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Peano Kernels of Non-Integer Order

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Abstract. We consider the representation of error functionals in numerical quadrature by the Peano kernel method. It is easily observed that the usual expressions for Peano kernels of order s still make sense if s is not a natural number. In this paper, we discuss how to interpret these Peano kernels, we state their main properties, and we compare them to the (classical) Peano kernels of integer order.

Keywords: *Quadrature formulas, error bounds, Peano kernels, fractional calculus*

AMS subject classification: Primary 41 A 55, secondary 26 A 33

1. Introduction

The Peano kernel theorem [5: Chapter 22] is one of the most important tools used for the estimation of errors in approximation processes. It can be stated in the following way.

Theorem 1.1 (Peano and Sard). *Let $s \in \mathbb{N}$, and let R be a continuous linear functional on $C[a,b]$. If $R[p] = 0$ for every $p \in \mathcal{P}_{s-1}$, then for every $f \in \mathcal{A}^s[a,b]$*

$$R[f] = \int_a^b K_s(x) f^{(s)}(x)\,dx \tag{1}$$

where

$$K_s(x) = \frac{1}{(s-1)!} R[(\cdot - x)_+^{s-1}].$$

Here $\mathcal{P}_m$ denotes the set of all polynomials of degree less or equal m, $\mathcal{A}^m[a,b]$ is the set of all functions with absolutely continuous derivative of order $m-1$, and $(\cdot)_+^m$ is the truncated power function given by

$$x_+^m = \begin{cases} 0 & \text{if } x \le 0 \\ x^m & \text{if } x > 0. \end{cases}$$

The function K_s is called the s-th *Peano kernel* of the functional R.

A proof of this theorem can be found in [7: Chapter 1/Theorem 43]. This theorem gives an extremely useful method for obtaining error bounds and for the purpose of

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comparison of different approximation methods (see, e.g., Powell [5: Subsections 22.1, 22.3 and 22.4] or Brass [1]. The reader who is interested in this general theory may also consult the recent survey by Brass and Förster [2].

In this paper, however, we will now focus our attention to the most important special case of this general theory, namely the case that

$$R[f] = \int_a^b f(x)\,dx - \sum_{\nu=1}^{n} a_\nu f(x_\nu) \tag{2}$$

($a_\nu \in \mathbb{R}$, $x_\nu \in [a,b]$) is the error functional of a quadrature formula. This is the case that has been considered by Peano in his classical paper [4]. In this situation, we have (see, e.g., Brass [1: Theorem 16/(ii)]):

Theorem 1.2. *Let $s \in \mathbb{N}$, let the functional R be given by (2), and let a_ν and x_ν be such that $R[p] = 0$ for every $p \in \mathcal{P}_{s-1}$. Then, the s-th Peano kernel of R is given by*

$$K_s(x) = \frac{(b-x)^s}{\Gamma(s+1)} - \frac{1}{\Gamma(s)} \sum_{\nu=1}^{n} a_\nu (x_\nu - x)_+^{s-1}. \tag{3}$$

This representation is the starting point for our investigation. We can immediately see that this expression can be used as a definition for a function on $[a,b]$ whenever $s > 0$, where we now drop the restriction that s must be an integer. To be precise, we give the following

Definition 1.1. Let $s > 0$, and let

$$R[f] = \int_a^b f(x)\,dx - \sum_{\nu=1}^{n} a_\nu f(x_\nu)$$

where $a_\nu \in \mathbb{R}$ and $a < x_1 < x_2 < \cdots < x_n \le b$. The function K_s given by

$$K_s(x) = \frac{1}{\Gamma(s)} R[(\cdot - x)_+^{s-1}]$$

is called the s-th *Peano kernel* of R.

It is a simple consequence of this definition that K_s is still given by (3). We note, however, that we have imposed a restriction on the quadrature formula, namely that a must not be among its nodes. The reason for this restriction will become clear in Theorem 2.1 below.

In the following sections, we shall see how the statement of Theorem 1.1 must be modified and interpreted if s is not an integer. Since the term $f^{(s)}(x)$ appears in the Peano kernel representation (1) of R in Theorem 1.1, it is not surprising that we have to deal with derivatives of non-integer order as considered in the theory of fractional calculus [6]. Because we assume that some readers are unfamiliar with this theory, we shall state some important facts and results from fractional calculus in the Appendix (Section 6).

For the sake of completeness, we mention that (as usual in fractional calculus) we actually do not need to assume that $s \in \mathbb{R}^+$, but we may even take $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$. Most of the results stated below will remain valid in this situation, but since the case $s \in \mathbb{R}^+$ is probably more interesting as far as applications are concerned, we shall not go into details on the complex case.

2. Main Results

The main result is the following theorem which can be seen as the fractional version of Theorem 1.1. The proof as well as the proofs of the other results will be given in Section 4. Here, by $\mathcal{D}^s_{a+}f$, we denote the s-th (left-handed) fractional derivative of f in the sense of Riemann and Liouville with respect to the point a (see Appendix). Moreover, $\lfloor s \rfloor$ is the largest integer not exceeding s.

Theorem 2.1. *Let $0 < s \notin \mathbb{N}$, let the functional R be given by* (2), *and let $a_\nu, x_\nu \in \mathbb{R}$ be such that*

$$R[(\cdot - a)^{s-j}] = 0, \qquad (j = 1, 2, \ldots, \lfloor s \rfloor + 1). \tag{4}$$

Assuming $f \in \mathcal{A}^s[a,b]$ and, if $s < 1$, additionally that $\mathcal{D}^s_{a+}f \in L_q(a,b)$ for some $q > \frac{1}{s}$, we have

$$R[f] = \int_a^b K_s(x)(\mathcal{D}^s_{a+}f)(x)\,dx.$$

We note that in Theorem 1.1 we had the hypothesis that R vanishes on a certain space of polynomials. This assumption is replaced by condition (4) now. In particular, condition (4) implies that $R[(\cdot - a)^{s-\lfloor s \rfloor - 1}] = 0$. This statement contains the implicit assumption that R is defined for this argument. Since the exponent $s - \lfloor s \rfloor - 1$ is negative, this means that the point a must not be a node of the quadrature formula.

The most important application of the Peano kernel representation in the classical case is the calculation of error bounds (see, e.g., [1: Theorem 18] or [5: Subsections 22.3 and 22.4]). This is also true in the fractional case considered here. By Hölder's inequality, we can immediately deduce the following result from Theorem 2.1.

Corollary 2.2. *Let $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Under the assumptions of Theorem 2.1, we have*

$$|R[f]| \le c_{s,q}(R) \left\| \mathcal{D}^s_{a+}f \right\|_q \tag{5}$$

where

$$c_{s,q}(R) = \|K_s\|_p.$$

Moreover, the constant $c_{s,q}(R)$ in inequality (5) *may not be replaced by a smaller number.*

From Theorem 2.1 and the representation (3) of the Peano kernels, we can deduce some of the properties of K_s. A comparison with the classical case [1: Theorem 16] shows that Peano kernels of integer order have got similar properties.

Theorem 2.3. *Under the assumptions of Theorem* 2.1, *we have:*

(a) *For* $s > 1$, $K_s(x) = -\int_a^x K_{s-1}(t)\,dt = \int_x^b K_{s-1}(t)\,dt$.

(b) *For* $s > 1$, $K_s \in C^{\lfloor s-1 \rfloor}[a,b]$.

(c) *For* $s > 1$, $K_s^{(\lfloor s-1 \rfloor)}$ *fulfils a Lipschitz condition of order* $s - \lfloor s \rfloor$.

(d) *If* $s > 1$, *then* $K_s \in L_\infty(a,b)$; *if* $s < 1$, *then* $K_s \in L_p(a,b)$ *if and only if* $p < \frac{1}{1-s}$.

Theorem 2.4. *Let the assumptions of Theorem* 2.1 *hold. Then* $K_s(a) = K_s(b) = 0$.

In the classical case, there exists another representation for K_s [1: Theorem 16/(iii)] from which we can see that the behaviour of a Peano kernel of integer order near one of the end points of the interval $[a,b]$ is very similar to its behaviour near the other end point. This is not the case for the Peano kernels of non-integer order as considered here:

Theorem 2.5. *Let the assumptions of Theorem* 2.1 *hold. Then:*

(a) *There exists* $\varepsilon > 0$ *such that* K_s *is analytic in a neighbourhood of* $[a, a+\varepsilon)$.

(b) *For every* $\varepsilon > 0$, K_s *is not analytic in any neighbourhood of* $(b-\varepsilon, b]$.

The reason for this non-symmetric behaviour is that, in Theorem 2.1, we have got a representation involving the fractional derivative $\mathcal{D}^s_{a+}f$ with respect to the point a which cannot be expressed by a fractional derivative with respect to the point b. This is a phenomenon that does not occur if s is an integer. We could, of course, establish a completely symmetric theory involving fractional derivatives with respect to the point b. Then, we would have the second representation for the Peano kernels as stated in [1: Theorem 16/(iii)], but not the representation (3). Furthermore, in Theorem 2.1, we would have to replace (4) by $R[(b-\cdot)^{s-j}] = 0$ $(j = 1, 2, \ldots, \lfloor s \rfloor + 1)$.

3. An Example

Before we come to the proofs of the theorems, we look at a simple example. Let

$$Q[f] = \frac{19}{15} f\left(\frac{1}{4}\right) - \frac{13\sqrt{2}}{15} f\left(\frac{1}{2}\right) + \frac{3\sqrt{3}}{5} f\left(\frac{3}{4}\right) \quad \text{and} \quad R[f] = \int_0^1 f(x)\,dx - Q[f].$$

Then $R[(\cdot - a)^k] = 0$ for $k = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$. Thus, R has got Peano kernels of the orders $\frac{1}{2}$, $\frac{3}{2}$ and $\frac{5}{2}$. They are given by

$$K_{\frac{1}{2}}(x) = \frac{2(1-x)^{\frac{1}{2}}}{\sqrt{\pi}} - \frac{1}{\sqrt{\pi}}\left(\frac{19}{15}\left(\frac{1}{4}-x\right)_+^{-\frac{1}{2}} - \frac{13\sqrt{2}}{15}\left(\frac{1}{2}-x\right)_+^{-\frac{1}{2}} + \frac{3\sqrt{3}}{5}\left(\frac{3}{4}-x\right)_+^{-\frac{1}{2}}\right)$$

$$K_{\frac{3}{2}}(x) = \frac{4(1-x)^{\frac{3}{2}}}{3\sqrt{\pi}} - \frac{2}{\sqrt{\pi}}\left(\frac{19}{15}\left(\frac{1}{4}-x\right)_+^{\frac{1}{2}} - \frac{13\sqrt{2}}{15}\left(\frac{1}{2}-x\right)_+^{\frac{1}{2}} + \frac{3\sqrt{3}}{5}\left(\frac{3}{4}-x\right)_+^{\frac{1}{2}}\right)$$

$$K_{\frac{5}{2}}(x) = \frac{8(1-x)^{\frac{5}{2}}}{15\sqrt{\pi}} - \frac{4}{3\sqrt{\pi}}\left(\frac{19}{15}\left(\frac{1}{4}-x\right)_+^{\frac{3}{2}} - \frac{13\sqrt{2}}{15}\left(\frac{1}{2}-x\right)_+^{\frac{3}{2}} + \frac{3\sqrt{3}}{5}\left(\frac{3}{4}-x\right)_+^{\frac{3}{2}}\right).$$

Plots of these Peano kernels are provided in Figures 1 - 3. The properties described in Theorems 2.3 - 2.5 are clearly exhibited.

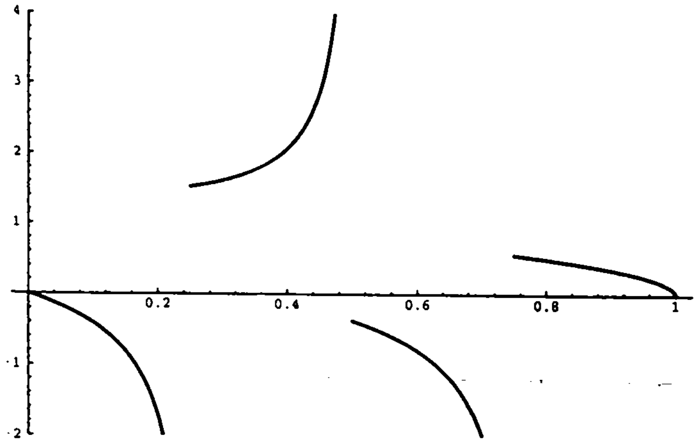

Figure 1: Plot of $K_{\frac{1}{2}}$

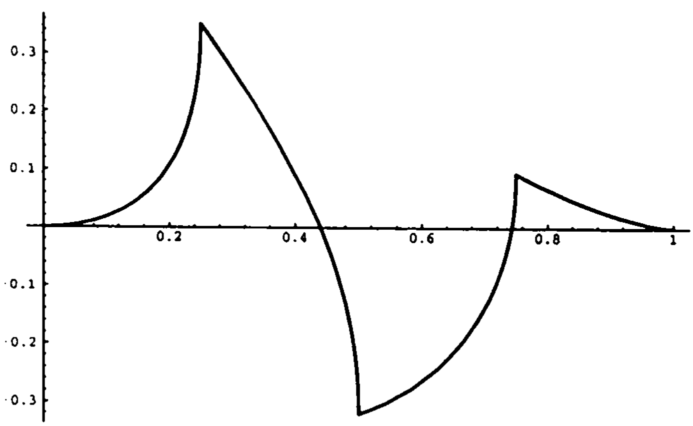

Figure 2: Plot of $K_{\frac{3}{2}}$

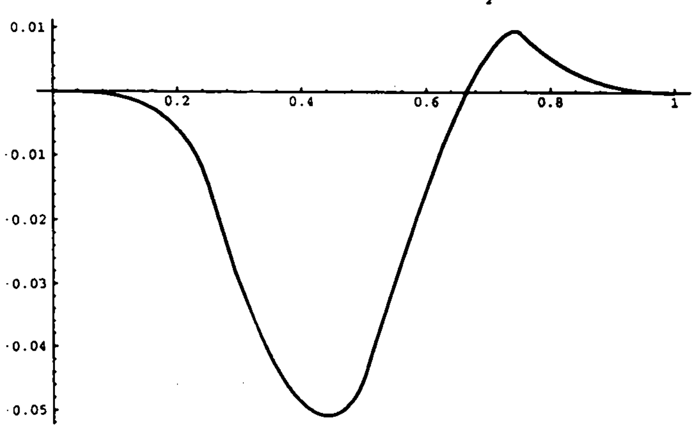

Figure 3: Plot of $K_{\frac{5}{2}}$

In order to obtain error bounds for the quadrature formula Q, we may now apply Corollary 2.2 with various different values of p. For example, we can deduce

$$
\begin{aligned}
|R[f]| &\le 0.0154992\|\mathcal{D}_{0+}^{\frac{5}{2}} f\|_\infty \\
|R[f]| &\le \Big[-14216 - 69255\ln 3 - 1300\sqrt{2}\ln\left(3-2\sqrt{2}\right) + 135\sqrt{3}\ln\left(3-2\sqrt{2}\right) \\
&\quad +2340\sqrt{6}\ln\left(5-2\sqrt{6}\right) - 54720\sqrt{3}\ln\left(2-\sqrt{3}\right) \\
&\quad +135\sqrt{3}\ln\left(21+14\sqrt{2}-12\sqrt{3}-8\sqrt{6}\right)\Big]^{\frac{1}{2}} (960\sqrt{15\pi})^{-1}\|\mathcal{D}_{0+}^{\frac{5}{2}} f\|_2 \\
&\le 0.02351\|\mathcal{D}_{0+}^{\frac{5}{2}} f\|_2 \\
|R[f]| &\le 0.05077\|\mathcal{D}_{0+}^{\frac{5}{2}} f\|_1 \\
|R[f]| &\le 0.1207712\|\mathcal{D}_{0+}^{\frac{3}{2}} f\|_\infty \\
|R[f]| &\le (40\sqrt{3\pi})^{-1}\Big[200 + 855\ln 3 + \left(-156\sqrt{2} - 15\sqrt{3} - 156\sqrt{6}\right)\ln\left(3-2\sqrt{2}\right) \\
&\quad +912\sqrt{3}\ln\left(2-\sqrt{3}\right) + 15\sqrt{3}\ln\left(35-24\sqrt{2}+20\sqrt{3}-14\sqrt{6}\right) \\
&\quad +\left(15\sqrt{3}+156\sqrt{6}\right)\ln\left(15-10\sqrt{2}-8\sqrt{3}+6\sqrt{6}\right)\Big]^{\frac{1}{2}}\|\mathcal{D}_{0+}^{\frac{3}{2}} f\|_2 \\
&\le 0.1613\|\mathcal{D}_{0+}^{\frac{3}{2}} f\|_2 \\
|R[f]| &\le \left(26\sqrt{2}+15\sqrt{3}-18\sqrt{6}\right)\left(30\sqrt{\pi}\right)^{-1}\|\mathcal{D}_{0+}^{\frac{3}{2}} f\|_1 \le 0.35092\|\mathcal{D}_{0+}^{\frac{3}{2}} f\|_1 \\
|R[f]| &\le \left(5+16\sqrt{2}+33\sqrt{3}-18\sqrt{6}\right)\left(15\sqrt{\pi}\right)^{-1}\|\mathcal{D}_{0+}^{\frac{1}{2}} f\|_\infty \le 1.53062\|\mathcal{D}_{0+}^{\frac{1}{2}} f\|_\infty .
\end{aligned}
$$

4. Proofs

First, we note that Theorem 2.3/(d) follows immediately from the representation (3) of the Peano Kernel.

Proof of Theorem 2.1. We point out that the integral mentioned in the claim of Theorem 2.1 exists because of the assumptions on f and Theorem 2.3/(d). Applying the definition of K_s, we then obtain

$$\int_a^b K_s(x)(\mathcal{D}_{a+}^s f)(x)\,dx = J_1 - J_2$$

where

$$J_1 = \frac{1}{\Gamma(s+1)}\int_a^b (b-x)^s(\mathcal{D}_{a+}^s f)(x)\,dx$$

and

$$J_2 = \frac{1}{\Gamma(s)} \sum_{\nu=1}^{n} a_\nu \int_a^b (x_\nu - x)_+^{s-1} (\mathcal{D}_{a+}^s f)(x)\, dx.$$

Now, using the definition of a fractional integral (see Appendix) and its semigroup property, we arrive at

$$J_1 = (I_{a+}^{s+1} \mathcal{D}_{a+}^s f)(b) = (I_{a+}^1 I_{a+}^s \mathcal{D}_{a+}^s f)(b) = \int_a^b (I_{a+}^s \mathcal{D}_{a+}^s f)(x)\, dx$$

and

$$J_2 = \sum_{\nu=1}^{n} a_\nu \frac{1}{\Gamma(s)} \int_a^{x_\nu} (x_\nu - x)^{s-1} (\mathcal{D}_{a+}^s f)(x)\, dx = \sum_{\nu=1}^{n} a_\nu (I_{a+}^s \mathcal{D}_{a+}^s f)(x_\nu)$$

which implies

$$\int_a^b K_s(x) (\mathcal{D}_{a+}^s f)(x)\, dx = R\left[I_{a+}^s \mathcal{D}_{a+}^s f\right]. \tag{6}$$

By [6: Equality (2.60)], we have

$$(I_{a+}^s \mathcal{D}_{a+}^s f)(x) = f(x) - \sum_{k=0}^{\lfloor s \rfloor} \gamma_k (x-a)^{s-k-1}$$

with certain coefficients γ_k that depend on f, k, a, and s, but not on x. An application of this relation to (6) yields

$$\int_a^b K_s(x) (\mathcal{D}_{a+}^s f)(x)\, dx = R[f] - \sum_{k=0}^{\lfloor s \rfloor} \gamma_k R[(\cdot - a)^{s-k-1}].$$

Since the last sum here is zero because of (4), the proof is complete ∎

Proof of Theorem 2.3/(a) - (c). By Theorem 2.1, the Peano kernels K_s and K_{s-1} exist. From (3), we can immediately see that K_{s-1} is integrable. Then, a simple explicit calculation yields

$$\int_x^b K_{s-1}(t)\, dt = K_s(x).$$

Furthermore, also by Theorem 2.1,

$$\int_a^x K_{s-1}(t)\, dt = \int_a^b K_{s-1}(t)\, dt - \int_x^b K_{s-1}(t)\, dt = R[\phi] - K_s(x)$$

where ϕ is a function with the property $(\mathcal{D}_{a+}^{s-1}\phi)(x) = 1$. From the theory of fractional calculus, we see that we may choose $\phi(x) = \frac{(x-a)^{s-1}}{\Gamma(s)}$. Now, by assumption (4), $R[\phi] = 0$ completing the proof of assertion (a).

For $s < 2$, assertion (a) immediately implies that K_s is continuous, i.e. $K_s \in C^0[a,b] = C^{\lfloor s-1 \rfloor}[a,b]$. Repeated application of assertion (a) gives

$$K_s^{(\lfloor s-1 \rfloor)} = (-1)^{\lfloor s-1 \rfloor} K_{s-\lfloor s \rfloor+1} \tag{7}$$

which is continuous because $s - \lfloor s \rfloor + 1 > 1$. Thus, we have shown assertion (b).

To prove assertion (c), let us first assume that $s < 2$. The function $(b - \cdot)^s$ is differentiable (recall that $s > 1$), therefore it fulfils a Lipschitz condition of order $s - \lfloor s \rfloor = s - 1$. The functions $(x_\nu - \cdot)_+^{s-1}$ obviously also fulfil the Lipschitz conditions of order $s - 1 = s - \lfloor s \rfloor$. Thus, K_s also fulfils such a Lipschitz condition. In the case $s > 2$, the claim follows using relation (7) and the previous considerations ∎

Proof of Theorem 2.4. We note that $K_s(b) = 0$ follows immediately from the representation (3) of K_s. Furthermore, since $a \notin \{x_1, \dots, x_n\}$, i.e. $a < \min_{1 \le \nu \le n} x_\nu$, we have $(x_\nu - a)_+^{s-1} = (x_\nu - a)^{s-1}$ for every ν. Thus,

$$\begin{aligned} K_s(a) &= \frac{(b-a)^s}{\Gamma(s+1)} - \frac{1}{\Gamma(s)} \sum_{\nu=1}^{n} a_\nu (x_\nu - a)_+^{s-1} \\ &= \int_a^b \frac{(t-a)^{s-1}}{\Gamma(s)}\, dt - \frac{1}{\Gamma(s)} \sum_{\nu=1}^{n} a_\nu (x_\nu - a)^{s-1} \\ &= \frac{1}{\Gamma(s)} R[(\cdot - a)^{s-1}] \\ &= 0 \end{aligned}$$

by assumption (4) ∎

Proof of Theorem 2.5. Looking at (3), we can see the following facts.

(a) The function $(b - \cdot)^s$ is analytic on $(-\infty, b) \supset [a, a+\varepsilon)$ for some $\varepsilon > 0$. Furthermore, since $x_\nu > a$ holds for every ν, we have that $(x_\nu - \cdot)_+^{s-1} = (x_\nu - \cdot)^{s-1}$ on $[a, \min_\nu x_\nu)$. All these functions are also analytic on $[a, \min_\nu x_\nu)$. Therefore, K_s is analytic there.

(b) Because of (7), it is sufficient to prove the claim for $0 < s < 1$. In this case, we can see that the function $(b - \cdot)^s$ is continuous but not differentiable at the point b. Now, if $\max_\nu x_\nu < b$, then we have $(x_\nu - \cdot)_+^{s-1} = 0$ in a neighbourhood of b for every ν. Thus, K_s is not differentiable in this case. If, on the other hand, $\max_\nu x_\nu = b$, then

$$\left(\max_\nu x_\nu - \cdot\right)_+^{s-1} = \left(\max_\nu x_\nu - \cdot\right)^{s-1} = (b - \cdot)^{s-1}$$

which is not even bounded in any neighbourhood of b. Therefore, in this case, K_s is not continuous at b ∎

5. Application to classical quadrature formulas

In the results described so far, we have developed a theory to investigate quadrature formulas satisfying the somewhat non-classical assumption (4). Quadrature formulas with this property seem to be quite natural, e.g., in the context of the numerical solution of Abel-type integral equations [3, 6]. However, the classical formulas fulfil $R[p] = 0$ for every $p \in \mathcal{P}_s$ with some natural number s, and they are of a very large practical importance. Therefore, we shall now state how our method may be transferred to these classical formulas, thus obtaining error representations and bounds of a new type involving fractional derivatives for them. It turns out that it is convenient to use weighted norms as described below.

Let us define the differential operator $\widehat{\mathcal{D}}^s_{a+}$ by

$$\widehat{\mathcal{D}}^s_{a+} f = \mathcal{D}^s_{a+} \left((\cdot - a)^{s - \lfloor s+1 \rfloor} f \right)$$

whenever the expression on the right-hand side exists. We can state the following result.

Theorem 5.1. *Let $s \in \mathbb{N}_0$ and $r \notin \mathbb{N}$ such that $0 < r < s$. Let R be the remainder of a quadrature formula given by (2) where $\min_\nu x_\nu > a$, and assume that $R[p] = 0$ whenever $p \in \mathcal{P}_{s-1}$. Define $\kappa = \lfloor r + 1 \rfloor - r$. Then, assuming $\widehat{\mathcal{D}}^{r-1}_{a+} f \in \mathcal{A}^1[a, b]$ and, if $r < 1$, additionally that $\widehat{\mathcal{D}}^r_{a+} f \in L_q(a, b)$ for some $q > \frac{1}{r}$, we have*

$$R[f] = \int_a^b \widehat{K}_r(x)(\widehat{\mathcal{D}}^r_{a+} f)(x)\, dx$$

where

$$\widehat{K}_r(x) = \frac{(x-a)^\kappa (b-x)^r}{\Gamma(r+1)} {}_2F_1\left(-\kappa, r; r+1; \frac{b-x}{a-x}\right) - \frac{1}{\Gamma(r)} \sum_{\nu=1}^n a_\nu (x_\nu - a)^\kappa (x_\nu - x)_+^{r-1}$$

where ${}_2F_1$ denotes the usual hypergeometric function.

Proof. Taking into consideration the identity

$$\int_a^b (t-a)^\kappa (t-x)_+^{r-1} dt = \frac{1}{r}(x-a)^\kappa (b-x)^r {}_2F_1\left(-\kappa, r; r+1; \frac{b-x}{a-x}\right)$$

and observing that $\widehat{K}_r$ is continuous for $r > 1$ whereas, for $r < 1$, $\widehat{K}_r \in L_p(a, b)$ if and only if $p < \frac{1}{1-r}$, the proof of this theorem is very similar to that of Theorem 2.1. So we leave the details to the reader ∎

Now, we can again apply Hölder's inequality in the statement of Theorem 5.1 to obtain error bounds involving fractional derivatives for classical quadrature formulas. As an example, we use the classical midpoint formula with n nodes for the interval $[0, 1]$, given by

$$Q_n^{\mathrm{Mi}}[f] = \frac{1}{n}\sum_{\nu=1}^{n} f\left(\frac{2\nu-1}{2n}\right).$$

For its remainder R_n^{Mi}, we have calculated the constants $c_{n,r,p} = \|\widehat{K}_r\|_q$ in bounds of the form

$$|R_n^{\mathrm{Mi}}[f]| \le c_{n,r,p} \|\widehat{\mathcal{D}}_{a+}^r f\|_p$$

for various values of n, r and p. Here, $\widehat{K}_r$ is the Peano kernel of Q_n^{Mi} described in Theorem 5.1, and $\frac{1}{p} + \frac{1}{q} = 1$. In all cases, the numerical observations indicate that the relation

$$c_{n,r,p} = \|\widehat{K}_r\|_q = O(n^{-r}) \tag{8}$$

holds. Of course, this is also known to be true for the error constants of integer order given in terms of the classical Peano kernels [1]. Some of the numerical results are given in the following table.

n	$\|\widehat{K}_{\frac{1}{2}}\|_1$	$\|\widehat{K}_{\frac{2}{3}}\|_1$	$\|\widehat{K}_{\frac{2}{3}}\|_2$	$\|\widehat{K}_{\frac{3}{2}}\|_1$	$\|\widehat{K}_{\frac{3}{2}}\|_2$
4	2.28 E−1	1.49 E−1	2.43 E−1	6.89 E−3	9.60 E−3
8	1.61 E−1	9.32 E−2	1.53 E−1	2.40 E−3	3.25 E−3
16	1.13 E−1	5.86 E−2	9.62 E−2	8.49 E−4	1.12 E−3

6. Appendix: Some results from fractional calculus

For the convenience of the reader, we collect some definitions and theorems from fractional calculus which have been used above. We start with the following basic definitions (see [6: Subsection 2.3]).

Definition A.1. Let $s > 0$ and let $\phi \in L_1(a, b)$. Then the integral

$$(I_{a+}^s \phi)(x) = \frac{1}{\Gamma(s)} \int_a^x \phi(t)(x-t)^{s-1}\, dt$$

is called the (*left-handed*) *Riemann-Liouville fractional integral* of the order s for the function ϕ.

If $s \in \mathbb{N}$, this definition coincides with the classical s-fold integral of ϕ.

Definition A.2. Let $0 < s \notin \mathbb{N}$ and $\sigma = \lfloor s \rfloor + 1$. Then

$$(\mathcal{D}^s_{a+}\phi)(x) = \frac{1}{\Gamma(\sigma - s)} \frac{d^\sigma}{dx^\sigma} \int_a^x \phi(t)(x-t)^{\sigma-s-1}\, dt$$

is called the (*left-handed*) *Riemann-Liouville fractional derivative* of the order s for the function ϕ, if it exists.

Obviously, $(\mathcal{D}^s_{a+}\phi)(x) = \frac{d^\sigma}{dx^\sigma}(I^{\sigma-s}_{a+}\phi)(x)$. A sufficient condition for the existence of $\mathcal{D}^s_{a+}\phi$ is

$$\int_a^\cdot \phi(t)(\cdot - t)^{\lfloor s \rfloor - s}\, dt \in \mathcal{A}^{\lfloor s \rfloor}[a,b],$$

which is fulfilled if $\phi \in \mathcal{A}^{\lfloor s \rfloor}[a,b]$. Among the main properties of these integrals and derivatives, we have [6: pp. 34 - 35]:

Lemma A.1 (Semigroup property of fractional integration). *Let $s, \hat{s} > 0$, and assume that $\phi \in L_1[a,b]$. Then, almost everywhere on (a,b),*

$$I^s_{a+} I^{\hat{s}}_{a+} \phi = I^{s+\hat{s}}_{a+} \phi. \tag{9}$$

If, additionally, $s + \hat{s} \geq 1$, then (9) holds everywhere on (a,b).

Furthermore, as in the classical case, we have [6: Theorem 2.4]:

Lemma A.2. *Let $\phi \in L_1(a,b)$, and let $0 < s \notin \mathbb{N}$. Then, almost everywhere on (a,b), $\mathcal{D}^s_{a+} I^s_{a+}\phi = \phi$.*

Finally, to give the reader a little bit of a feeling for fractional differentiation and integration, we give some examples for fractional integrals and derivatives.

Lemma A.3. *Let $s, \hat{s} > 0$, and let $\phi(x) = (x-a)^{\hat{s}-1}$. Then,*

$$(I^s_{a+}\phi)(x) = \frac{\Gamma(\hat{s})}{\Gamma(\hat{s}+s)}(x-a)^{\hat{s}+s-1}$$

and

$$(\mathcal{D}^s_{a+}\phi)(x) = \begin{cases} \frac{\Gamma(\hat{s})}{\Gamma(\hat{s}-s)}(x-a)^{\hat{s}-s-1} & \text{if } s - \hat{s} \notin \mathbb{N}_0 \\ 0 & \text{if } s - \hat{s} \in \mathbb{N}_0. \end{cases}$$

We remark that in this lemma, it is not important whether s (the order of the derivative or integral) is an integer or not. The last assertion means that, for $s \notin \mathbb{N}$, the functions $(\cdot - a)^{s-k}$ $(k = 1, 2, \ldots, \lfloor s \rfloor + 1)$ play the same role for the fractional differentiation operator as the polynomials of degree less than s do for the usual differentiation operator $\left(\frac{d}{dx}\right)^s$ if $s \in \mathbb{N}$.

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