On Absolute Summability Factors

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Abstract. By using for $\delta \geq 0$ so-called $[N, p_n; \delta]_k$-boundedness of series $\sum_{n=1}^{\infty} a_n$ and sequences $(\lambda_n)_{n=1}^{\infty}$ we prove $[N, p_n; \delta]_k$-summability of the series $\sum_{n=1}^{\infty} a_n \lambda_n$. This result generalizes a known one related to $[N, p_n]_k$-summability of series.

Keywords: Absolute summability of series, summability factors, infinite series

AMS subject classification: 40 F 05, 40 D 15, 40 G 05

1. Introduction

Let $\sum_{n=1}^{\infty} a_n$ be a given series and $(s_n)$ its sequence of partial sums. We denote by $(u_n^\alpha)$ with $\alpha > -1$ the sequence of $n$-th Cesàro means of order $\alpha$ of $(s_n)$. Let $k \geq 1$ and $\delta \geq 0$. The series $\sum a_n$ is said to be $|C, \alpha|_k$-summable if (see [6])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty,$$

and it is said to be $|C, \alpha; \delta|_k$-summable if (see [7])

$$\sum_{n=1}^{\infty} n^{\delta k + k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty.$$

In the special case when $\delta = 0$ or $\alpha = 1$, the $|C, \alpha; \delta|_k$-summability is the same as the $|C, \alpha|_k$- or $|C, 1; \delta|_k$-summability, respectively.

Let $(p_n)$ be any sequence of positive numbers such that

$$P_n = \sum_{\nu=0}^{n} p_{\nu} \to \infty \quad \text{as} \quad n \to \infty.$$

The transformation defined by

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{\nu} s_{\nu}$$
gives the sequence \((t_n)\) of \((N, p_n)\)-means of a sequence \((s_n)\), generated by the sequence of coefficients \((p_n)\) (see [8]).

Let as before \(k \geq 1\) and \(\delta \geq 0\). The series \(\sum a_n\) is said to be \([N, p_n]_k\)-summable if (see [1])

\[
\sum_{n=1}^{\infty} \left(\frac{p_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty,
\]

and it is said to be \([N, p_n; \delta]_k\)-summable if (see [4, 5])

\[
\sum_{n=1}^{\infty} \left(\frac{p_n}{p_n}\right)^{\delta k + k-1} |t_n - t_{n-1}|^k < \infty.
\]

In the special cases when \(\delta \leq 0\) or \(k = 1\) and \(\delta \leq 0\), the \([N, p_n; \delta]_k\)-summability is the same as the \([N, p_n]_k\) or \([N, p_n]_\delta\)-summability, respectively. The \([N, p_n]_k\) and \([N, p_n; \delta]_k\)-summability methods are totally different from each other. As a matter of fact one can see that \([N, p_n; \delta]_k\)-summability methods are different for different values of \(\delta\). Also if we take \(p_n = 1\) for all values of \(n\), then \([N, p_n; \delta]_k\)-summability reduces to \([C, 1; \delta]_k\)-summability.

At last, let again \(k \geq 1\) and \(\delta \geq 0\). The series \(\sum a_n\) is said to be \([N, p_n]_k\)-bounded if (see [2])

\[
\sum_{n=1}^{n} p_n |s_n|^k = O(P_n) \quad \text{for } n \to \infty,
\]

and it is said to be \([N, p_n; \delta]_k\)-bounded if (see [4, 5])

\[
\sum_{\nu=1}^{\infty} \left(\frac{p_{\nu}}{p_\nu}\right)^{\delta k} p_\nu |s_{\nu}|^k = O(P_n) \quad \text{for } n \to \infty.
\]

The \([N, p_n]_k\)- and \([N, p_n; \delta]_k\)-boundedness are totally different from each other. In the special cases when \(\delta \leq 0\) or \(k = 1\) and \(\delta \leq 0\), the \([N, p_n; \delta]_k\)-boundedness is the same as the \([N, p_n]_k\) and \([N, p_n]_\delta\)-boundedness, respectively.

In [3] the following theorem for \([N, p_n]_k\)-summability factors of infinite series is proved.

**Theorem A.** Let the series \(\sum a_n\) be \([N, p_n]_k\)-bounded and let the sequences \((\lambda_n)\) and \((p_n)\) satisfy for \(n \to \infty\) the conditions

(i) \(p_{n+1} = O(p_n)\)

(ii) \(\sum_{\nu=1}^{n} p_\nu |\lambda_\nu| = O(1)\)

(iii) \(P_n |\Delta \lambda_n| = O(p_n |\lambda_n|)\).

Then the series \(\sum a_n P_n \lambda_n\) is \([N, p_n]_k\)-summable for \(k \geq 1\).
2. The main result

Our aim is to generalize Theorem A to the case of $|N, p_n; \delta|_k$-summability. Thus we shall prove the following theorem.

**Theorem B.** Let the series $\sum a_n$ be $|N, p_n; \delta|_k$-bounded and let the sequences $(\lambda_n)$ and $(p_n)$ satisfy the conditions (i) - (iii) of Theorem A. If

$$\sum_{n=\nu+1}^{\infty} \left( \frac{p_n}{p_{n-1}} \right)^{\delta k - 1} \frac{1}{P_{n-1}} = O \left( \left( \frac{p_{\nu}}{p_{\nu-1}} \right)^{\delta k} \frac{1}{P_{\nu}} \right),$$

then the series $\sum a_n P_n \lambda_n$ is $|N, p_n; \delta|_k$-summable for $k \geq 1$ and $\delta \geq 0$.

Note that for $\delta < 0$ Theorem B implies Theorem A. Because, in this case the $|N, p_n; \delta|_k$-boundedness reduces to the $|N, p_n|_k$-boundedness and condition (1) reduces to

$$\sum_{n=\nu+1}^{\infty} \frac{p_n}{P_n P_{n-1}} = O \left( \frac{1}{P_{\nu}} \right)$$

which always holds.

We need the following lemma for the proof of Theorem B.

**Lemma (see [3]).** If the sequences $(\lambda_n)$ and $(p_n)$ satisfy conditions (ii) and (iii) of Theorem A, then $P_n |\lambda_n| = O(1)$ for $n \to \infty$.

3. Proof of Theorem B

Without any loss of generality we can assume that $a_0 = s_0 = 0$. Let $(T_n)$ denote the sequence of $(N, p_n)$-means of the series $\sum a_n P_n \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{\nu=0}^{n} \sum_{i=0}^{\nu} P_{\nu} a_i \lambda_i = \frac{1}{P_n} \sum_{\nu=0}^{n} (P_n - P_{\nu-1}) P_{\nu} a_{\nu} \lambda_{\nu}.$$

Then, for $n \geq 1$,

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{\nu-1} P_{\nu} a_{\nu} \lambda_{\nu}.$$

Using the Abel transformation, we get

$$T_n - T_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_{\nu} P_{\nu} s_{\nu} \lambda_{\nu} + \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_{\nu} P_{\nu} \Delta \lambda_{\nu} s_{\nu}
$$

$$- \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_{\nu} P_{\nu+1} s_{\nu} \lambda_{\nu+1} + p_n s_n \lambda_n
$$

$$=: T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.$$
To complete the proof of the theorem, by the Minkowski inequality for $k > 1$, it is sufficient to show that
\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{p_{n-1}} \right)^{\delta k + k - 1} |T_{n,r}|^k < \infty
\]
for $1 \leq r \leq 4$. Now applying the Hölder inequality with indices $k$ and $k'$ where $\frac{1}{k} + \frac{1}{k'} = 1$, we get
\[
\sum_{n=2}^{m+1} \left( \frac{p_n}{p_{n-1}} \right)^{\delta k + k - 1} |T_{n,1}|^k
\]
\[
\leq \sum_{n=2}^{m+1} \left( \frac{p_n}{p_{n-1}} \right)^{\delta k - 1} \frac{1}{p_{n-1}} \left\{ \sum_{\nu=1}^{n-1} (p_\nu |\lambda_\nu|^k p_\nu |s_\nu|^k) \right\} \left\{ \frac{1}{p_{n-1}} \sum_{\nu=1}^{n-1} p_\nu \right\}^{1-k}
\]
\[
= O(1) \sum_{\nu=1}^{m} (p_\nu |\lambda_\nu|^k p_\nu |s_\nu|^k) + O(1) |\lambda_m| \sum_{\nu=1}^{m} \left( \frac{p_\nu}{p_{n-1}} \right)^{\delta k} p_\nu |s_\nu|^k
\]
\[
= O(1) \sum_{\nu=1}^{m} |\lambda_\nu| \left( \frac{p_\nu}{p_{n-1}} \right)^{\delta k} p_\nu |s_\nu|^k
\]
\[
= O(1) \sum_{\nu=1}^{m} \Delta |\lambda_\nu| \sum_{i=1}^{\nu} \left( \frac{p_i}{p_{n-1}} \right)^{\delta k} p_i |s_i|^k + O(1) |\lambda_m| \sum_{\nu=1}^{m} \left( \frac{p_\nu}{p_{n-1}} \right)^{\delta k} p_\nu |s_\nu|^k
\]
\[
= O(1) \sum_{\nu=1}^{m} p_\nu |\lambda_\nu| + O(1) |\lambda_m| P_m
\]
\[
= O(1) \quad \text{for } m \to \infty
\]
by virtue of the hypotheses of the theorem and the Lemma. Since $P_\nu |\Delta \lambda_\nu| = O(p_\nu |\lambda_\nu|)$, by condition (iii) of Theorem A, as for $T_{n,1}$, we get
\[
\sum_{n=2}^{m+1} \left( \frac{p_n}{p_{n-1}} \right)^{\delta k + k - 1} |T_{n,2}|^k = O(1) \sum_{\nu=1}^{m} |\lambda_\nu| \left( \frac{p_\nu}{p_{n-1}} \right)^{\delta k} p_\nu |s_\nu|^k = O(1)
\]
for $m \to \infty$. Again, since $p_{n+1} = O(p_n)$, by condition (i) of Theorem A, as for $T_{n,1}$, we have
\[
\sum_{n=2}^{m+1} \left( \frac{p_n}{p_{n-1}} \right)^{\delta k + k - 1} |T_{n,3}|^k = O(1) \sum_{\nu=1}^{m} |\lambda_{\nu+1}| \left( \frac{p_\nu}{p_{n-1}} \right)^{\delta k} p_\nu |s_\nu|^k = O(1)
\]
for $m \to \infty$. Finally, as for $T_{n,1}$ we get
\[
\sum_{n=1}^{m} \left( \frac{p_n}{p_{n-1}} \right)^{\delta k + k - 1} |T_{n,4}|^k = O(1) \sum_{n=1}^{m} |\lambda_n| \left( \frac{p_n}{p_{n-1}} \right)^{\delta k} p_n |s_n|^k = O(1)
\]
for \( m \to \infty \). Summarizing we get \( \sum_{n=1}^{m} \left( \frac{P_n}{P_{n+1}} \right)^{6k+k-1} |T_{n,r}|^k = O(1) \) as \( m \to \infty \), for 
\( 1 \leq r \leq 4 \). This completes the proof of Theorem B.

**Acknowledgement.** The author is grateful to the referees for their valuable suggestions for the improvement of this paper.

**References**


Received 04.09.1995; in revised form 14.03.1996