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The Character-Automorphic Nehari Problem: Non-Uniqueness Criterion and some Extremal Solutions

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Abstract. A non-uniqueness criterion for the character-automorphic Nehari problem is given. Certain subclass of solutions, connected with "the entropy functional" of the problem, is described. The description yields a character-automorphic counterpart of the Adamyan-Arov-Krein theorem.

Keywords: *Fuchsian group of Widom type, direct Cauchy theorem, character-automorphic Hardy classes, Nehari problem*

AMS subject classification: Primary 30 F 35, 47 A 45, secondary 30 H 05, 30 E 05, 47 A 57

1. Introduction

In this work we continue the study of the character-automorphic Nehari problem [7]. First we would like to recall some basic concepts and notation.

Let Γ be a Fuchsian group, that is a discontinuous group of linear-fractional transformations of the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ onto itself. Let Γ^* be the group of unitary characters of the group Γ. We assume throughout the paper that Γ has no elliptic and parabolic elements.

An analytic function $f = f(\zeta)$ $(\zeta \in \mathbb{D})$ is called to be of *bounded characteristic* if

$$\sup_{0<r<1} \int_{\mathbb{T}} \log^+ |f(rt)|\, dm(t) < \infty$$

where $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$ and dm is the Lebesgue measure on $\mathbb{T}$. Any function of this class possesses an inner-outer factorization (see, for example, [6, 8]). We denote by f_{in} and f_{out} the inner and the outer factors of the function f, $f(\zeta) = f_{in}(\zeta) f_{out}(\zeta)$. We denote by H^p $(1 \le p \le \infty)$ the Hardy spaces of analytic functions $f = f(\zeta)$ $(\zeta \in \mathbb{D})$ with

$$\|f\|_p = \sup_{0<r<1} \left\{ \int_{\mathbb{T}} |f(rt)|^p dm(t) \right\}^{\frac{1}{p}} < \infty \qquad (1 \le p < \infty)$$

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and $\|f\|_\infty = \sup\{|f(\zeta)| : \zeta \in \mathbb{D}\}$. A function f of bounded characteristic has boundary values almost everythere on $\mathbb{T}$ and one can identify it with the function given by $f(t) = \lim_{r\to 1} f(rt)$, $t \in \mathbb{T}$. From this point of view, there is another description of the Hardy space H^p: it consists of L^p-functions on $\mathbb{T}$ with vanishing negative Fourier coefficients. We denote by H^p_- the space of L^p-functions with vanishing non-negative Fourier coefficients.

One can find a detailed presentation of the theory of Hardy spaces at infinitely-connected Riemann surfaces of Parreau–Widom type in the monograph of M. Hasumi [5]. Following the paper of Ch. Pommerenke [9], we consider the Fuchsian groups of Widom type and Hardy spaces of character–automorphic functions with respect to a group of this type.

One can consider the action of the group Γ on the unit circle $\mathbb{T}$. We associate with an arbitrary character $\alpha \in \Gamma^*$ spaces of character-automorphic functions

$$L^p(\alpha) = \left\{f \in L^p : f\circ\gamma = \alpha(\gamma) f \quad \text{for all} \quad \gamma \in \Gamma\right\}$$

and

$$H^p(\alpha) = L^p(\alpha) \cap H^p.$$

The group Γ is said to be of *Widom type* [9, 10, 12] if for any $\alpha \in \Gamma^*$ the space $H^\infty(\alpha)$ is not trivial, i.e. $H^\infty(\alpha) \neq \{\text{const}\}$ for all $\alpha \in \Gamma^*$. Let us note that any group of Widom type is necessary a group of convergent type, i.e. $\sum_{\gamma\in\Gamma}(1-|\gamma(0)|^2) < \infty$. The Blaschke product

$$b(\zeta) = \prod_{\gamma\in\Gamma} \frac{\gamma(0)-\zeta}{1-\zeta\overline{\gamma(0)}}\frac{|\gamma(0)|}{\gamma(0)}$$

is called the *Green function* of the group Γ with respect to the origin. The group Γ is of Widom type if and only if the derivative of the Green function b' is of bounded characteristic [9]. Moreover, the inner part of b' is a Blaschke product $\Delta = (b')_{in}$. It solves the extremal problem

$$\inf_{\alpha\in\Gamma^*} m^\infty(\alpha) = \Delta(0)$$

where

$$m^p(\alpha) = \sup_{f\in H^p(\alpha)} \frac{|f(0)|}{\|f\|_p} \qquad (1 \le p \le \infty).$$

In what follows we denote by $\alpha[f]$ the character of a character-automorphic function f, i.e. $\alpha[f](\gamma)f = f\circ\gamma$ for all $\gamma \in \Gamma$.

The following conditions are equivalent for groups of Widom type (see [5, 10]):

- The direct Cauchy theorem holds, i.e. $\int_{\mathbb{T}} \frac{f}{\Delta}\, dm = \frac{f(0)}{\Delta(0)}$ for every $f \in H^1(\alpha[\Delta])$.
- For all $1 \le p < \infty$ and all $\alpha \in \Gamma^*$ the annihilator of the space $H^p(\alpha)$ has the form

$$H^p_\perp(\alpha) := \left\{f \in L^q(\alpha) : \int_{\mathbb{T}} \bar{g} f\, dm = 0 \ \text{ for all } g \in H^p(\alpha)\right\} = \Delta H^q_-(\{\alpha[\Delta]\}^{-1}\alpha)$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $H^q_-(\alpha) = L^q(\alpha) \cap H^q_-$.

- Every invariant subspace of $H^p(\alpha)$ ($\alpha \in \Gamma^*$) is of the form $\Theta H^p(\{\alpha[\Theta]\}^{-1}\alpha)$ for some character-automorphic inner function Θ (this is an analog of Beurling's theorem).
- The functions $m^p = m^p(\alpha)$ are continuous on Γ^* for all $1 \le p \le \infty$.

We would like to stress that the direct Cauchy theorem is not true for an arbitrary group of Widom type and is an additional condition on the group Γ. The conditions of the direct Cauchy theorem hold, for example, if the zeros of Δ satisfy Carleson's condition [11]. In the following, we suppose that Γ is a group of Widom type and that the conditions of the direct Cauchy theorem hold.

We introduce the character-automorphic Nehari problem in the following way:

(N) Let $f_\perp \in H^2_\perp(\beta)$ for fixed $\beta \in \Gamma^*$. Describe all functions $f \in L^\infty(\beta)$ such that $f = f_+ + f_\perp$, where $f_+ \in H^2(\beta)$ and $\|f\|_\infty \le 1$.

Let us denote by $\mathcal{N}(f_\perp)$ the set of solutions of the problem (N), associated with a given function $f_\perp$. We say that the problem is *indeterminate* if it has at least two different solutions.

In [7], following Abrahamse [2] (see also [3]), we gave a solvability criterion for the scalar character-automorphic Nehari problem. Let $f_\perp \in H^2_\perp(\beta)$. We denote by $P_+(\alpha)$ and $P_\perp(\alpha)$ the orthoprojectors from $L^2(\alpha)$ onto the spaces $H^2(\alpha)$ and $H^2_\perp(\alpha)$, respectively. We define an operator $F(\alpha)$ from $H^2(\alpha)$ to $H^2_\perp(\alpha\beta)$ for an arbitrary $\alpha \in \Gamma^*$ by

$$F(\alpha)x = P_\perp(\alpha\beta)(f_\perp x) \qquad (x \in H^\infty(\alpha)).$$

Theorem (see [7]). *A function $f_\perp \in H^2_\perp(\beta)$ is a projection of a function $f \in L^\infty(\beta)$ with $\|f\|_\infty \le 1$ onto $H^2_\perp(\beta)$ if and only if $\sup_{\alpha\in\Gamma^*} \|F(\alpha)\| \le 1$.*

In this paper we use a vector-valued analog of the previous theorem. Let $L^p(\mathbb{C}^n)$ be the space of $\mathbb{C}^n$-valued functions on $\mathbb{T}$ with

$$\|f\|_{L^p(\mathbb{C}^n)} = \left\{ \int_{\mathbb{T}} \|f(t)\|^p_{\mathbb{C}^n} \, dm(t) \right\}^{\frac{1}{p}}.$$

We associate the following spaces with an arbitrary unitary representation β of a group Γ ($\beta(\gamma)$ is a unitary $(n \times n)$-matrix):

$$L^p(\beta, \mathbb{C}^n) = \left\{ f \in L^p(\mathbb{C}^n) : \, f \circ \gamma = \beta(\gamma) f \;\text{ for all }\; \gamma \in \Gamma \right\}$$

and

$$H^p(\beta, \mathbb{C}^n) = L^p(\beta, \mathbb{C}^n) \cap H^p.$$

Theorem. *Let β be an n-dimensional unitary representation of a Fuchsian group Γ. A vector-valued function $f_\perp \in H^2_\perp(\beta, \mathbb{C}^n)$ is a projection of a vector-valued function $f \in L^\infty(\beta, \mathbb{C}^n)$ with $\|f\|_\infty \le 1$ onto $H^2_\perp(\beta, \mathbb{C}^n)$ if and only if $\sup_{\alpha\in\Gamma^*} \|F(\alpha)\| \le 1$ where $F(\alpha)x = P_\perp(\alpha\beta, \mathbb{C}^n)(f_\perp x)$ for $x \in H^\infty(\alpha)$.*

A proof could be given as word for word repetition of the proof in [7]. We have just to mention that in the vector-valued case, as well as in the scalar case, any function $f \in H^1(\mathbb{C}^n)$ with $\|f\| \leq 1$ possesses the factorization $f(\zeta) = g_1(\zeta)g_2(\zeta)$, with $g_1 \in H^2(\mathbb{C}^n)$, $\|g_1\| \leq 1$ and $g_2 \in H^2$, $\|g_2\| \leq 1$.

In this paper we propose a non-uniqueness criterion, which looks like a natural character-automorphic counterpart of the classical one [1, 4]. Assume that $\mathcal{N}(f_\perp) \neq \emptyset$ or, equivalently,

$$D(\alpha) = I - F^*(\alpha)F(\alpha) \geq 0 \qquad \text{for all } \alpha \in \Gamma^*.$$

Let us associate with the system of non-negative operators $\{D(\alpha)\}_{\alpha\in\Gamma^*}$ a system of spaces $\operatorname{clos}_{L^2}\{\sqrt{D(\alpha)}H^2(\alpha)\}$. The criterion states that the solution of the problem (N) is not unique if and only if the norms of the functionals $\sqrt{D(\alpha)}x \mapsto x(0)$ $(x \in H^2(\alpha))$ are uniformly bounded with respect to α (see Criterion 2).

Another purpose was to evaluate the extremum of the "entropy functional" [4] for the problem (N). Our result (see Proposition 5.1) has the form

$$\inf_{f\in\mathcal{N}(f_\perp)} \int_{\mathbb{T}} \log\{1-|f|^2\}^{-\frac{1}{2}} dm = \inf_{\chi\in\Gamma^*} \sup_{\alpha\in\Gamma^*} \sup_{x\in H^2(\alpha)} \log \frac{|x(0)|}{m^2(\alpha\chi)\,\|\sqrt{D(\alpha)}x\|}.$$

Both results mainly follow from some duality principle, which states Theorem 1. Theorem 1 leads to a notion of χ-extremal solution. Theorem 2 describes some properties of such solutions. From these properties we deduce, for example, the existence of a unimodular solution to the indeterminate problem (N) (character-automorphic counterpart of the Adamyan-Arov-Krein theorem [1, 6, 8]). We should say here that before this work was done S. Kupin had shown us another proof of this proposition. The proof was a character-automorphic counterpart of the proof of [6: Theorem 4.3].

2. Statement of main results

We start with the following evident

Criterion 1. *The problem* (N) *for a given function* $f_\perp$ *is indeterminate if and only if there exists a solution* $f_0 \in \mathcal{N}(f_\perp)$ *such that*

$$\log(1-|f_0|^2) \in L^1. \tag{2.1}$$

Proof. Let $f_1, f_2 \in \mathcal{N}(f_\perp)$ and $f_1 \not\equiv f_2$. Set $f_0 = \frac{f_1+f_2}{2}$. It is obvious that $f_0 \in \mathcal{N}(f_\perp)$ and

$$1 - \left|\frac{f_1+f_2}{2}\right|^2 \geq \left|\frac{f_1-f_2}{2}\right|^2.$$

Since $0 \not\equiv \frac{f_1-f_2}{2} \in H^\infty$, we have

$$\int \log\left(1-\left|\frac{f_1+f_2}{2}\right|^2\right) dm \geq \int \log\left|\frac{f_1-f_2}{2}\right|^2 dm > -\infty.$$

Conversely, by virtue of (2.1) we can define a function

$$\phi(\zeta) = \exp\left(\frac{1}{2}\int_{\mathbb{T}} \frac{t+\zeta}{t-\zeta} \log(1-|f_0|^2)\, dm\right) \tag{2.2}$$

which is a character-automorphic function, lying in H^∞, and

$$|f_0|^2 + |\phi|^2 = 1 \qquad \text{a.e. on } \mathbb{T}. \tag{2.3}$$

Let $\mathcal{E} \in H^\infty(\beta\{\alpha[\phi]\}^{-2})$ and $\|\mathcal{E}\| \le \frac{1}{2}$. Define the function $f_{\mathcal{E}}$ by

$$f_{\mathcal{E}} = f_0 + \mathcal{E}\phi^2. \tag{2.4}$$

Let us verify that any function of this form belongs to $\mathcal{N}(f_\perp)$. Since $\mathcal{E}\phi^2 \in H^\infty(\beta)$, we have only to check that $\|f_{\mathcal{E}}\| \le 1$. The last inequality follows from the straightforward computation

$$\begin{aligned} 1 - |f_{\mathcal{E}}|^2 &= 1 - |f_0|^2 - \overline{\phi^2\mathcal{E}} f_0 - \overline{f_0}\phi^2\mathcal{E} - |\mathcal{E}|^2|\phi|^4 \\ &= |\phi|^2\left\{1 - \frac{\bar\phi}{\phi} f_0 \overline{\mathcal{E}} - \overline{\frac{\bar\phi}{\phi} f_0} \mathcal{E} - |\mathcal{E}|^2|\phi|^2\right\} \\ &= |\phi|^2\left\{\left|1 - \frac{\bar\phi}{\phi} f_0\overline{\mathcal{E}}\right|^2 - |\mathcal{E}|^2\right\} \end{aligned}$$

and the trivial inequalities $\left|1 - \frac{\bar\phi}{\phi} f_0\overline{\mathcal{E}}\right| \ge 1 - |\mathcal{E}| \ge |\mathcal{E}|$ ∎

Remark. One can reformulate Criterion 1 in the form that the problem (N) is indeterminate if and only if there exists a pair of functions (f, ϕ) such that

$$f \in \mathcal{N}(f_\perp), \qquad 0 \not\equiv \phi \in H^\infty(\chi), \qquad |f|^2 + |\phi|^2 \le 1 \text{ on } \mathbb{T}. \tag{2.5}$$

Any function (2.4) with $\mathcal{E} \in H^\infty(\beta\chi^{-2})$ and $\|\mathcal{E}\| \le \frac{1}{2}$ lies in $\mathcal{N}(f_\perp)$.

Definition. A pair of functions (f, ϕ) with properties (2.5) will be called a *χ-pair.*

The following non-uniqueness criterion looks like a natural character-automorphic counterpart of the classical one [1, 4].

Criterion 2. *Let $\mathcal{N}(f_\perp) \ne \emptyset$. Then the character-automorphic Nehari problem* (N) *is indeterminate if and only if*

$$\sup_{\alpha\in\Gamma^*} \sup_{x \in H^2(\alpha)} \frac{|x(0)|}{\|\sqrt{D(\alpha)}x\|} < \infty.$$

We denote by k^α the reproducing kernel of the space $H^2(\alpha)$ with respect to the origin, i.e. $k^\alpha \in H^2(\alpha)$ and $\langle x, k^\alpha\rangle = x(0)$ for all $x \in H^2(\alpha)$. We note that $\|k^\alpha\| = m^2(\alpha)$.

The following theorem describes the connection between the Criteria 1 and 2.

Theorem 1. *For a fixed* $\chi \in \Gamma^*$, *let*

$$\Phi(\chi) = \sup\left\{|\phi(0)| : |\phi|^2 + |f|^2 \le 1 \text{ for } \phi \in H^\infty(\chi) \text{ and } f \in \mathcal{N}(f_\perp)\right\} \tag{2.6}$$

and

$$M(\chi) = \sup_{\alpha\in\Gamma^*} \sup_{x\in H^2(\alpha)} \frac{|x(0)|}{\|k^\alpha x\| \, \|\sqrt{D(\alpha)}x\|}. \tag{2.7}$$

Then $\Phi(\chi) = \frac{1}{M(\chi)}$ *and the supremum* (2.6) *is attained.*

Remark. As follows from well-known estimates $1 \ge \|k^\alpha\| \ge \Delta(0)$ of the norm of k^α for a group of Widom type [5, 12], the boundedness of $M(\chi)$ for some χ implies that of $M(\chi)$ for all $\chi \in \Gamma^*$.

Definition. A χ-pair (f, ϕ) will be called *χ-extremal* if $|\phi(0)| = \Phi(\chi)$ and the function f will be called *χ-extremal solution.*

Theorem 1 asserts that χ-extremal pairs exist. The next theorem describes their properties.

Theorem 2. *Let* $\chi \in \Gamma^*$ *and let* (f, ϕ) *be a* χ*-extremal pair.*

(i) *If* $(\tilde{f}, \tilde{\phi})$ *is a* χ*-extremal pair, then* $\tilde{f} = f$ *and* $\tilde{\phi} = \frac{\tilde{\phi}(0)}{\phi(0)}\phi$.

(ii) $|f|^2 + |\phi|^2 = 1$ *a.e. on* $\mathbb{T}$.

(iii) $\frac{\overline{\phi_{out}}}{\phi_{out}} f = -\frac{\overline{s}}{\tilde{\Delta}}$.

Here $s \in H^\infty, s(0) = 0$ *and* $\tilde{\Delta}$ *is an inner character-automorphic function.*

Remark. For an appropriate choice of χ, $\tilde{\Delta}$ is a divisor of the function Δ (see Proposition 5.2). In the general case, $\tilde{\Delta}$ is a divisor of the function k^α_{in} for a certain $\alpha \in \Gamma^*$ (see the proof of Theorem 2).

Corollary of Theorem 2. *Let* (f, ϕ) *be a* χ*-extremal pair. Then any function of the form*

$$f_\mathcal{E} = f + \tilde{\Delta}\frac{\phi_{out}^2 \mathcal{E}}{1 - s\mathcal{E}} \qquad (\mathcal{E} \in H^\infty(\{\alpha[s]\}^{-1}), \|\mathcal{E}\| \le 1) \tag{2.8}$$

is a solution of the problem (N). *In particular, there exists a unimodular solution of the problem* (N).

Remark. Formula (2.8) is a straight forward corollary of the properties (ii) and (iii) of χ-extremal pairs given in Theorem 2. Even the existence of a solution of the character-automorphic Nehari problem (N) with the property $|f| = |s|$ ($s \in H^\infty$) looks non-trivial. Moreover, the existence of a unimodular solution for the character-automorphic Nehari problem follows from (2.8). To obtain such a solution it is sufficient to take any inner function as $\mathcal{E}$. Nevertheless we doubt that our approach is fruitful in a question of a parametrization of the set of all solutions.

3. Proof of Theorem 1

Let (f, ϕ) be a χ-pair, i.e. $|\phi|^2 + |f|^2 \leq 1$ with $f \in \mathcal{N}(f_\perp)$ and $\phi \in H^\infty(\chi)$, and suppose that $\phi(0) \neq 0$. Let us consider a system of operators

$$\tilde{F}(\alpha) : H^2(\alpha) \to \begin{bmatrix} H^2_\perp(\alpha\beta) \\ H^2_\perp(\alpha\chi\bar{\mu}) \end{bmatrix}$$

defined by

$$\tilde{F}(\alpha)x = \begin{bmatrix} P_\perp(\alpha\beta)(fx) \\ P_\perp(\alpha\chi\bar{\mu})(\bar{b}\phi x) \end{bmatrix}$$

where $\mu = \alpha[b]$. Using the evident decomposition

$$\frac{\phi x}{b} = \left(\frac{\phi x}{b} - \frac{\phi(0)x(0)}{b}\frac{k^{\alpha\chi}}{k^{\alpha\chi}(0)}\right) + \frac{\phi(0)x(0)}{b}\frac{k^{\alpha\chi}}{k^{\alpha\chi}(0)}$$

where the first term belongs to $H^2(\alpha\chi\bar{\mu})$ and the second one to $H^2_\perp(\alpha\chi\bar{\mu})$, we obtain

$$\tilde{F}(\alpha)x = \begin{bmatrix} F(\alpha)x \\ \dfrac{\phi(0)x(0)}{b}\dfrac{k^{\alpha\chi}}{k^{\alpha\chi}(0)} \end{bmatrix}.$$

Since $\tilde{F}(\alpha)$ is a contraction, we get

$$\langle x, x\rangle - \langle \tilde{F}(\alpha)x, \tilde{F}(\alpha)x\rangle = \langle D(\alpha)x, x\rangle - \frac{|\phi(0)|^2|x(0)|^2}{\|k^{\alpha\chi}\|^2} \geq 0.$$

Therefore,

$$\frac{|x(0)|}{\|\sqrt{D(\alpha)}x\|\,\|k^{\alpha\chi}\|} \leq \frac{1}{|\phi(0)|}. \tag{3.1}$$

Let $\{(f_n, \phi_n)\}$ be a sequence of χ-pairs such that $\phi_n(0) \to \Phi(\chi)$ as $n \to \infty$. Substituting ϕ_n into (3.1) and then passing to the limit give the estimate

$$\frac{|x(0)|}{\|\sqrt{D(\alpha)}x\|\,\|k^{\alpha\chi}\|} \leq \frac{1}{\Phi(\chi)}.$$

Passing to the supremum over $x \in H^2(\alpha)$ and $\alpha \in \Gamma^*$ we get

$$M(\chi) \leq \frac{1}{\Phi(\chi)}. \tag{3.2}$$

To prove the inverse inequality we use the vector version of the solvability criterion for the character-automorphic Nehari problem (N). Let us associate with $M(\chi)$ a vector-valued function

$$\tilde{f}_\perp = \begin{bmatrix} f_\perp \\ \dfrac{1}{M(\chi)}\dfrac{k^\chi}{b\,k^\chi(0)} \end{bmatrix} \in \begin{bmatrix} H^2_\perp(\beta) \\ H^2_\perp(\bar{\mu}\chi) \end{bmatrix}. \tag{3.3}$$

It generates a system of operators

$$\tilde{F}(\alpha)x = \begin{bmatrix} P_\perp(\alpha\beta)(f_\perp x) \\ P_\perp(\alpha\bar{\mu}\chi)\Big(\frac{1}{M(\chi)}\frac{k^\chi}{b\,k^\chi(0)}x\Big) \end{bmatrix} = \begin{bmatrix} F(\alpha)x \\ \frac{1}{M(\chi)}\frac{k^{\alpha\chi}}{b\,k^{\alpha\chi}(0)}\,x(0) \end{bmatrix} \qquad (x \in H^\infty(\alpha)).$$

Let us verify that the above-defined operators are contractions. In fact,

$$\langle x,x\rangle - \langle \tilde{F}(\alpha)x, \tilde{F}(\alpha)x\rangle = \langle D(\alpha)x,x\rangle - \frac{1}{M^2(\chi)}\frac{|x(0)|^2}{\|k^{\alpha\chi}\|^2}.$$

The last value is non-negative by the definition of $M(\chi)$. Hence there exist functions $f_+ \in H^2(\beta)$ and $g_+ \in H^2(\bar{\mu}\chi)$ such that

$$f = f_\perp + f_+, \qquad g = \frac{1}{M(\chi)}\frac{k^\chi}{b\;k^\chi(0)} + g_+, \qquad |f|^2 + |g|^2 \le 1.$$

In other words, $f \in \mathcal{N}(f_\perp)$, $\phi = bg \in H^\infty(\chi)$, $|f|^2 + |\phi|^2 \le 1$ and $\phi(0) = \frac{1}{M(\chi)}$. It means that (f,ϕ) is a χ-pair, and thus $\Phi(\chi) \ge \frac{1}{M(\chi)}$. Together with (3.2) it proves that $\Phi(\chi) = \frac{1}{M(\chi)}$, and moreover, there exists a χ-pair such that $\phi(0) = \Phi(\chi)$.

4. Proof of Theorem 2

To prove Theorem 2 we need the following known lemmas (see, for example, [11]).

Lemma 1. *Let a sequence $\{\alpha_n\}$, $\alpha_n \in \Gamma^*$, tends to the unit character $\iota \in \Gamma^*$, i.e. $\iota(\gamma) = 1$ for all $\gamma \in \Gamma$. Then there exists a sequence of functions $\{\epsilon_n\} \subset H^\infty(\alpha_n)$ with $\|\epsilon_n\|_\infty \le 1$ such that $\epsilon_n \to 1$ with respect to the Lebesgue measure on $\mathbb{T}$.*

Proof. Let us use one of the characteristic properties of groups of Widom type with the conditions of the direct Cauchy theorem. Namely, $m^\infty(\alpha_n) \to 1$ as $\alpha_n \to \iota$. Let ϵ_n be the extremal function from $H^\infty(\alpha_n)$, normalized by the conditions $\|\epsilon_n\| = 1$ and $\epsilon_n(0) > 0$. Hence $\epsilon_n(0) = m^\infty(\alpha_n)$. The inequality $\int_{\mathbb{T}} |1-\epsilon_n|^2\,dm \le 2 - 2\epsilon_n(0)$ shows that $\epsilon_n \to 1$ in the L^2-metric, and therefore with respect to the measure ∎

Lemma 2. *Let a sequence $\{\alpha_n\}$, $\alpha_n \in \Gamma^*$, tends to a character $\tilde{\alpha}$. Then $k^{\alpha_n} \to k^{\tilde{\alpha}}$ in L^2.*

Proof. Let $\{\epsilon_n\}$ be a sequence of functions from Lemma 1, constructed with respect to the sequence of characters $\{\alpha_n\tilde{\alpha}^{-1}\}$, i.e. $\epsilon_n \in H^\infty(\alpha_n\tilde{\alpha}^{-1})$. First we show that $k^{\alpha_n} - \epsilon_n k^{\tilde{\alpha}} \to 0$ in L^2. In fact,

$$\begin{aligned}\|k^{\alpha_n} - \epsilon_n k^{\tilde{\alpha}}\|^2 &\le \|k^{\alpha_n}\|^2 - 2\epsilon_n(0)k^{\tilde{\alpha}}(0) + \|k^{\tilde{\alpha}}\|^2 \\ &= \|k^{\alpha_n}\|^2 - 2\epsilon_n(0)\|k^{\tilde{\alpha}}\|^2 + \|k^{\tilde{\alpha}}\|^2.\end{aligned}$$

The last value tends to zero, because $\|k^{\alpha_n}\| = m^2(\alpha_n) \to \|k^{\tilde{\alpha}}\| = m^2(\tilde{\alpha})$. Next,

$$\|k^{\alpha_n} - k^{\tilde{\alpha}}\| \le \|k^{\alpha_n} - \epsilon_n k^{\tilde{\alpha}}\| + \|(1-\epsilon_n)k^{\tilde{\alpha}}\|.$$

The function $[(1-\epsilon_n)k^{\tilde{\alpha}}]^2$ has the absolutely integrable majorant $|2k^{\tilde{\alpha}}|^2$ and since $\epsilon_n \to 1$ with respect to the Lebesgue measure on $\mathbb{T}$, the Lebesgue dominated convergence theorem finishes the proof ∎

Lemma 3. *Let* $\nu = \alpha[\Delta]$. *Then* $\Delta\overline{k^{\alpha}} = \text{const}\, k^{\nu\bar{\alpha}}$ *for all* $\alpha \in \Gamma^*$.

Proof. The proof follows immediately from the formula for the orthogonal complement of $H^2(\alpha)$ ∎

Lemma 4. *Let* $\tilde{\Delta} \in H^{\infty}(\tilde{\nu})$ *be an inner divisor of the inner function* $(k^{\alpha})_{in}$. *Then:*

(i) $\frac{k^{\alpha}}{\tilde{\Delta}} = \text{const}\, k^{\alpha\tilde{\nu}^{-1}}$.

(ii) $\tilde{\Delta}$ *is a divisor of* $(k^{\nu\overline{\alpha_{out}}})_{in}$, *where* $\alpha_{out} = \alpha[(k^{\alpha})_{out}]$.

Proof. The first statement is a direct corollary of the definition. Lemma 3 yields the second statement ∎

Proof of Theorem 2. Let (f, ϕ) be a χ-extremal pair. In what follows we normalize the pair by the condition $\phi(0) > 0$, or $\phi(0) = \Phi(\chi)$. Let $\{\alpha_n\}$ and $\{x_n\}$ be extremal sequences,

$$\lim_{n\to\infty} \frac{|x_n(0)|}{\|k^{\alpha_n\chi}\|\,\|\sqrt{D(\alpha_n)}x_n\|} = M(\chi). \tag{4.1}$$

Since Γ^* is compact, passing to a subsequence if necessary we may assume the sequence $\{\alpha_n\}$ is convergent. Let $\tilde{\alpha} = \lim \alpha_n$ and $\tilde{\chi} = \tilde{\alpha}\chi$. Let us normalize x_n by the conditions $\|\sqrt{D(\alpha_n)}x_n\| = \|k^{\alpha_n\chi}\|$ and $x_n(0) > 0$. First, we show that

$$\begin{bmatrix} P_+(\alpha_n\beta)(fx_n) \\ \phi x_n \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ k^{\tilde{\chi}} \end{bmatrix} \quad \text{in } L^2(\mathbb{C}^2) \tag{4.2}$$

and

$$\sqrt{1-|f|^2-|\phi|^2}\,x_n \to 0 \quad \text{in } L^2. \tag{4.3}$$

In fact,

$$\begin{aligned}
&\|\sqrt{1-|f|^2-|\phi|^2}\,x_n\|^2 + \left\| \begin{bmatrix} P_+(\alpha_n\beta)(fx_n) \\ \phi x_n \end{bmatrix} - \begin{bmatrix} 0 \\ k^{\alpha_n\chi} \end{bmatrix} \right\|^2 \\
&= \langle (1-|f|^2-|\phi|^2)x_n, x_n\rangle + \langle P_+(\alpha_n\beta)fx_n, fx_n\rangle \\
&\quad + \langle \phi x_n, \phi x_n\rangle - 2\phi(0)x_n(0) + k^{\alpha_n\chi}(0) \\
&= \langle D(\alpha_n)x_n, x_n\rangle - 2\phi(0)x_n(0) + k^{\alpha_n\chi}(0) \\
&= k^{\alpha_n\chi}(0) - 2\frac{1}{M(\chi)}x_n(0)\frac{\|k^{\alpha_n\chi}\|}{\|\sqrt{D(\alpha_n)}x_n\|} + k^{\alpha_n\chi}(0) \\
&= 2\frac{k^{\alpha_n\chi}(0)}{M(\chi)}\left\{ M(\chi) - \frac{|x_n(0)|}{\|k^{\alpha_n\chi}\|\,\|\sqrt{D(\alpha_n)}x_n\|} \right\}.
\end{aligned}$$

Combining with (4.1) we obtain that the last value tends to zero. Since $k^{\alpha_n\chi} \to k^{\tilde{\chi}}$ in L^2 (see Lemma 2) we get (4.2) and (4.3).

Now we are in the position to prove the third property of a χ-extremal pair. Note that the convergence of $\{\phi x_n\}$ implies that of $\{\phi_{out}x_n\}$, therefore ϕ_{in} is a divisor of $k^{\tilde{\chi}}_{in}$ and

$$\phi_{out}x_n \to \frac{k^{\tilde{\chi}}}{\phi_{in}} \in H^2. \tag{4.4}$$

Let $\{\epsilon_n\}$ be a sequence of functions from Lemma 1, constructed with respect to the character sequence $\{\alpha_n\tilde{\alpha}^{-1}\}$. Consider functions of the form $\overline{\epsilon_n\phi_{out}f x_n} \in L^2(\overline{\chi_{out}\tilde{\alpha}\beta})$. Arguing as in the proof of Lemma 2, we show that

$$\overline{\epsilon_n\phi_{out}f x_n} = \left(\frac{\overline{\phi_{out}}}{\phi_{out}}f\right)(\overline{\epsilon_n}\phi_{out}x_n) \to \left(\frac{\overline{\phi_{out}}}{\phi_{out}}f\right)\left(\frac{k^{\tilde{\chi}}}{\phi_{in}}\right)$$

in L^2 (see (4.4)). At the same time

$$P_+(\overline{\chi_{out}\tilde{\alpha}\beta})\overline{\epsilon_n\phi_{out}f x_n} = P_+(\overline{\chi_{out}\tilde{\alpha}\beta})\overline{\epsilon_n\phi_{out}P_+(\alpha_n\beta)f x_n}$$

and (4.2) yields that

$$\|P_+(\overline{\chi_{out}\tilde{\alpha}\beta})\overline{\epsilon_n\phi_{out}}P_+(\alpha_n\beta)f x_n\| \le \|P_+(\alpha_n\beta)f x_n\| \to 0.$$

Therefore,

$$\left(\frac{\overline{\phi_{out}}}{\phi_{out}}f\right)\left(\frac{k^{\tilde{\chi}}}{\phi_{in}}\right) \in H^2_\perp(\overline{\chi_{out}\tilde{\alpha}\beta}).$$

It means that this function may be represented as

$$\left(\frac{\overline{\phi_{out}}}{\phi_{out}}f\right)\left(\frac{k^{\tilde{\chi}}}{\phi_{in}}\right) = \Delta\bar{g} \qquad (g \in H^2,\, g(0)=0) \tag{4.5}$$

To complete this part of the proof we have just to use Lemmas 3 and 4, and introduce some notation. Since $k^{\tilde{\chi}} = k_{in}^{\tilde{\chi}}k_{out}^{\tilde{\chi}}$ and

$$\bar{\Delta}k_{out}^{\tilde{\chi}} = \text{const}\,\bar{\Delta}k^{\tilde{\chi}_{out}} = \text{const}\,\overline{k^{\nu\overline{\tilde{\chi}_{out}}}} = \overline{k_{in}^{\nu\overline{\tilde{\chi}_{out}}}k_{out}^{\tilde{\chi}}}$$

we get from (4.5)

$$\left(\frac{\overline{\phi_{out}}}{\phi_{out}}f\right)\left(\frac{k_{in}^{\tilde{\chi}}}{\phi_{in}}\right)\overline{k_{in}^{\nu\overline{\tilde{\chi}_{out}}}k_{out}^{\tilde{\chi}}} = \bar{g} \qquad (g \in H^2,\, g(0)=0).$$

Let us define the function

$$\tilde{\Delta} = \frac{[k^{\nu\overline{\tilde{\chi}_{out}}}]_{in}}{[(k^{\tilde{\chi}})_{in}/\phi_{in}]}. \tag{4.6}$$

This function is an inner one in H^∞, because $(k^{\tilde{\chi}})_{in}$ is a divisor of $(k^{\nu\overline{\tilde{\chi}_{out}}})_{in}$ (see Lemma 4). Denote $s = -\frac{g}{k_{out}^{\tilde{\chi}}}$ with $s(0)=0$. This function belongs to H^∞, because the denominator of the fraction is an outer function and its boundary values are bounded by 1 a.e. on $\mathbb{T}$.

Using (4.3), we prove the second property of χ-extremal pairs. Since

$$\phi\sqrt{1-|f|^2-|\phi|^2}x_n \to 0,$$

according to (4.2) we have $\sqrt{1-|f|^2-|\phi|^2}k^{\tilde{\chi}} = 0$, or $1-|f|^2-|\phi|^2 = 0$ a.e. on $\mathbb{T}$.

Let us turn now to the first property. Let (f,ϕ) and $(\tilde{f},\tilde{\phi})$ be χ-extremal pairs with $\phi(0)>0$ and $\tilde{\phi}(0)>0$. Then $(\frac{f+\tilde{f}}{2}, \frac{\phi+\tilde{\phi}}{2})$ is χ-extremal as well. With the help of Theorem 2/(ii) we get

$$1 \ge \left\|\left(\frac{f+\tilde{f}}{2}, \frac{\phi+\tilde{\phi}}{2}\right)\right\|^2 + \left\|\left(\frac{f-\tilde{f}}{2}, \frac{\phi-\tilde{\phi}}{2}\right)\right\|^2 = 1 + \left\|\left(\frac{f-\tilde{f}}{2}, \frac{\phi-\tilde{\phi}}{2}\right)\right\|^2.$$

Therefore $f = \tilde{f}$ and $\phi = \tilde{\phi}$, and the proof is completed ∎

Proof of Corollary of Theorem 2. A straightforward computation shows that $\|f_{\mathcal{E}}\| \le 1$. Since $f_{\mathcal{E}} - f$ is a function of bounded characteristic and the denominator in (2.8) is an outer function $1 - s\mathcal{E}$, we have $f_{\mathcal{E}} - f \in H^\infty$. Hence $f_{\mathcal{E}} \in \mathcal{N}(f_\perp)$. Moreover, $|f_{\mathcal{E}}| = 1$ a.e. on $\mathbb{T}$ whenever $|\mathcal{E}| = 1$ a.e. on $\mathbb{T}$. It was proved in [7] that there always exists such a function $\mathcal{E}$ in $H^\infty(\alpha)$ for all $\alpha \in \Gamma^*$ (any extremal solution of the finite Nevanlinna-Pick problem is a unimodular function) ∎

5. Maximal and minimal χ-extremal solutions

We prove the existence of maximal and minimal solutions of the character-automorphic Nehari problem among all the χ-extremal solutions and prove some of their properties. In particular, the maximal χ-extremal solution gives the extremum to "entropy functional".

Proposition 5.1. *There exists a χ-extremal pair $(\tilde{f}, \tilde{\phi})$ such that*

$$\tilde{\phi}(0) = \sup_{\chi\in\Gamma^*} \Phi(\chi). \tag{5.1}$$

The function $\tilde{f}$ is a solution of the extremal problem

$$\inf_{f\in\mathcal{N}(f_\perp)} I(f) = I(\tilde{f}) \qquad \textit{where} \;\; I(f) = \int_{\mathbb{T}} \log(1-|f|^2)^{-\frac{1}{2}}\, dm. \tag{5.2}$$

Proof. Let $\{f_n\}_{n\ge0}$ be an extremal sequence, $\inf_{f\in\mathcal{N}(f_\perp)} I(f) = \lim_{n\to\infty} I(f_n)$. We define a sequence of outer character-automorphic functions by the conditions $1-|f_n|^2 = |\phi_n|^2$ a.e. on $\mathbb{T}$ and $\phi_n(0) > 0$. Note that $\phi_n(0) \to \sup_{f\in\mathcal{N}(f_\perp)} \exp\{-I(f)\}$. Consider the harmonic continuation of the pairs (f_n, ϕ_n) inside the disk $\mathbb{D}$. Since $f_n \in \mathcal{N}(f_\perp)$, we have $\epsilon_n = f_n - f_0 \in H^\infty(\beta)$ and $\|\epsilon_n\|_\infty \le 2$. Let us pass to subsequences $\{\epsilon_{n_k}\}$ and $\{\phi_{n_k}\}$ which converge uniformly on compact subsets of $\mathbb{D}$, and let

$$(\tilde{\epsilon}, \tilde{\phi}) = \lim_{n_k\to\infty} (\epsilon_{n_k}, \phi_{n_k}).$$

We claim that $\tilde{f} = f_0 + \tilde{\epsilon}$ is an extremal function. Note that $\|(\tilde{f}, \tilde{\phi})(\zeta)\| \le 1$ $(\zeta \in \mathbb{D})$ because

$$\|(f_n, \phi_n)(\zeta)\| = \left\| \int_{\mathbb{T}} \frac{1-|\zeta|^2}{|t-\zeta|^2} (f_n, \phi_n)\, dm \right\| \le \int_{\mathbb{T}} \frac{1-|\zeta|^2}{|t-\zeta|^2} \|(f_n, \phi_n)\|\, dm = 1.$$

Besides, $P_\perp(\beta)\tilde{f} = P_\perp(\beta)(f_0 + \tilde{\epsilon}) = P_\perp(\beta) f_0 = f_\perp$. Therefore $(\tilde{f}, \tilde{\phi})$ is a χ-pair. Using this fact, we have

$$\sup_{f\in\mathcal{N}(f_\perp)} \exp\{-I(f)\} = \tilde{\phi}(0) \le \sup_{\chi\in\Gamma^*} \Phi(\chi). \tag{5.3}$$

On the other hand, for any χ there always exists a χ-extremal pair (f, ϕ), and hence

$$\Phi(\chi) = \phi(0) \le \phi_{out}(0) = \exp\{-I(f)\} \le \sup_{f\in\mathcal{N}(f_\perp)} \exp\{-I(f)\}. \tag{5.4}$$

Passing to the supremum in (5.4) over χ and comparing with (5.3), we get (5.1) and (5.2) ∎

Proposition 5.2. *There exists a χ-extremal pair $(\tilde{f}, \tilde{\phi})$ such that*

$$\tilde{\phi}(0) = \inf_{\chi \in \Gamma^*} \Phi(\chi).$$

In this case, the function $\tilde{\Delta}$ from the property (iii) *of heorem 2 is a divisor of the function Δ, i.e. there is a function $\tilde{s} \in H^\infty$ with $\tilde{s}(0) = 0$ such that*

$$\Delta \tilde{\phi}_{out} \overline{\tilde{f}} + \overline{\tilde{\phi}_{out}} \tilde{s} = 0.$$

Proof. First, we define the character $\tilde{\alpha}$ as a limit point for an extremal sequence $\{\alpha_n\}$:

$$\sup_{\alpha \in \Gamma^*} \sup_{x \in H^2(\alpha)} \frac{|x(0)|}{\|\sqrt{D(\alpha)} x\|} = \lim_{n \to \infty} \sup_{x \in H^2(\alpha_n)} \frac{|x(0)|}{\|\sqrt{D(\alpha_n)} x\|}$$

$$\lim_{n \to \infty} \alpha_n = \tilde{\alpha}.$$

We are going to prove that

$$\sup_{\chi \in \Gamma^*} M(\chi) = \lim_{n \to \infty} \sup_{x \in H^2(\alpha_n)} \frac{|x(0)|}{\|k^{\nu \tilde{\alpha}^{-1} \alpha_n}\| \, \|\sqrt{D(\alpha_n)} x\|} = M(\nu \tilde{\alpha}^{-1}).$$

Indeed,

$$\begin{aligned}
\sup_{\chi \in \Gamma^*} M(\chi) &= \sup_{\chi \in \Gamma^*} \sup_{\alpha \in \Gamma^*} \sup_{x \in H^2(\alpha)} \frac{|x(0)|}{\|k^{\alpha \chi}\| \, \|\sqrt{D(\alpha)} x\|} \\
&\le \frac{1}{\Delta(0)} \sup_{\alpha \in \Gamma^*} \sup_{x \in H^2(\alpha)} \frac{|x(0)|}{\|\sqrt{D(\alpha)} x\|} \\
&= \frac{1}{\Delta(0)} \lim_{n \to \infty} \sup_{x \in H^2(\alpha_n)} \frac{|x(0)|}{\|\sqrt{D(\alpha_n)} x\|} \\
&= \lim_{n \to \infty} \sup_{x \in H^2(\alpha_n)} \frac{|x(0)|}{\|k^{\nu \tilde{\alpha}^{-1} \alpha_n}\| \, \|\sqrt{D(\alpha_n)} x\|} \\
&\le \sup_{\alpha \in \Gamma^*} \sup_{x \in H^2(\alpha)} \frac{|x(0)|}{\|k^{\nu \tilde{\alpha}^{-1} \alpha}\| \, \|\sqrt{D(\alpha)} x\|} \\
&= M(\nu \tilde{\alpha}^{-1}) \\
&\le \sup_{\chi \in \Gamma^*} M(\chi).
\end{aligned}$$

Hence $(\tilde{f}, \tilde{\phi})$ is a χ-extremal pair as $\chi = \nu \tilde{\alpha}^{-1}$. Let us show that $\tilde{\Delta}$, defined by (4.6), is a divisor of Δ. In this case $\tilde{\chi} = \nu \tilde{\alpha}^{-1} \tilde{\alpha} = \nu$, and therefore $k^{\tilde{\chi}} = k^{\nu} = \Delta \Delta(0)$, i.e. $(k^{\tilde{\chi}})_{in} = \Delta$ and $\tilde{\chi}_{out} = \iota$ is the unit character. Thus $\tilde{\Delta} = \phi_{in}$ and ϕ_{in} is a divisor of Δ ∎

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