Strong Duality for Transportation Flow Problems

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Abstract. This paper is a supplement and correction to the author's article "Optimal transportation flows" [2]. By new methods the existence of optimal transportation flows and the strong duality to deposit problems is proved.

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1. Introduction

In conformity with [2] we consider the following transportation flow problem:

$$K(\mu) := \int_{\Omega} r(x, d\mu(x)) \to \min \quad \text{on } Y$$

where

$$Y := \{\mu \in L^\infty(\Omega)^* \mid \langle \nabla \sigma, \mu \rangle = K_D(\sigma) \quad \forall \sigma \in W^{1,n}_\infty(\Omega)\}$$

and

$$K_D(\sigma) := \int_{\Omega} \sigma(x)^T d\alpha(x) \quad \text{on } W^{1,n}_\infty(\Omega).$$

We assume, $\Omega$ is a bounded strongly Lipschitz domain of $E^m$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a given vector of finite Borel measures $\alpha_k$ on the $\sigma$-algebra $\mathcal{B}$ of all Lebesgue-measurable subsets of $\mathcal{B}$ which satisfy the assumption

$$\int_{\Omega} d\alpha_k = 0 \quad (k = 1, \ldots, n);$$

$r$ is a given local cost rate on $\Omega \times E^{mn}$ with the following basic properties:

$$r(\cdot, v) \text{ is summable on } \Omega$$

$$r(x, \cdot) \text{ is positive homogeneous of degree one and convex on } E^{mn} \forall x \in \Omega$$

$$\gamma_1 |v| \leq r(x, v) \leq \gamma_2 |v| \quad (v \in E^{mn}, x \in \Omega)$$

for some constants $\gamma_1, \gamma_2 > 0$. 

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The objective functional of (1) is defined by

$$\int_{\Omega} r(x, d\mu(x)) := \sup_u \left\{ (u, \mu) \mid u \in L^m_{\infty}(\Omega), u^T(x)v \leq r(x, v) \forall v \in E^{mn} \right\}. \quad (6)$$

Every element $\mu = (\mu_1, \ldots, \mu_n) \in Y$ is said to be a feasible flow and $\mu_k$ the flow of the $k$-th transportation good.

Referring to [2], between the transportation flow problem (1) and the deposit problem

$$K_D(S) = \int_{\Omega} S(x)^T d\alpha(x) \longrightarrow \max \text{ on } \mathcal{S}'.$$  

there exists duality, i.e.

$$K(\mu) \geq K_D(S) \quad \forall \mu \in Y, \ S \in \mathcal{S}',$$  

if we define $\mathcal{S}'$ by

$$\mathcal{S}' := \left\{ S \in W^{1,m}_{\infty}(\Omega) \mid \nabla S(x) \in \mathcal{F}(x) \text{ for a.e. } x \in \Omega \right\} \quad (9)$$

with

$$\mathcal{F}(x) := \left\{ z \in E^{mn} \mid z^T v \leq r(x, v) \forall v \in E^{mn} \right\}. \quad (10)$$

The restrictions of (9) characterize slope restrictions in the sense that $\nabla S(x)$ belongs to the convex figuratrix set $\mathcal{F}(x)$ for a.e. $x \in \Omega$.

Since (4), the linear functional $K_D$ has the property $K_D(S) = K_D(S + C)$ for any constant vector $C \in E^n$. Therefore, without loss of generality we can reduce the deposit problem (7) on the restricted class $\mathcal{S} := \{ S \in \mathcal{S} \mid S(\hat{x}) = 0 \}$ where $\hat{x}$ is an arbitrary fixed point in $\hat{\Omega}$.

We know from [2] the following theorem.

**Theorem 1.** The deposit problem (7) has an optimal solution $S_0$.

### 2. The existence of optimal flows

In $L^m_{\infty}(\Omega)^*$ the standardized norm is defined by

$$\|\mu\| := \sup_u \left\{ \langle u, \mu \rangle \mid u \in L^m_{\infty}(\Omega), |u(x)| \leq 1 \text{ a.e. on } \Omega \right\}. \quad (11)$$

We introduce in this Banach space an equivalent norm by

$$\|\mu^*\| := \sup_u \left\{ \langle u, \mu \rangle \mid u \in L^m_{\infty}(\Omega), u(x) \in \mathcal{F}(x) \text{ a.e. on } \Omega \right\}. \quad (12)$$
The equivalence of both norms is obvious under consideration of the third property of assumption (5):

\[
\sup_{u} \left\{ (u, \mu) \left| u \in L_{\infty}^{m,n}(\Omega), u(x)^{T}v \leq \gamma_{1}|v| \forall v \in E^{m,n}, \text{a.e. on } \Omega \right\} 
\leq \sup_{u} \left\{ (u, \mu) \left| u \in L_{\infty}^{m,n}(\Omega), u(x)^{T}v \leq r(x,v) \forall v \in E^{m,n}, \text{a.e. on } \Omega \right\} 
\leq \sup_{u} \left\{ (u, \mu) \left| u \in L_{\infty}^{m,n}(\Omega), u(x)^{T}v \leq \gamma_{2}|v| \forall v \in E^{m,n}, \text{a.e. on } \Omega \right\},
\]

and this means \( \gamma_{1}\|\mu\| \leq \|\mu\|^{*} \leq \gamma_{2}\|\mu\| \) thus equivalence of both norms.

Now, let \( \mathcal{S}_{0} = \{ \sigma \in W_{0}^{1,n}(\Omega) | \sigma(\bar{x}) = 0 \} \) and \( U \) be a subspace of \( L_{\infty}^{m,n}(\Omega) \), characterized by

\[
U := \left\{ u \in L_{\infty}^{m,n}(\Omega) \left| u = \nabla \sigma, \sigma \in \mathcal{S}_{0} \right. \right\}.
\]

In virtue of Sóbolev's embedding theorems [3: p. 60], the mapping \( f : U \to \mathbb{R} \) is a linear continuous functional \( \mu_{0} \) on \( U \), if we define \( f(\nabla \sigma) := K_{D}(\sigma) \) for all \( \sigma \in \mathcal{S}_{0} \). Namely, there is a constant \( M > 0 \) such that for every \( \sigma \) of this type

\[
\|\sigma\|_{C^{n}(\Omega)} \leq M \text{esssup}_{\Omega} |\nabla \sigma|
\]

holds and therefore

\[
|f(\nabla \sigma)| = |K_{D}(\sigma)| \leq M \int_{\Omega} d|\alpha| \|\nabla \sigma\|_{L_{\infty}^{m,n}(\Omega)}.
\]

(14)

The linearity of \( f \) is obvious. Together with the boundedness (14) of \( f \) it follows that \( f \) is a linear continuous functional \( \mu_{0} \) on \( U \). By the Hahn-Banach extension theorem [1: p. 109] we can extend \( \mu_{0} \) as a continuous linear functional on the whole space \( L_{\infty}^{m,n}(\Omega) \) with the same norm. That means, for each \( u \in U \) there is uniquely a \( \sigma \in \mathcal{S}_{0} \) such that \( u = \nabla \sigma \),

\[
f(u) = K_{D}(\sigma) = (\nabla \sigma, \mu_{0}),
\]

(15)

and, with (12),

\[
\|\mu_{0}\|^{*} = \sup_{\Omega} \left\{ (\nabla \sigma, \mu_{0}) \left| \sigma \in \mathcal{S} \right. \right\} = \sup_{\mathcal{S}} K_{D} = K_{D}(S_{0})
\]

(16)

hold.

After the extension of \( \mu_{0} \) on the totality of \( L_{\infty}^{m,n}(\Omega) \), it holds again, according to (12),

\[
\|\mu_{0}\|^{*} = \sup_{u} \left\{ (u, \mu_{0}) \left| u \in L_{\infty}^{m,n}(\Omega), u(x) \in \mathcal{F}(x) \text{ a.e. on } \Omega \right. \right\}
\]

and since (6), (10) and (16)

\[
\|\mu_{0}\|^{*} = K(\mu_{0}) = K_{D}(S_{0}).
\]

(17)

From (15) \( \mu_{0} \in Y \) follows such that (8) and (17) lead to the optimality of \( \mu_{0} \) with respect to problem (1). So we can summarize:

Theorem 2. The transportation flow problem (1) has an optimal solution \( \mu_{0} \).
3. Conclusions and generalizations

The existence of optimal solutions $S_0$ of the deposit problem (7) and $\mu_0$ of the transportation flow problem (1) has in connection with (8) and (17) the following consequence.

**Theorem 3.** *Between the dual problems* (1) *and* (7) *there exists strong duality in the sense that* \( \min_y K = \max K_D \).

From this theorem we obtain under consideration of (3), (12), (15) and (17)

\[
K_D(S_0) = \int_\Omega S_0(x)^T \alpha(x) = K(\mu_0) = \langle \nabla S_0, \mu_0 \rangle \geq \langle u, \mu_0 \rangle
\]

for all $u \in L^{m,n}_\infty(\Omega), u(x) \in \mathcal{F}(x)$ a.e. This leads to the following conclusion.

**Theorem 4.** *An element* $S_0 \in S$ *is an optimal solution of the deposit problem* (7) *if and only if there is a vectorial set function* $\mu_0 \in L^{m,n}_\infty(\Omega)^*$ *which satisfies the continuity equation*

\[
\langle \nabla \sigma, \mu_0 \rangle = \int_\Omega \sigma(x)^T \alpha(x) \quad \forall \sigma \in W^{1,n}_\infty(\Omega)
\]

*and the maximum condition*

\[
\langle \nabla S_0, \mu_0 \rangle \geq \langle u, \mu_0 \rangle \quad \forall u \in L^{m,n}_\infty(\Omega), u(x) \in \mathcal{F}(x) \text{ a.e. on } \Omega.
\]

**Remark.** Theorems 3 and 4 coincide essentially with Theorems 4 and 3 from [2]. However, unfortunately the proof of Theorem 3 in that paper was not correct because of a mistake in identifying weak* compactness and sequentially weak* compactness by the application of Alaoglu's theorem. Finally, we mention that all results proved here hold also for the case in which $W^{1,n}_\infty(\Omega)$ in (2) and (9) is replaced by $\dot{W}^{1,n}_\infty(\Omega)$. Then we can omit even assumption (4).

**References**


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