Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 15 (1996), No. 2, 299–307

On Measures of Non-Compactness in Regular Spaces

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Abstract. Previous results on non-compactness obtained in [11-13] are extended to regular spaces of measurable functions, and new criteria for the μ-compactness of sets and operators are proved. An application of the abstract results to elliptic boundary problems is given as well.

Keywords: *Regular spaces, measures of non-compactness, measures of non-equiboundedness, Lebesgue spaces, Sobolev spaces, embedding operators, elliptic boundary value problems*

AMS subject classification: 47 H 09, 46 E 30, 46 E 35, 47 B 07, 47 B 38

1. Introduction

Let Ω be some subset of $\mathbb{R}^n$ and μ a non-negative continuous measure on a σ-algebra of subsets of Ω such that $\mu(\Omega) < \infty$. Throughout this paper, P_D denotes the operator of multiplication by the characteristic function of a measurable subset $D \subseteq \Omega$.

Definition 1 (see [5, 14]). A Banach space E of measurable functions is called *regular* if

(a) $\|x\|_E \le \|y\|_E$ for all $y \in E$ and measurable function x with $|x(t)| \le |y(t)|$

(b) $\lim_{n\to\infty} \|P_{D_n} x\| = 0$ for every $x \in E$ and decreasing sequence of measurable sets $\{D_n\}$ with empty intersection.

is fulfilled.

Remark 1. It is well-known (see [5, 14]) that all Lebesgue spaces, Lorentz spaces and Orlicz spaces whose generating N-function satisfies a Δ_2-condition are regular spaces.

Definition 2 (see [5, 14]). A set U in a Banach space E of μ-measurable functions is called *μ-compact* if it is compact in the topology induced by μ-convergence, i.e. convergence in the measure μ.

Definition 3 (see [1, 9]). Given a bounded subset U of a normed space E, the (Hausdorff) *measure of non-compactness* $\chi_E(U) = \chi(U)$ is defined as the infimum of all $\varepsilon > 0$ such that there exists a finite ε-net for U in E.

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ISSN 0232-2064 / \$ 2.50 © Heldermann Verlag

Definition 4 (see [1, 9]). Let E and G be Banach spaces. The *upper χ-norm* of a bounded linear map $A: E \to G$ is defined by

$$\|A\|^{(\chi)} = \inf \left\{ k > 0 \middle| \chi_G(A(U)) \leq k\chi_E(U) \text{ for every bounded } U \subset E \right\}.$$

Remark 2. In [6] the equality $\|A\|^{(\chi)} = \chi(AS)$ was proved, where $S = S(E) = \{x \in E : \|x\|_E = 1\}$ is the unit sphere in E.

We denote the measure

$$\nu(U) = \limsup_{\mu(D)\to 0} \sup_{u \in U} \|P_D u\|_E \tag{1}$$

for $U \subset E$, where E is a regular space.

Remark 3. The characteristic (1) was introduced and studied for Lebesgue spaces E in [10] and, independently, in [3] (see also [4]). For regular spaces E the characteristic (1) was considered first in [2].

A bounded subset U of a regular space E is compact if and only if it is μ-compact and $\nu(U) = 0$. Check of μ-compactness presents a real challenge. In this paper we shall propose a necessary and sufficient criterion of μ-compactness for all normed spaces of μ-measurable functions which can be embedded into the Lebesgue space. The criterion is reduced to the equality $\nu = \chi$. The theory of measures of non-compactness has a lot of applications. There exists a large amount of literature devoted to this subject (see, e.g., [1, 9] and the references therein).

2. The results

We shall research a conjunction between the Hausdorff measure of non-compactness χ and the measure ν defined by (1) for any sets and operators.

Lemma 1. *The measure ν has the following properties:*

(a) $\nu(U) \leq \nu(V)$ *if* $U \subseteq V$.

(b) $\nu(U \cup V) = \max\{\nu(U), \nu(V)\}$.

(c) $\nu(U) = \nu(\overline{U})$, *where* $\overline{U}$ *denotes the closure of* U.

(d) $\nu(tU) = |t|\nu(U)$ *for all* $t \in \mathbb{R}$.

(e) $\nu(\text{conv}U) = \nu(U)$, *where* $\text{conv}\, U$ *denotes the convex hull of* U.

(f) $\nu(U + V) \leq \nu(U) + \nu(V)$, *where* $U + V = \{u + v : u \in U \text{ and } v \in V\}$.

(g) $|\nu(U) - \nu(V)| \leq k\,\text{dist}\,(U, V)$, *where* $\text{dist}\,(U, V)$ *denotes the Hausdorff distance, and the constant k does not depend on U and V.*

(h) $\nu(U) = 0$ *if U is relatively compact, but $\nu(U) = 0$ does not imply that U is relatively compact.*

Proof. The proof follows directly from the definition (1) and a well-known compactness criterion in regular spaces (see, e.g., [14]) ∎

Lemma 2. *Let U be a bounded subset of a regular space E. Then $\nu(U) \le \chi(U)$.*

Proof. For the unit ball $B = B(E) = \{x \in E : \|x\|_E \le 1\}$ in E we have $\nu(B) = 1$, since $\|P_D x\|_E^{-1} P_D x \in B$ for every measurable $D \subset \Omega$ and $x \in E$. Let $\varepsilon > 0$. Applying Lemma 1 to any $[\chi(U) + \varepsilon]$-net $C_\varepsilon = \{c_1, \dots, c_m\}$ for the set U we obtain

$$\nu(U) \le \nu(C_\varepsilon + [\chi(U) + \varepsilon]B) \le \chi(U) + \varepsilon$$

which proves the assertion ∎

Let $u_0 = u_0(t)$ be a *unit* in E, i.e. a fixed non-negative function such that $\operatorname{supp} u_0 = \operatorname{supp} E$ (see [14]), and $T > 0$. In what follows, we denote by $[x]_{u_0,T}$ $(x \in E)$ the *truncation*

$$[x]_{u_0,T}(t) = \min\{|x(t)|, Tu_0(t)\} \operatorname{sgn} x(t).$$

Lemma 3. *Let U be a bounded and μ-compact subset of a regular space E. Then*

$$\chi(U) \le \sup_{x \in U} \left\| x - [x]_{u_0,T} \right\|_E \tag{2}$$

and $\chi(U) \le \nu(U)$.

Proof. Let $D(x, u_0, T) = \{t \in \Omega : |x(t)| \ge Tu_0(t)\}$. By [14: Theorems 1 and 3] we know that

$$\lim_{T \to \infty} \sup_{x \in U} \mu[D(x, u_0, T)] = 0. \tag{3}$$

Furthermore, for every $T > 0$ we have

$$\limsup_{\mu(D) \to 0} \sup_{x \in U} \left\| P_D [x]_{u_0,T} \right\|_E \le \limsup_{\mu(D) \to 0} T \|P_D u_0\|_E = 0.$$

Since U is μ-compact, from [14: Theorem 15] it follows that the set $\{[x]_{u_0,T} : x \in U\}$ is compact in E. Consequently,

$$\chi(U) \le \sup_{x \in U} \left\| x - [x]_{u_0,T} \right\|_E \le \sup_{x \in U} \left\| P_{D(x,u_0,T)} x \right\|_E$$

which together with (3) proves the statement ∎

Combining Lemmas 1-3 we arrive at the following

Theorem 1. *Let U be a bounded subset of a regular space E. Then $\nu(U) \le \chi(U)$, and $\nu(U) = \chi(U)$ if U is μ-compact.*

We are now going to apply Theorem 1 to a particularly important class of regular spaces.

Theorem 2. *Let U be a bounded subset of $L^p(\Omega, \mu)$, where $L^p(\Omega, \mu)$ is the space of μ-measurable functions with the usual norm*

$$\|x\|_{L^p(\Omega,\mu)} = \left(\int_\Omega |x|^p d\mu \right)^{1/p} \qquad (1 \le p < \infty).$$

Then U is μ-compact if and only if $\chi(V) = \nu(V)$ for every $V \subseteq U$.

Proof. Let $U \subset L^p(\Omega,\mu)$ and $\chi(V) = \nu(V)$ for every $V \subseteq U$. We shall show that U is μ-compact. If $\chi(U) = 0$, the assertion is certainly true. So let $\chi(U) = \nu(U) > 0$. Obviously, U is μ-compact if the set $[U]_T = \{[x]_T : x \in U\}$ is μ-compact for every $T > 0$, where

$$[x]_T(t) = \begin{cases} x(t) & \text{if } |x(t)| \le T \\ 0 & \text{if } |x(t)| > T. \end{cases}$$

Suppose that the set $[U]_{T_1}$ is not μ-compact for some $T_1 > 0$. Then there exists a sequence $\{x_n\} \subset U$ such that, for all $n \neq m$,

$$\rho\big([x_n]_{T_1},[x_m]_{T_1}\big) \ge c \qquad \text{where } \rho(x,y) = \int_\Omega \frac{|x(t)-y(t)|}{1+|x(t)-y(t)|}\,d\mu$$

and the constant c is independent of n and m. By hypothesis, for the set $V = \{x_1, x_2, x_3, \ldots\}$ we have $\chi(V) = \nu(V)$. Let $0 < \varepsilon < \chi(V)$, and let $\{c_1,\ldots,c_m\}$ be a finite $[\chi(V)+\varepsilon]$-net for V. For $T_2 > T_1$ large enough we get then

$$\sup_{x\in U}\sup_{1\le i\le m} \big\|P_{D(x,T_2)}c_i\big\|_{L^p(\Omega,\mu)} < \varepsilon$$

where $D(x,T) = \{t \in \Omega : |x(t)| > T\}$. Choose $n \neq m$ such that, for $l = n$ or $l = m$,

$$\big\|P_{D(x_l,T_2)}x_l\big\|_{L^p(\Omega,\mu)} \ge \nu(V) - \varepsilon = \chi(V) - \varepsilon$$

and $\|x_l - c_i\|_{L^p(\Omega,\mu)} \le \chi(V) + \varepsilon$ for a suitable $i \in \{1,\ldots,m\}$. Consequently,

$$\begin{aligned} \big\|[x_l]_{T_2} - c_i\big\|^p_{L^p(\Omega,\mu)} &\le \big\|x_l - c_i\big\|^p_{L^p(\Omega,\mu)} - \big\|P_{D(x_l,T_2)}(x_l - c_i)\big\|^p_{L^p(\Omega,\mu)} + \varepsilon^p \\ &\le [\chi(V)+\varepsilon]^p - [\chi(V)-2\varepsilon]^p + \varepsilon^p \\ &\le k_1\varepsilon^p. \end{aligned}$$

From this we further obtain

$$\big\|[x_n]_{T_2} - [x_m]_{T_2}\big\|_{L^p(\Omega,\mu)} \le k_2\varepsilon$$

and

$$\begin{aligned} c &\le \rho\big([x_n]_{T_1},[x_m]_{T_1}\big) \le \rho\big([x_n]_{T_2},[x_m]_{T_2}\big) \\ &\le \int_\Omega \big|[x_n]_{T_2} - [x_m]_{T_2}\big|\,d\mu \le k_3 \big\|[x_n]_{T_2} - [x_m]_{T_2}\big\|_{L^p(\Omega,\mu)} \\ &\le k_2k_3\varepsilon \end{aligned}$$

where the constants k_i $(i = 1,2,3)$ do not depend on ε. This contradiction shows that U is μ-compact. Conversely, if U is μ-compact, then $\chi(V) = \nu(V)$ for every $V \subseteq U$, by Theorem 1 ∎

Theorem 3. *Let G be a Banach space and $A : G \to L^p(\Omega,\mu)$ $(1 \le p < \infty)$ a bounded linear operator. Let $S = S(G) = \{x \in G : \|x\|_G = 1\}$ be the unit sphere in G, and suppose that $\nu(AV) = \chi(AV)$ for every $V \subseteq S$. Then A is μ-compact, i.e. if U is an arbitrary bounded subset in G, then AU is μ-compact in $L^p(\Omega,\mu)$.*

Proof. It is enough to show that rAB is μ-compact for any r, where $B = B(G)$ is the unit ball in G. Using Lemma 1 and the properties of χ (see [1: Subsection 1.1.4]), we have

$$\begin{aligned}\nu(rAB) &= \nu(rA\,\mathrm{conv}S) = r\nu(\mathrm{conv}AS)\\ &= r\nu(AS) = r\chi(AS) = \chi(r\,\mathrm{conv}AS) = \chi(rAB)\end{aligned}$$

which by Theorem 2 implies the μ-compactness of rAB ∎

Now let Ω be a domain in $\mathbb{R}^n$ and m the Lebesgue measure on Ω. For $1 \le p < \infty$ and $s \in \mathbb{N}$ we consider the following function spaces:

$L^p(\Omega) = L^p$	is the Lebesgue space
$W^{s,p}(\Omega) = W^{s,p}$	is the Sobolev space
$L^{s,p}(\Omega) = L^{s,p}$	is a space of generalized functions on Ω defined by the seminorm $\|u\|_{L^{s,p}} = \|\nabla_s u\|_{L^p} = \big(\int_\Omega \big[\sum_{\lvert i\rvert=s} \lvert D^i u(x)\rvert^2\big]^{p/2} dx\big)^{1/p}$
$\tilde{L}^{s,p}(\Omega) = \tilde{L}^{s,p}$	is $L^{s,p}$ with the norm $\|\nabla_s u\|_{L^p} + \|u\|_{L^p(\omega)}$ (see [8]), where ω is some (non-empty) open set with compact closure $\overline{\omega} \subset \Omega$
$C^{0,1}(\Omega) = C^{0,1}$	is the space of all Lipschitz functions on an arbitrary compact subset $Q \subset \Omega$
$W_0^{s,p}(\Omega) = W_0^{s,p}$	is the closures of $C_0^\infty(\Omega)$ in the norm of $W^{s,p}$
$L_0^{s,p}(\Omega) = L_0^{s,p}$	is the closure of $C_0^\infty(\Omega)$ in the norm of $L^{s,p}$
C	is the closed subspace of all constant functions on Ω.

Theorem 4. *Let E be a regular space of m-measurable functions on a domain $\Omega \subset \mathbb{R}^n$ with $m(\Omega) < \infty$. Then for $S^{1,p} \in \{W^{1,p},\ W_0^{1,p},\ L_0^{1,p},\ L^{1,p}\}$ the equality*

$$\|I\|^{(\chi)} = \limsup_{m(D)\to 0}\ \sup_{x \in U_D} \frac{\|x\|_E}{\|x\|_{S^{1,p}}}$$

holds, where $U_D = \{x \in C^{0,1} \cap S^{1,p} : x = 0$ outside $D\}$, $I : S^{1,p} \to E$ is the embedding map for $S^{1,p} \in \{W^{1,p}, W_0^{1,p}, L_0^{1,p}\}$, and $I : L^{1,p}/C \to E/C$ is the embedding map modulo constant functions for $S^{1,p} = L^{1,p}$.

Proof. We shall consider only the case $I : L^{1,p}/C \to E/C$, since the proof is analogous for $S^{1,p} \in \{W^{1,p}, W_0^{1,p}, L_0^{1,p}\}$. The existence of the embedding $I : L^{1,p}/C \to E/C$ means that

$$\inf_{c \in C} \|u - c\|_E \le k\|\nabla u\|_{L^p} \qquad (u \in L^{1,p})$$

where the constant k does not depend on u.

Let $S = \{x \in L^{1,p} : \|\nabla x\|_{L^p} = 1\}$. By [8: Theorem 1.1.2 and Lemma 1.1.11] there exists a set $B_0' \subset S$ such that B_0' is bounded in $\tilde{L}^{1,p}$ and $S = B_0' + C$. By [8: Theorem 4.8.4], B_0' is m-compact. Therefore, since the embedding map $I : L^{1,p}/C \to E/C$ is bounded, we can choose any bounded m-compact set $B_0 \subset S$ such that $S = B_0 + C$.

Obviously, $\chi_{E/C}(S) = \chi_E(B_0)$. By Theorem 1 we have

$$\chi_E(B_0) = \nu_E(B_0) = \limsup_{m(D)\to 0} \sup_{x\in B_0} \|P_D x\|_E. \tag{4}$$

In the last limit, by [8: Theorem 1.1.5/1], we may assume without loss of generality that $B_0 \subset C^\infty(\Omega)$. In addition, we use the inequality

$$\chi_E(B_0) \le \sup_{x\in B_0} \big\|x - [x]_T\big\|_E \qquad (T > 0)$$

which follows from (2). In view of (3) we obtain

$$\chi_E(B_0) \le \limsup_{m(D)\to 0} \sup_{x\in U_D} \frac{\|x\|_E}{\|\nabla x\|_{L^p}}$$

since

$$x - [x]_T \in U_{D(x,T)} = \Big\{x \in C^{0,1}(\Omega) \cap L^{1,p}(\Omega) : x = 0 \ \text{ outside } \ D(x,T)\Big\}.$$

On the other hand, for every $x \in U_D$ there exists a constant $c_x \in C$ such that $\|\nabla x\|_{L^p}^{-1}(x - c_x) \in B_0$, i.e.

$$k_0 \|\nabla x\|_{L^p} \ge \|x - c_x\|_E \ge \|P_{\Omega\setminus D} c_x\|_E$$

where k_0 is independent of x. From this it follows that

$$|c_x| \le k_0 \frac{\|\nabla x\|_{L^p}}{\|P_{\Omega\setminus D} 1\|_E}$$

and

$$\limsup_{m(D)\to 0} \sup_{x\in U_D} \frac{\|P_D c_x\|_E}{\|\nabla x\|_{L^p}} \le \limsup_{m(D)\to 0} k_0 \frac{\|P_D 1\|_E}{\|P_{\Omega\setminus D} 1\|_E} = 0.$$

Thus from (4) we conclude that

$$\chi_E(B_0) \ge \limsup_{m(D)\to 0} \sup_{x\in U_D} \frac{\|x - c_x\|_E}{\|\nabla x\|_{L^p}} = \limsup_{m(D)\to 0} \sup_{x\in U_D} \frac{\|x\|_E}{\|\nabla x\|_{L^p}}.$$

The proof is complete ∎

Remark 4. It follows from Theorem 4 that the upper χ-norm of the embedding map $I : L^{1,p}/C \to L^q/C$ is a characteristic of non-compactness which has been assumed as a basic criterion of compactness in [8: Lemmas 4.2 and 4.4.1, and Subsections 4.8.1 and 4.8.2].

3. Applications

As an example of an application of our results we consider now the solvability of the *Neumann problem* for the linear operator

$$Ju = \sum_{|i|,|j|\le s} (-1)^{|i|} D^i(a_{ij} D^j u)$$

in the space $W^{s,2}(\Omega)$. Here the coefficients $a_{ij} \in L^\infty$ are assumed to satisfy the *boundedness condition*

$$\sup_{i,j} \|a_{ij}\|_{L^\infty} \le c_1 \tag{5}$$

as well as the *ellipticity condition*

$$Re \int_\Omega \sum_{|i|=|j|=s} a_{ij} D^i u \overline{D^j u}\, dx \ge c \|\nabla_s u\|^2_{L^2} \qquad (u \in L^{s,2}). \tag{6}$$

We say that $u \in W^{s,2}(\Omega)$ is a *generalized solution* of the Neumann problem $Au = f$ if

$$\int_\Omega \bar{v} Au\, dx = \int_\Omega \sum_{|i|,|j|\le s} a_{ij} D^i u \overline{D^j v}\, dx = \int_\Omega f \bar{v}\, dx$$

for any $v \in W^{s,2}$ and $f \in L^2$.

Lemma 4. *Let the embedding operator $I : L^{1,2}/C \to L^2/C$ be bounded. Then for all $u \in L^2 \cap L^{s,2}$ the estimate*

$$\sum_{k=0}^{s-1} \|\nabla_k u\|_{L^2} \le C(U(\varepsilon), n) \|\nabla_s u\|_{L^2} + C(\varepsilon)\|u\|_{L^2}$$

is true, where

$$C(U(\varepsilon), n) \le \frac{aU(\varepsilon)}{1 - aU(\varepsilon)}, \qquad a = n^{1/2} + 1, \qquad U(\varepsilon) = \sup_{m(D)<\varepsilon} \sup_{u \in U_D} \frac{\|u\|_{L^2}}{\|\nabla u\|_{L^2}}.$$

Proof. Given $\varepsilon > 0$ and $u \in L^2 \cap L^{s,2}$, and putting $T = \inf\{t : m(D(u,t)) \le \varepsilon\}$, by Theorem 4 we get

$$\begin{aligned}\|u\|_{L^2} &\le \big\||u| - T\big\|_{L^2_{D(u,T)}} + \big\||u| - T\big\|_{L^2(\Omega \setminus D(u,T))} + T[m(\Omega)]^{1/2} \\ &\le U(\varepsilon)\|\nabla u\|_{L^2} + 2T[m(\Omega)]^{1/2}.\end{aligned}$$

Now, following a similar reasoning as in [8: Proof of Theorem 4.8.2], we denote by Ω_ε any bounded subdomain Ω with a $C^{0,1}$-boundary such that $m(\Omega \setminus \Omega_\varepsilon) < \frac{\varepsilon}{2}$. Since $m[D(u,T)] \ge \varepsilon$ we have $m[D(u,T) \cap \Omega_\varepsilon] \ge \frac{\varepsilon}{2}$. Hence $\|u\|_{L^r(\Omega_\varepsilon)} \ge 2^{-1/r} T \varepsilon^{1/r}$ for any $r \ge 1$, and therefore

$$\|u\|_{L^2(\Omega)} \le U(\varepsilon)\|\nabla u\|_{L^2(\Omega)} + 2^{1-1/r}[m(\Omega)]^{1/2}\|u\|_{L^r(\Omega)}$$

where the embedding map from $L^{1,2}(\Omega_\varepsilon)$ into $L^r(\Omega_\varepsilon)$ is compact. The remaining part of the proof goes precisely along the line of [8: Proof of Lemma 4.10.2] ∎

Theorem 5. *Let*

$$\|I\|^{(\chi)} < \frac{\sqrt{c}}{a[\sqrt{4c_1} + \sqrt{c}]}$$

where I is the embedding map from $L^{1,2}/C$ into L^2/C, $a = \sqrt{n} + 1$, c and c_1 are the constants from (5) *and* (6). *Then, for* Reλ *large enough, the equation $Au + \lambda u = f$ has a unique generalized solution for each $f \in L^2$.*

Proof. By (6),

$$\mathrm{Re} \int_\Omega \sum_{|i|,|j| \le s} a_{ij} D^i u \overline{D^j u}\, dx \ge c \|\nabla_s u\|_{L^2}^2 - c_1 \sum_{k=0}^{s-1} \|\nabla_k u\|_{L^2}^2.$$

By Theorem 4 and the assumption on $\|I\|^{(\chi)}$ there exists an $\varepsilon > 0$ such that

$$2 \frac{a^2 U(\varepsilon)^2}{[1 - aU(\varepsilon)]^2} \le \frac{c}{2c_1}.$$

Consequently,

$$\mathrm{Re} \int_\Omega \Bigg(\sum_{|i|,|j| \le s} a_{ij} D^i u \overline{D^j u} + \lambda |u|^2 \Bigg) dx \ge \frac{1}{2c_1} \|\nabla_s u\|_{L^2(\Omega)}^2 + (\mathrm{Re}\lambda - c_2) \|u\|_{L^2(\Omega)}^2,$$

i.e. the "coercivity" condition

$$\sum_{k=0}^{s} \|\nabla_k u\|_{L^2}^2 \le \mathrm{const}\, \mathrm{Re} \int_\Omega \Bigg(\sum_{|i|,|j| \le s} a_{ij} D^i u \overline{D^j u} + \lambda |u|^2 \Bigg) dx$$

is fulfilled for sufficiently large Reλ. But this implies (see, e.g., [7: Theorem 2.9.1]) the assertion of our theorem ∎

Remark 5. In the special case when $I : L^{1,2}/C \to L^2/C$ is compact, i.e. $\|I\|^{(\chi)} = 0$, Theorem 5 implies [8: Theorem 4.10.2].

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Received 04.09.1995; in revised form 27.11.1995