and antisymmetric. That is, our notations are those of [3].

The essential idea for the derivation of a minimal point theorem (cf. [2, 8]) in general product spaces $X \times Y$, as well as of the vectorial Ekeland principle, consists in including the ordering cone $K \subset Y$ in a "larger" cone $B \subset Y$: $K \setminus \{0\} \subset \text{int} B$. We will use $B$ to define a suitable functional $z_B : Y \to \mathbb{R}$. Moreover, we will replace the usual boundedness condition of the projection $P_Y A$ of $A$ onto $Y$ by a weaker one.
Theorem 1. Assume that there exists a proper convex cone \( B \subset Y \) such that \( K \setminus \{0\} \subset \text{int} B \). Suppose that the set \( A \subset X \times Y \) satisfies the condition

\[(H1) \text{ for every } \leq_k^0 \text{-decreasing sequence } ((x_n, y_n)) \subset A \text{ with } x_n \to z \in X \text{ there exists } y \in Y \text{ such that } (x, y) \in A \text{ and } (x, y) \leq_k^0 (x_n, y_n) \text{ for every } n \in \mathbb{N} \]

and that \( P_Y(A) \cap (\tilde{y} - \text{int} B) = \emptyset \) for some \( \tilde{y} \in Y \). Then for every \((x_0, y_0) \in A\) there exists \((\bar{x}, \bar{y}) \in A\), minimal with respect to \( \leq_k^0 \), such that \((\bar{x}, \bar{y}) \leq_k^0 (x_0, y_0)\).

Proof. Let \( z_B : Y \to \mathbb{R} \), \( z_B(y) = \inf \{ t \in \mathbb{R} : y \leq t k^0 - \text{cl} B \} \).

By [3: Lemma 7], \( z_B \) is a continuous sublinear function such that \( z_B(y + t k^0) = z_B(y) + t \) for all \( t \in \mathbb{R} \) and \( y \in Y \), and for every \( \lambda \in \mathbb{R} \)

\[
\{ y \in Y : z_B(y) \leq \lambda \} = \lambda k^0 - \text{cl} B \\
\{ y \in Y : z_B(y) < \lambda \} = \lambda k^0 - \text{int} B.
\]

Moreover, if \( y_2 - y_1 \in K \setminus \{0\} \), then \( z_B(y_1) < z_B(y_2) \). Observe that for \((x, y) \in A\) we have that \( z_B(y - \tilde{y}) \geq 0 \). Otherwise for some \((x, y) \in A\) we have \( z_B(y - \tilde{y}) < 0 \). It follows that there exists \( \lambda > 0 \) such that \( y - \tilde{y} \leq -\lambda k^0 - \text{cl} B \). Hence

\[ y \in \tilde{y} - (\lambda k^0 + \text{cl} B) \subset \tilde{y} - (\text{int} B + \text{cl} B) \subset \tilde{y} - \text{int} B \]

which is a contradiction. Since \( 0 \leq z_B(y - \tilde{y}) \leq z_B(y) + z_B(-\tilde{y}) \), it follows that \( z_B \) is bounded from below on \( P_Y(A) \). Let us construct a sequence \(((x_n, y_n))_{n \geq 0} \subset A\) as follows: having \((x_n, y_n) \in A\) we take \((x_{n+1}, y_{n+1}) \in A\), \((x_{n+1}, y_{n+1}) \leq_k^0 (x_n, y_n)\), such that

\[ z_B(y_{n+1}) \leq \inf \{ z_B(y) : (x, y) \in A \text{ and } (x, y) \leq_k^0 (x_n, y_n) \} + \frac{1}{n+1}. \]

Of course, the sequence \(((x_n, y_n))\) is \( \leq_k^0 \)-decreasing. It follows that

\[ y_{n+p} + k^0 d(x_{n+p}, x_n) \leq_k y_n \quad \forall \, n, p \in \mathbb{N}^* \]

so that

\[ d(x_{n+p}, x_n) \leq z_B(y_n) - z_B(y_{n+p}) \leq \frac{1}{n} \quad \forall \, n, p \in \mathbb{N}^*. \]

It follows that \((x_n)\) is a Cauchy sequence in the complete metric space \((X, d)\), and so \((x_n)\) is convergent to some \( \bar{x} \in X \). By condition \((H1)\) there exists \( \bar{y} \in Y \) such that \((\bar{x}, \bar{y}) \in A \) and \((\bar{x}, \bar{y}) \leq_k^0 (x_n, y_n) \) for every \( n \in \mathbb{N} \).

Let us show that \((\bar{x}, \bar{y})\) is the desired element. Indeed, \((\bar{x}, \bar{y}) \leq_k^0 (x_0, y_0) \). Suppose that \((x', y') \in A \) is such that \((x', y') \leq_k^0 (\bar{x}, \bar{y}) \) \((\leq_k^0 (x_n, y_n) \) for every \( n \in \mathbb{N} \)). Thus \( z_B(y') + d(x', \bar{x}) \leq z_B(\bar{y}) \), whence

\[ d(x', \bar{x}) \leq z_B(\bar{y}) - z_B(y') \leq z_B(y_n) - z_B(y') \leq \frac{1}{n} \quad \forall \, n \geq 1. \]

It follows that \( d(x', \bar{x}) = z_B(\bar{y}) - z_B(y') = 0 \). Hence \( x' = \bar{x} \). As \( y' \leq_k \bar{y} \), if \( y' \neq \bar{y} \), then \( \bar{y} - y' \in K \setminus \{0\} \), whence \( z_B(y') < z_B(\bar{y}) \), which is a contradiction. Therefore \((x', y') = (\bar{x}, \bar{y}) \) ■
Comparing with [3: Theorem 4], note that the present condition on $K$ is stronger (because in this case $K \neq \emptyset$), while the condition on $A$ is weaker ($A$ may be not contained in a half-space). Note that when $K$ and $k^0$ are as in Theorem 1, Corollaries 2 and 3 from [3] may be improved. In the next result $Y^* = Y \cup \{\infty\}$ with $\infty \notin Y$; we consider that $y \leq_K \infty$ for every $y \in Y$. We consider also a function $f : X \to Y^*$ and $\text{dom} f = \{x \in X : f(x) \neq \infty\}$.

In the following corollary we derive a variational principle of Ekeland's type for objective functions which take values in a general space $Y$ (cf. [2, 3, 5 - 7]) under a weaker assumption with respect to the usual lower semicontinuity. For the case $Y = \mathbb{R}$, assumption (H4) in Corollary 2 is fulfilled for decreasingly semicontinuous real-valued functions as in the paper [4].

**Corollary 2.** Let $f : X \to Y^*$. Assume that there exists a proper convex cone $B \subset Y$ such that $K \setminus \{0\} \subset \text{int} B$ and $f(X) \cap (\hat{y} - B) = \emptyset$ for some $\hat{y} \in Y$. Also, suppose that

(H3) \{x' \in X : f(x') + k^0 d(x', x) \leq_K f(x)\} is closed for every $x \in X$

or

(H4) for every sequence $(x_n) \subset \text{dom} f$ with $x_n \to x$ and $(f(x_n)) \leq_K \text{-decreasing}, f(x) \leq_K f(x_n)$ for every $n \in \mathbb{N}$, and $K$ is closed in the direction $k^0$.

Then for every $x_0 \in \text{dom} f$ there exists $\bar{x} \in X$ such that

$$f(\bar{x}) + k^0 d(\bar{x}, x_0) \leq_K f(x_0)$$

and

$$\forall x \in X : f(x) + k^0 d(x, x) \leq_K f(\bar{x}) \implies x = \bar{x}.$$  

We say that $K$ is closed in the direction $k^0$ if $K \cap (y - \mathbb{R}^+ k^0)$ is closed for every $y \in K$. The proof of Corollary 2 is similar to those of Corollaries 2 and 3 in [3].

As mentioned in [3], condition (H1) is verified if $K$ is a well based convex cone, $Y$ is a Banach space and $A$ is closed. As usually (cf. [1]), a convex set $S$ is said to be a base for a convex cone $K \subset Y$ if

$$K = \mathbb{R}_+ S = \{\lambda y : \lambda \geq 0 \text{ and } y \in S\} \quad \text{and} \quad 0 \notin \text{cl} S.$$  

The cone $K$ is called well based if $K$ has a bounded base $S$. Concerning well based convex cones in normed spaces we have the following characterization.

**Proposition 3.** Let $Y$ be a normed vector space and $K \subset Y$ a proper convex cone. Then $K$ is well based if and only if there exist $k^0 \in K$ and $z^* \in K^+$ such that $\langle k^0, z^* \rangle > 0$ and

$$K \cap S_1 \subset k^0 + \{y \in Y : \langle y, z^* \rangle > 0\}$$

where $S_1 = \{y \in Y : \|y\| = 1\}$ is the unit sphere in $Y$.

**Proof.** Suppose first that $K$ is well based with bounded base $S$; therefore $0 \notin \text{cl} S$ and $K = [0, \infty) \cdot S$. Then there exists $z^* \in Y^*$ such that $1 \leq \langle y, z^* \rangle$ for all $y \in S$. Consider $\hat{S} := \{k \in K : \langle k, z^* \rangle = 1\}$. It follows that $\hat{S}$ is a base of $K$; moreover, since
\[ S \subset [0, 1] \cdot S, \tilde{S} \text{ is also bounded. Taking } k^1 \in K \setminus \{0\} \text{ we have } K \cap S_1 \subset \lambda k^1 + B_+ \text{ for some } \lambda > 0, \text{ where } B_+ = \{ y \in Y : \langle y, z^* \rangle > 0 \}. \text{ Otherwise}
\]

\[ \forall n \in \mathbb{N}^* \exists k_n \in K \cap S_1 : \quad k_n \notin \frac{1}{n} k^1 + B_+.
\]

Therefore \( (k_n, z^*) \leq \frac{1}{n} (k^1, z^*) \) for every \( n \geq 1 \). But, because \( \tilde{S} \) is a base, \( k_n = \lambda_n b_n \) with \( \lambda_n > 0 \) and \( b_n \in S \); it follows that \( 1 = ||k_n|| = \lambda_n ||b_n|| \leq \lambda_n M \) with \( M > 0 \) (because \( S \) is bounded). Therefore

\[
M^{-1} \leq \lambda_n = \langle \lambda_n b_n, z^* \rangle = \langle k_n, z^* \rangle \leq n^{-1} (k^1, z^*) \quad \forall n \in \mathbb{N}^*
\]

whence \( M^{-1} \leq 0 \), which is a contradiction. Thus there exists \( \lambda > 0 \) such that \( K \cap S_1 \subset \lambda k^1 + B_+ \). Taking \( k^0 := \lambda k^1 \) the conclusion follows.

Suppose now that \( K \cap S_1 \subset k^0 + B_+ \) for some \( k^0 \in K \) and \( z^* \in K^+ \) with \( \langle k^0, z^* \rangle = c > 0 \), where \( B_+ \) is defined as above. Consider \( S = \{ k \in K : \langle k, z^* \rangle = 1 \} \). Let \( k \in K \setminus \{0\} \). Then \( \|k\|^{-1} k = k^0 + y \) for some \( y \in B_+ \). It follows that \( \langle k, z^* \rangle > c \|k\| > 0 \); therefore \( z^* \in K^# \) and so \( k \in (0, \infty) \cdot S \). Since \( \text{cl} S \subset \{ y \in Y : \langle k, z^* \rangle = 1 \} \), we have that \( S \) is a base of \( K \). Let now \( y \in S \subset (0, \infty) \cdot S \). There exists \( z \in B_+ \) such that \( \|y\|^{-1} y = k^0 + z \). We get

\[ 1 = \langle y, z^* \rangle = \|y\| (k^0 + z, z^*) \geq c \|y\|
\]

whence \( \|y\| \leq c^{-1} \). Therefore \( S \) is bounded, and so \( K \) is well-based \( \blacksquare \)

References


Received 21.10.1998