On Some Uniform Convexities and Smoothness in Certain Sequence Spaces

Y. Cui, H. Hudzik and R. Pluciennik

Abstract. It is proved that any Banach space $X$ with property $A_2$ has property $A_2^*$ and that a Banach space $X$ is nearly uniformly smooth if and only if it is nearly uniformly $*$-smooth and weakly sequentially complete. It is shown that if $X$ is a Köthe sequence space the dual of which contains no isomorphic copy of $l_1$ and has property $A_2^*$, then $X$ has the uniform Kadec-Klee property. Criteria for nearly uniformly convexity of Musielak-Orlicz spaces equipped with the Orlicz norm are presented. It is also proved that both properties nearly uniformly smoothness and nearly uniformly convexity for Musielak-Orlicz spaces equipped with the Luxemburg norm coincide with reflexivity. Finally, an interpretation of those results for Nakano spaces $l^{(p_i)} (1 < p_i < \infty)$ is given.

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1. Introduction

Let $(X, \| \cdot \|)$ be a real Banach space and $X^*$ be the dual space of $X$. By $B(X)$ and $S(X)$ we denote the closed unit ball and the unit sphere of $X$, respectively. For any subset $A$ of $X$ by $\text{conv}(A)$ we denote the convex hull of $A$. In the sequel $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_+^*$ stand for the set of natural numbers, the set of reals, the set of non-negative reals and the interval $[0, +\infty)$, respectively.

The following notions used in the paper can be found in [14; Chapter 1].

A sequence $(x_n)$ in a real Banach space $X$ is called a Schauder basis of $X$ (or basis for short) if for each $x \in X$ there exists a unique sequence $(a_n)$ of reals such that

$$\left\| x - \sum_{n=1}^{k} a_n x_n \right\| \to 0 \quad \text{as} \ k \to \infty.$$
A sequence \((x_n)\) which is a Schauder basis of its closed linear span is called a basic sequence.

A basis \((x_n)\) of \(X\) is said to be an unconditional basis if every convergent series \(\sum_{n=1}^{\infty} a_n x_n\) with \(a_n \in \mathbb{R}\) is unconditionally convergent, i.e. for any permutation \((\pi(n))\) of \(\mathbb{N}\) the series \(\sum_{n=1}^{\infty} a_{\pi(n)} x_{\pi(n)}\) converges.

For a basis \((x_n)\) of \(X\), its basic constant is defined by \(K = \sup_{n} \|P_n\|\), where \(P_n : X \to X\) are projections defined by

\[
P_n \left( \sum_{i=1}^{\infty} a_i x_i \right) = \sum_{i=1}^{n} a_i x_i.
\]

If \((x_n)\) is a basis of \(X\) such that the series \(\sum_{n=1}^{\infty} a_n x_n\) converges whenever \((a_n)\) is a sequence of reals such that

\[
\sup_{n} \left\| \sum_{i=1}^{n} a_i x_i \right\| < \infty,
\]

then \((x_n)\) is said to be a boundedly complete basis. It is known that \((x_n)\) is a boundedly complete basis of a Banach space \(X\) if and only if \((x_n)\) is an unconditional basis and \(X\) is weakly sequentially complete.

Recall that \(X\) is said to be weakly sequentially complete if for any sequence \((y_n)\) in \(X\) such that \(\lim_n x^*(y_n)\) exists for every \(x^* \in X^*\), there is \(y \in X\) such that \(y_n \to y\) weakly.

Clarkson [5] introduced the concept of uniform convexity. The norm \(\| \cdot \|\) is called uniformly convex if for each \(\varepsilon > 0\) there is \(\delta > 0\) such that for \(x, y \in S(X)\) the inequality \(\|x - y\| > \varepsilon\) implies \(\|\frac{1}{2}(x + y)\| < 1 - \delta\).

A Banach space \(X\) is said to have the Kadec-Klee property if every sequence from \(S(X)\) weakly convergent to an element \(x \in S(X)\) is convergent to \(x\) in norm. Recall that for a given \(\varepsilon > 0\) a sequence \((x_n)\) is said to be \(\varepsilon\)-separated if

\[
\text{sep}(x_n) = \inf_{m \neq n} \{\|x_n - x_m\|\} > \varepsilon.
\]

A Banach space \(X\) is said to have the uniform Kadec-Klee property if for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \(x\) is a weak limit of an \(\varepsilon\)-separated sequence in \(S(X)\), then \(\|x\| < 1 - \delta\).

The notion of nearly uniformly convexity for Banach spaces was introduced in [11]. It is an infinite dimensional counterpart of the classical uniform convexity. A Banach space is said to be nearly uniformly convex if for every \(\varepsilon > 0\) there exists \(\delta \in (0, 1)\) such that for every sequence \((x_n) \subset B(X)\) with \(\text{sep}(x_n) > \varepsilon\), there holds

\[
\text{conv}(\{x_n\}) \cap (1 - \delta)B(X) \neq \phi.
\]

It is easy to see that every nearly uniformly convex space has the uniform Kadec-Klee property, and every Banach space with the uniform Kadec-Klee property has the Kadec-Klee property. Huff [11] proved that \(X\) is nearly uniformly convex if and only if \(X\) is reflexive and has the uniform Kadec-Klee property.
A Banach space $X$ is said to be nearly uniformly smooth if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each basic sequence $(x_n)$ in $B(X)$ there is $k > 1$ such that

$$\|x_1 + tx_k\| \leq 1 + t\varepsilon$$

for each $t \in [0, \delta]$ (see [17, 18]). Originally, this property was defined in [20] in a different way. Prus [17] showed that a Banach space $X$ is nearly uniformly convex if and only if $X^*$ is nearly uniformly smooth.

For $x \in S(X)$ and a positive number $\delta$, denote

$$S^*(x, \delta) = \{x^* \in B(X^*) : x^*(x) \geq 1 - \delta\}.$$ 

Let $A$ be a bounded subset of $X$. Its Kuratowski measure of non-compactness $\alpha(A)$ is defined as the infimum of all numbers $d > 0$ such that $A$ may be covered by finitely many sets of diameter smaller than $d$ (see [1, 2]).

A Banach space $X$ is said to be nearly uniformly $*$-smooth provided that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in S(X)$, then

$$\alpha(S^*(x, \delta)) \leq \varepsilon.$$ 

A Banach space $X$ is said to have property $A_2$ if there exists $\Theta \in (0,2)$ such that for each weakly null sequence $(x_n)$ in $S(X)$, there are $n_1, n_2 \in \mathbb{N}$ satisfying $\|x_{n_1} + x_{n_2}\| < \Theta$. It is well known that if $X$ has property $A_2$, then it has the weak Banach-Saks property (see [7]).

A Banach space $X$ is said to have property $A_\xi$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each weakly null sequence $(x_n)$ in $B(X)$, there is $k \in \mathbb{N} \setminus \{1\}$ satisfying $\|x_1 + tx_k\| < 1 + t\varepsilon$ whenever $t \in [0, \delta]$. Prus [18] proved that $X$ is nearly uniformly $*$-smooth if and only if $X$ has property $A_\xi$ and contains no copy of $l_1$. Moreover, he also showed that if $X$ is nearly uniformly $*$-smooth, then it has the weak Banach-Saks property.

The space of all real sequences $x = (x(i))$ is denoted by $l^0$. A Banach space $X$ is called a Köthe sequence space if it is a subspace of $l^0$ equipped with a norm $\|\cdot\|$ such that for every $x = (x(i)) \in l^0$ and $y = (y(i)) \in X$ satisfying $|x(i)| \leq |y(i)|$ for all $i \in \mathbb{N}$, there hold $x \in X$ and $\|x\| \leq \|y\|$.

$X$ is said to have the Fatou property, if $0 \leq x_n \uparrow x$ with $x_n \in X$, $x \in l^0$, $\sup_{n \in \mathbb{N}}\|x_n\| < \infty$ imply $x \in X$ and $\lim_{n \to \infty}\|x_n\| = \|x\|$.

We say an element $x \in X$ is order continuous if for any sequence $(x_n)$ in $X$ such that $|x_n(i)| \to 0$ and $|x_n(i)| \leq |x(i)|$ (i.e. $x_n \in X$) we have $\lim_{n \to \infty}\|x_n\| = 0$. It is easy to see that $x$ is order continuous if and only if $\lim_{n \to \infty}\sum_{i=1}^{\infty}x(i)e_i = 0$. The space $X$ is called order continuous if every $x \in X$ is order continuous.

A mapping $\Phi : \mathbb{R} \to \mathbb{R}_+^*$ is said to be an Orlicz function if $\Phi$ is vanishing only at 0, even, convex and left continuous on the whole $\mathbb{R}_+$ (see [13, 16, 19]). An Orlicz function $\Phi$ is said to be an $N$-function if $\lim_{u \to 0} \frac{\Phi(u)}{u} = 0$ and $\lim_{u \to \infty} \frac{\Phi(u)}{u} = \infty$. A sequence $\Phi = (\Phi_i)$ of Orlicz functions is called a Musielak-Orlicz function. By $\Psi = (\Psi_i)$ we denote the complementary function of $\Phi$ in sense of Young, i.e.

$$\Psi_i(u) = \sup\{|v|u - \Phi_i(u) : u \geq 0\} \quad (i \in \mathbb{N}).$$
For a given Musielak-Orlicz function $\Phi$, we define a convex modular

$$I_{\Phi}(x) = \sum_{i=1}^{\infty} \Phi_i(x_i)$$

for any $x \in l^0$. A linear space $l_{\Phi}$ defined by

$$l_{\Phi} = \{ x \in l^0 : I_{\Phi}(cx) < \infty \text{ for some } c > 0 \}$$

is called the Musielak-Orlicz sequence space generated by $\Phi$. We consider $l_{\Phi}$ equipped with the Luxemburg norm

$$\|x\| = \inf \{ \varepsilon > 0 : I_{\Phi}(\frac{x}{\varepsilon}) \leq 1 \}$$

or with the Amemiya-Orlicz norm

$$\|x\|_0 = \inf \{ \frac{1}{k}(1 + I_{\Phi}(kx)) : k > 0 \}.$$

To simplify notations, we assume $l_{\Phi} = (l_{\Phi}, \| \cdot \|)$ and $l^0_{\Phi} = (l_{\Phi}, \| \cdot \|_0)$. Both $l_{\Phi}$ and $l^0_{\Phi}$ are Banach spaces (see [3, 16]).

We say a Musielak-Orlicz function $\Phi$ satisfies the $\delta_2$-condition ($\Phi \in \delta_2$ for short) if there exist constants $k \geq 2$ and $\alpha > 0$ and a sequence $(c_i)$ in $\mathbb{R}_+$ such that $\sum_{i=1}^{\infty} c_i < \infty$ and the inequality

$$\Phi_i(2u) \leq k\Phi_i(u) + c_i$$

holds for every $i \in \mathbb{N}$ and every $u \in \mathbb{R}$ satisfying $\Phi_i(u) \leq \alpha$.

In the sequel $h_{\Phi}$ stands for the space $\{ x \in l^0 : I_{\Phi}(lx) < \infty \text{ for any } l > 0 \}$ equipped with the norm induced from $l_{\Phi}$. To indicate that it is considered with the Orlicz norm, we write $h^0_{\Phi}$.

Let us recall three results which will be used in the following.

**Lemma 1** (see [9]). If $\Phi = (\Phi_i)$ is a Musielak-Orlicz function with all $\Phi_i$ being finitely valued, $\Phi$ satisfies the $\delta_2$-condition and $(x_n)$ is a sequence in $l_{\Phi}$ such that $I_{\Phi}(x_n) \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$ as $n \to \infty$.

**Lemma 2** (see [6]). If a Musielak-Orlicz function $\Psi = (\Psi_i)$ satisfies the $\delta_2$-condition, then for each $\lambda, \varepsilon \in (0,1)$ there is $\theta \in (0,1)$ and a sequence $(h_i)$ in $\mathbb{R}_+$ with $\sum_{i=1}^{\infty} \Phi_i(h_i) = \varepsilon$ such that

$$\Phi_i(\lambda u) \leq \lambda \theta \Phi_i(u)$$

holds for every $i \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $\Phi_i(h_i) \leq \Phi_i(u) \leq 1$.

**Lemma 3** (see [3, 8, 21]). If $\Phi = (\Phi_i)$ is a Musielak-Orlicz function with all $\Phi_i$ being finitely-valued $N$-functions, then for each $x \neq 0$ in $l^0_{\Phi}$ there is $k > 0$ such that

$$\|x\|_0 = \frac{1}{k}(1 + I_{\Phi}(kx)).$$

For more details on Musielak-Orlicz spaces we refer to [3] or [16].
2. Results

We start with some general results which improve the result of Prus [18] that nearly uniformly *-smooth Banach spaces have the weak Banach-Saks property.

Theorem 1. If a Banach space $X$ has property $A_2^*$, then $X$ has property $A_2$.

Proof. For $\varepsilon = \frac{1}{2}$ there is $\delta \in (0,1)$ such that for each weakly null sequence $(x_n)$ in $S(X)$ there is $k > 1$ such that

$$\|x_1 + tx_k\| < 1 + \frac{t}{2} \quad (t \in [0,\delta]).$$

Hence

$$\|x_1 + x_k\| = \|x_1 + \delta x_k + (1-\delta)x_k\| < 1 + \frac{\delta}{2} + (1-\delta) = 2 - \frac{\delta}{2} = \Theta < 2$$

and the statement is proved.

Now we will present the following useful remark.

Remark 1. A Banach space $X$ is reflexive if and only if $X$ contains no isomorphic copy of $l_1$ and $X$ is weakly sequentially complete.

Indeed, since $l_1$ is not reflexive, a reflexive Banach space cannot contain an isomorphic copy of $l_1$. Moreover, any reflexive Banach space $X$ is weakly sequentially complete. If $X$ contains no isomorphic copy of $l_1$, by the well known Rosenthal theorem, for every sequence $(x_n)$ in $B(X)$ there exists a subsequence $(x_{n_k})$ of $(x_n)$ which is a weakly Cauchy sequence. So, if $X$ is additionally weakly sequentially complete, we get that $(x_n)$ is relatively weakly sequentially compact. Hence $X$ is reflexive.

Corollary 1. A Banach space $X$ is nearly uniformly smooth if and only if $X$ is nearly uniformly *-smooth and weakly sequentially complete.

Proof. It is obvious that $X$ is nearly uniformly *-smooth and weakly sequentially complete if it is nearly uniformly smooth. Assume now that $X$ is nearly uniformly *-smooth and weakly sequentially complete. Since nearly uniformly *-smoothness of $X$ implies that $X$ contains no copy of $l_1$, by Remark 1, $X$ is reflexive, whence nearly uniformly *-smoothness coincides with nearly uniformly smoothness.

So, we can now easily understand why $c_0$ is not nearly uniformly smooth although it is nearly uniformly *-smooth.

Theorem 2. Let $X$ be a Köthe sequence space. If $X^*$ contains no isomorphic copy of $l^1$ and has property $A_2^*$, then $X$ has the uniform Kadec-Klee property.

Proof. Since $X^*$ contains no isomorphic copy of $l^1$, for every sequence $(x_n^*)$ in $B(X^*)$ there is a weak Cauchy subsequence $(x_{n_k}^*)$. It is obvious that the sequence $(x_{n_k}^* - x_{n_i}^*)$ is weakly null. By the assumption that $X^*$ has property $A_2^*$, there are $n > k > 1$ such that

$$\|x_1^* + tx_k^* - x_n^*\| < 1 + \frac{t\varepsilon}{32} \quad (t \in [0,\delta]).$$
Let \((x_n)\) be a sequence in \(S(X)\) with \(\text{sep}(x_n) > \varepsilon\) and \(x_n \to x \in X\) weakly. Then \(\text{sep}(x_n - x) > \varepsilon\). We need to show that \(\|x\| < 1 - \eta(\varepsilon)\), where \(\eta(\varepsilon)\) depends only on \(\varepsilon\). Let \(K = \frac{32 + 26\varepsilon}{32 + 8\varepsilon}\). By the Bessaga-Pelczyński selection principle, there exists a subsequence \((z_n)\) of \(\{x_n - x, x : n \in \mathbb{N}\}\) with \(z_1 = x\) being a basic sequence with basic constant less or equal to \(K\). Let \(X_0 = \overline{\text{sp}}\{z_n : n \in \mathbb{N}\}\). Let us consider the sequence \((z^*_n)\) of the norm preserving extensions from \(X_0\) to the whole \(X\) of the coefficient functionals for the basic sequence \((z_n)\). Then \(\langle z, z^*_n \rangle \to 0\) as \(n \to \infty\) for any \(z \in X_0\). Indeed, \(x = \sum_{i=1}^{\infty} z^*_i(z)z_i\) for any \(z \in X_0\), whence

\[
\|z^*_n(z)z_n\| = \left\| \sum_{i=n}^{\infty} z^*_i(z)z_i - \sum_{i=n+1}^{\infty} z^*_i(z)z_i \right\|
\leq \left\| \sum_{i=n}^{\infty} z^*_i(z)z_i \right\| + \left\| \sum_{i=n+1}^{\infty} z^*_i(z)z_i \right\|
\to 0 \quad \text{as} \quad n \to \infty.
\]

Since \(\|z_n\| > \frac{\varepsilon}{2}\) for all \(n\), this yields \(z^*_n(z) \to 0\) as \(n \to \infty\).

Let us write \((x, z^*_k)\) for \(z^*_k(x)\) and take \(n > k > 1\) large enough such that \(|\langle x, z^*_k \rangle| < \frac{\varepsilon}{32}\) and \(|\langle x, z^*_n \rangle| < \frac{\varepsilon}{32}\). Notice that \(\|z^*_i\| \leq K\) and \(\|z^*_k\| \leq 2K\) for \(k > 1\). Hence, taking into account that \(\|x + z_k\| = 1\) for \(k > 1\) and applying property \(A_2\) for \(X^*\), we get

\[
\left\| z^*_1 + \frac{\delta}{2} (z^*_k - z^*_n) \right\| \leq K \left( 1 + \frac{\delta \varepsilon}{32} \right)
\]

and consequently

\[
\|x\| = \langle x, z^*_1 \rangle
= \langle x + z_k, z^*_1 \rangle + \frac{\delta}{2} \langle x + z_k, z^*_k - z^*_n \rangle - \frac{\delta}{2} \langle x + z_k, z^*_k - z^*_n \rangle
= \langle x + z_k, z^*_1 + \frac{\delta}{2} (z^*_k - z^*_n) \rangle - \frac{\delta}{2} \langle x + z_k, z^*_k - z^*_n \rangle
\leq \left\| z^*_1 + \frac{\delta}{2} (z^*_k - z^*_n) \right\| - \frac{\delta}{2} \|z_k\| + |\langle x, z^*_k \rangle| + |\langle x, z^*_n \rangle|
\leq K \left( 1 + \frac{\delta \varepsilon}{32} \right) - \frac{\delta \varepsilon}{4} + \frac{\delta \varepsilon}{16}
= K \left( 1 + \frac{\delta \varepsilon}{32} \right) - \frac{3\delta \varepsilon}{16}
\leq \left( 1 + \frac{\delta \varepsilon}{16} \right) - \frac{3\delta \varepsilon}{16}
= 1 - \frac{\delta \varepsilon}{8}
\]

which finishes the proof \(\blacksquare\).
Lemma 4. Let $\Phi = (\Phi_i)$ be a finitely-valued Musielak-Orlicz function such that $\Phi^*$ satisfies the $\delta_2$-condition. Then for every $\varepsilon > 0$, $\lambda \in (0,1)$ and $K \geq 1$ there are $(h_i)_{i=1}^\infty \subset \mathbb{R}_+$ and $\theta \in (0,1)$ such that $\sum_{i=1}^\infty \Phi_i(h_i) < \varepsilon$ and the inequality

$$\Phi_i(\gamma u) \leq \gamma \theta \Phi_i(u)$$

holds for all $i \in \mathbb{N}$ and $u \geq 0$ satisfying the inequalities $\Phi_i(h_i) \leq \Phi_i(u) \leq K$ and all $\gamma \in (0,\lambda]$.

**Proof.** It is known from [6] that our lemma is true for $K = 1$ under the additional assumption that $\Phi_i(1) = 1$ for all $i \in \mathbb{N}$. Let $a_i > 0$ be such that $\Phi_i(a_i) = K$ for all $i \in \mathbb{N}$ and define $\phi_i(u) = \frac{1}{K} \Phi_i(a_i u)$ for all $u \in \mathbb{R}$ and $i \in \mathbb{N}$. Then $\phi = (\phi_i)$ is a Musielak-Orlicz function such that $\phi_i(1) = 1$ for all $i \in \mathbb{N}$. Since, denoting by $\phi^*_i$ and $\Phi^*_i$ the complementary functions of $\phi_i$ and $\Phi_i$, respectively, there holds

$$\phi^*_i(u) = \frac{1}{K} \Phi^*_i \left( \frac{K}{a_i} u \right)$$

for all $i \in \mathbb{N}$ and $u \in \mathbb{R}$, we know that $\phi^*$ satisfies the $\delta_2$-condition. By the above mentioned result from [6] there are $(h^*_i)_{i=1}^\infty \subset \mathbb{R}_+$ and $\theta \in (0,1)$ such that

$$\sum_{i=1}^\infty \phi_i(h^*_i) < \varepsilon$$

and

$$\phi_i(\gamma u) \leq \gamma \theta \phi_i(u)$$

for all $\gamma \in (0,\lambda]$, $i \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $\phi_i(h^*_i) \leq \phi_i(u) \leq 1$. Setting $h_i = a_i h^*_i$ for each $i \in \mathbb{N}$, we easily see that it is just the desired result.\[\square\]

**Theorem 3.** If $\Phi = (\Phi_i)$ is a Musielak-Orlicz function with all $\Phi_i$ being finitely-valued $N$-functions, then $l^\theta_{\Phi}$ is nearly uniformly convex if and only if $\Phi$ and $\Psi$ satisfy the $\delta_2$-condition.

**Proof.** We need only to prove the sufficiency. Since $l^\theta_{\Phi}$ is reflexive, it suffices to prove that $l^\theta_{\Phi}$ has the uniform Kadec-Klee property. Let $\varepsilon > 0$ be given and take any sequence $\{x_n\} \subset S(l^\theta_{\Phi})$ with $\text{sep}(x_n) > 2\varepsilon$ and $x_n \xrightarrow{w} x$. It is clear that for any $m \in \mathbb{N}$ there is $n_m \in \mathbb{N}$ such that

$$\text{sep} \left( \left( \sum_{i=m+1}^\infty x_n(i) e_i \right)_{n=n_m}^\infty \right) > 2\varepsilon.$$

This follows by the fact that $x_n \xrightarrow{w} x$ implies that $x_n \rightarrow x$ coordinatewise. Hence for any $m \in \mathbb{N}$ there is $n_m \in \mathbb{N}$ such that

$$\left\| \sum_{i=m+1}^\infty x_n(i) e_i \right\|_0 \geq \varepsilon \quad (n \geq n_m). \quad (1)$$

By Lemma 3 there are $k_n \geq 1$ and $k \geq 1$ such that

$$\|x_n\|_0 = \frac{1}{k_n} (1 + I_{\Phi}(k_n x_n)) \quad (n \in \mathbb{N})$$
and
\[\|x\|_0 = \frac{1}{k}(1 + I_\Phi(kx)).\]

Then \( K = \sup_n k_n < \infty \). Indeed, since \( \|x\|_0 > 1 - \delta \), there is \( i_0 \in \mathbb{N} \) such that \( x_0(i_0) \neq 0 \).

If \( K = \infty \), we can assume without loss of generality that \( \lim_{n \to \infty} k_n = \infty \). Hence
\[
1 = \frac{1}{k_n}(1 + I_\Phi(k_n x_n)) = \lim_{n \to \infty} \frac{1}{k_n} I_\Phi(k_n x_n) \\
\geq \liminf_{n \to \infty} \frac{1}{k_n} \Phi_{i_0}(k_n x_n(i_0)) \\
\rightarrow \infty
\]

which is a contradiction.

By the \( \delta_2 \)-condition of \( \Phi \) and inequality (1) there is \( \delta > 0 \) such that
\[
\sum_{i=m+1}^{\infty} \Phi_i(x_n(i)) \geq \delta \quad (n \geq n_m).
\]

(2)

Put \( \lambda = \frac{K}{K + 1} \). Then, by Lemma 4, there is \( h = (h_i)_{i=1}^{\infty} \) with \( \sum_{i=1}^{\infty} \Phi_i(h_i) \leq \frac{K}{2} \) and a number \( \theta \in (0, 1) \) such that
\[
\Phi_i(\gamma u) \leq \gamma(1 - \theta) \Phi_i(u)
\]
for all \( \gamma \in [0, \lambda] \) and \( u \in \mathbb{R} \) satisfying \( \Phi_i(h_i) \leq \Phi_i(u) \leq K \). Take \( m \) large enough such that
\[
\left\| \sum_{i=m+1}^{\infty} x(i) e_i \right\|_0 < \frac{\delta \theta}{8}
\]

(3)

and
\[
\left\| \sum_{i=m+1}^{\infty} h_i e_i \right\|_0 < \frac{\delta \theta}{8}.
\]

(4)

Since \( \frac{k}{k+n} \leq \frac{k}{k+n+1} \leq \frac{K}{K+n+1} \) for any \( n \in \mathbb{N} \), we have
\[
\Phi_i \left( \frac{k k_n}{k + k_n} x_n(i) \right) \leq \frac{1 - \theta}{k + k_n} k \Phi_i(k_n x_n(i))
\]
whenever \( |x_n(i)| \geq h_i \). Therefore,
\[
\sum_{i=m+1}^{\infty} \Phi_i \left( \frac{k k_n}{k + k_n} x_n(i) \right) \leq \sum_{i=1}^{\infty} \Phi_i(h_i) + \frac{1 - \theta}{k + k_n} k \sum_{i=1}^{\infty} \Phi_i(k_n x_n(i)).
\]

(5)
It is obvious that

\[
\|x_n + x\|_0 = \left\| \sum_{i=1}^{m} x(i)e_i + \sum_{i=m+1}^{\infty} x(i)e_i + x_n \right\|_0 \leq \left\| \sum_{i=1}^{m} x(i)e_i + x_n \right\|_0 + \left\| \sum_{i=m+1}^{\infty} x(i)e_i \right\|_0
\]

(6)

for \(m\) large enough. Moreover, by (3) - (5), we get for \(n \geq n_m\)

\[
\left\| \sum_{i=1}^{m} x(i)e_i + x_n \right\|_0 \leq \frac{k_n + k}{k_n k} \left(1 + \sum_{i=1}^{m} \Phi_i \left(\frac{kk_n}{k_n + k} x(i) + x_n(i)\right)\right) + \sum_{i=m+1}^{\infty} \Phi_i \left(\frac{kk_n}{k_n + k} x_n(i)\right) + \frac{1 - \theta}{k_n + k} \sum_{i=m+1}^{\infty} \Phi_i (h_i) \\
\leq \frac{1}{k} + \frac{1}{k_n} + \frac{1}{k} \sum_{i=1}^{m} \Phi_i (kx(i)) + \frac{1}{k_n} \sum_{i=1}^{m} \Phi_i (k_n x_n(i)) + \frac{1}{k_n} \sum_{i=m+1}^{\infty} \Phi_i (k_n x_n(i)) + \sum_{i=m+1}^{\infty} \Phi_i (h_i) - \frac{1}{k_n} \sum_{i=m+1}^{\infty} \Phi_i (k_n x_n(i)) \\
\leq \frac{1}{k} \left(1 + \sum_{i=1}^{m} \Phi_i (kx(i))\right) + \frac{1}{k_n} \left(1 + I\Phi(k_n x_n)\right) + \sum_{i=m+1}^{\infty} \Phi_i (h_i) - \frac{1}{k_n} \sum_{i=m+1}^{\infty} \Phi_i (k_n x_n(i)) \\
\leq 2 + \frac{\delta \theta}{8} - \delta \theta.
\]

Therefore, combining (6) and (7), we obtain

\[
\|x_n + x\|_0 \leq 2 + \frac{\delta \theta}{8} - \delta \theta + \frac{\delta \theta}{8} = 2 - \frac{3}{4} \theta \quad (n > n_m).
\]
Hence, by $x_n \to x$ and the lower semicontinuity of the norm with respect to the weak topology, we deduce that

$$\|x\|_0 \leq \lim_{n \to \infty} \left\| \frac{x_n + x}{2} \right\|_0 \leq \frac{1}{2} \left( 2 - \frac{3}{4} \theta \right) = 1 - \frac{3}{8} \theta.$$  

This contradiction finishes the proof.

**Theorem 4.** For any Musielak-Orlicz function $\Phi = (\Phi_i)$ with all $\Phi_i$ being finitely-valued $N$-functions the following statements are equivalent:

(a) $l_\Phi$ is nearly uniformly smooth.

(b) $l_\Phi$ is nearly uniformly $\ast$-smooth.

(c) $\Phi$ and $\Psi$ satisfy the $\delta_2$-condition.

**Proof.** (c) $\Rightarrow$ (a): By Theorem 3, $l_\Phi^0$ is nearly uniformly convex, so its dual $l_\Phi$ is nearly uniformly smooth. Therefore, we need only to prove that (b) $\Rightarrow$ (c). We will show that (b) implies the $\delta_2$-condition for $\Phi$. If $\Phi$ does not satisfy the $\delta_2$-condition, we can construct $x \in S(l_\Phi)$ such that $I_\Phi(x) \leq 1$ and $I_\Phi((1 + \frac{1}{n})x) = \infty$ for every $n \in \mathbb{N}$ (see [12]). Take a sequence $(i_k)$ of natural numbers such that $i_k \uparrow$ and

$$\sum_{i=i_k+1}^{i_k+1} \Phi_i \left( \left( 1 + \frac{1}{k} \right) x(i) \right) \geq 1 \quad (k \in \mathbb{N}).$$

Put

$$x_k = \left( 0, 0, \ldots, 0, x(i_k + 1), x(i_k + 2), \ldots, x(i_k + 1), 0, 0, \ldots \right) \quad (k \in \mathbb{N}).$$

Then it is obvious that

$$\frac{k}{k+1} \leq \|x_k\|_0 \leq 1 \quad (k \in \mathbb{N}).$$

Moreover,

$$x_k \to 0 \quad \text{weakly.} \tag{8}$$

Indeed, for every $y \in (l_\Phi)^*$ we have $y^* = y_0^* + y_1^*$ uniquely, where $y_0^*$ is the regular part of $y^*$ and $y_1^*$ is the singular part of $y^*$, i.e. $y_1^*(x) = 0$ for any $x \in h_\Phi$ (see [10]). The functional $y_0^*$ is generated by some $y_0 \in l_\Psi$ by the formula

$$y_0^*(x) = \langle x, y_0 \rangle = \sum_{i=1}^{\infty} x(i)y_0(i) \quad (x \in l_\Phi).$$

Let $\lambda > 0$ be such that $\sum_{i=1}^{\infty} \Psi_i(\lambda y_0(i)) < \infty$. Since $x_k \in h_\Phi$ for any $k \in \mathbb{N}$, we have

$$\langle x_k, y^* \rangle = \langle x_k, y_0^* \rangle$$

$$= \sum_{i=i_k+1}^{i_k+1} x(i)y_0(i)$$

$$\leq \frac{1}{\lambda} \left( \sum_{i=i_k+1}^{i_k+1} \Phi_i(x(i)) + \sum_{i=i_k+1}^{i_k+1} \Psi_i(\lambda y_0(i)) \right)$$

$$\to 0 \quad \text{as } k \to \infty,$$
i.e. (8) holds.

Since the space $l_\Phi$ is nearly uniformly $*$-smooth, it has property $A_2^\varepsilon$, i.e. for any $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that for each weakly null sequence $(z_n)$ in $B(l_\Phi)$ there is $m > 1$ such that

$$\|z_1 + tz_m\| \leq 1 + t\varepsilon$$

whenever $t \in [0,\delta]$ (see [18]). Take $k_0 \in \mathbb{N}$ such that $\frac{2}{k+1} < (1-\varepsilon)\delta$ if $k \geq k_0$. We have for $k \geq k_0$

$$1 + \delta \varepsilon \geq \|x + \delta x_k\| \geq \|(1 + \delta)x_k\| \geq (1 + \delta)\frac{k}{k+1}$$

$$= (1 + \delta) \left(1 - \frac{1}{k+1}\right) > 1 + \delta - \frac{2}{k+1}$$

whence $\frac{2}{k+1} > (1-\varepsilon)\delta$. This is a contradiction which finishes the proof of the fact that (b) implies the $\delta_2$-condition for $\Phi$.

Next, we will show that (b) implies the $\delta_2$-condition for $\Psi$. By the above part of the proof, we can assume that $l_\Phi$ is nearly uniformly $*$-smooth and $\Phi$ satisfies the $\delta_2$-condition. So, $l_\Phi$ is order continuous. Moreover, any Musielak-Orlicz space $l_\Phi$ has the Fatou property and consequently, it is weakly sequentially complete. So, in view of Corollary 1, $l_\Phi$ is nearly uniformly smooth and consequently reflexive. This yields the $\delta_2$-condition for $\Psi$.

**Theorem 5.** Let $\Phi = (\Phi_i)$ be a Musielak-Orlicz function with all $\Phi_i$ being finitely-valued $N$-functions. Then $\Phi$ and $\Psi$ satisfy the $\delta_2$-condition whenever $l_\Phi^0$ is nearly uniformly $*$-smooth.

**Proof.** Since $l_\Phi^0$ is nearly uniformly $*$-smooth, it has property $A_2^\varepsilon$, i.e. for any $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that for each weakly null sequence $(z_n)$ in $B(l_\Phi^0)$ there is $m \in \mathbb{N} \setminus \{1\}$ such that

$$\|z_1 + tz_m\|_0 \leq 1 + t\varepsilon$$

for all $t \in [0,\delta]$. Let $\theta \in (0,1)$ be such that $1 + \delta \varepsilon < (1 + \delta)\theta$. If $\Phi$ does not satisfy the $\delta_2$-condition, then there exists $x \in S(l_\Phi^0)$ and a sequence $\{n_i\}$ of natural numbers $n_i \uparrow$ such that $n_1 = 1$ and

$$\left\|\sum_{i=n_k}^{n_{k+1}} x(i)\right\|_0 \geq \theta \quad (k \in \mathbb{N})$$

(see [4]). Define

$$x_k = \sum_{i=n_k}^{n_{k+1}} x(i) \quad (k \in \mathbb{N}).$$

Then we can prove in the same way as for the Luxemburg norm (see the proof of Theorem 4) that $(x_k)$ is a weakly null sequence. Therefore, there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$1 + \delta \varepsilon \geq \|x + \delta x_k\|_0 \geq \|(1 + \delta)x_k\|_0 \geq (1 + \delta)\theta.$$

This is a contradiction which shows the necessity of the $\delta_2$-condition of $\Phi$ for the nearly uniformly $*$-smoothness of $l_\Phi^0$.
The necessity of the $\delta_2$-condition of $\Psi$ can be proved in the same way as for the Luxemburg norm in Theorem 4, since the Amemiya-Orlicz norm has the Fatou property.

Recall that the Nakano space $l^{(p_i)}$ is the Musielak-Orlicz space $l_\Phi$ with $\Phi = (\Phi_i)$, where $\Phi_i(u) = |u|^{p_i} \quad (1 < p_i < +\infty, i \in \mathbb{N})$.

**Corollary 2.** For both the Luxemburg and the Amemiya-Orlicz norms the following statements are equivalent:

(a) $l^{(p_i)}$ is nearly uniformly convex.
(b) $l^{(p_i)}$ is nearly uniformly smooth.
(c) $l^{(p_i)}$ is nearly uniformly $*$-smooth.
(d) $1 < \liminf_{i \to \infty} p_i \leq \limsup_{i \to \infty} p_i < +\infty$.

**Proof.** If $\Phi_i(u) = |u|^{p_i}$ for all $u \in \mathbb{R}$ and $i \in \mathbb{N}$, then the complementary functions $\Psi_i$ of $\Phi_i$ are defined by the formula

$$\Psi_i(u) = c_i |u|^{q_i}$$

where $\frac{1}{p_i} + \frac{1}{q_i} = 1$ and $c_i = (p_i)^{1/p_i} (q_i)^{1/q_i}$ for all $i \in \mathbb{N}$. It is easy to see that $\Phi = (\Phi_i)$ satisfies the $\delta_2$-condition if and only if $\limsup_{i \to \infty} p_i < +\infty$. Moreover, $\Psi = (\Psi_i)$ satisfies the $\delta_2$-condition if and only if $\liminf_{i \to \infty} p_i > 1$.

Now, we prove the equivalence of the conditions.

(d) $\Rightarrow$ (a): Assume first that $l^{(p_i)}$ is equipped with the Amemiya-Orlicz norm. Then, by Theorem 4, $l_\Psi$ is nearly uniformly smooth. So $l^{(p_i)}$ is nearly uniformly convex as well. It follows in the same way that condition (d) implies that $l_\Psi$ is nearly uniformly convex. Therefore, by the fact that a Banach space $X$ is nearly uniformly convex if and only if $X^*$ is nearly uniformly smooth and that if both Musielak-Orlicz functions $\Phi$ and $\Psi$ satisfy the $\delta_2$-condition, then $(l_\Phi)^* \cong l_\Psi^0$ and $(l_\Psi)^* \cong l_\Psi$ (see [3, 15, 16, 19]), we deduce that (a) and (b) are equivalent for both norms. By Theorem 4, conditions (b), (c) and (d) are pairwise equivalent. The implication (b) $\Rightarrow$ (c) holds in general and, by Theorem 5, (c) $\Rightarrow$ (d) in the case of the Amemiya-Orlicz norm. This completes the proof.

**References**


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