A Note on the Bonnet-Myers Theorem

V. Boju and L. Funar

Abstract. The aim of this note is to derive a compactness result for complete manifolds whose Ricci curvature is bounded from below. The classical result, usually stated as Bonnet-Myers theorem, provides an estimation of the diameter of a manifold whose Ricci curvature is greater than a strictly positive constant. Weaker assumptions that the Ricci curvature function tends slowly to zero (when the distance from a fixed point goes to infinity) were already considered in [2, 3]. We shall improve here their results.

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We will be concerned with the following analytic problem. Given a function $a: [r_0, +\infty) \to (0, +\infty)$ we consider positive solutions $y = y(r)$ of the differential equation

$$y'' + ay = 0$$

satisfying $y(r_0) = 0$. Obviously, $y$ has to be concave. We have to determine the functions $a = a(r)$ for which $y$ has a further zero $r_1 > r_0$ which may be bounded from above.

It is clear that there is such a bound in case $a$ is a positive constant, but this bound tends to infinity as $a(r) \to 0$. The above problem seems to be interesting for functions $a$ satisfying $\lim_{r \to +\infty} a(r) = 0$. It turns out that the right asymptotic is $a(r) \sim cr^{-2}$, with critical value $c = \frac{1}{4}$. In fact, for $c = \frac{1}{4} + v^2$ one gets the solution $y(r) = r^\frac{1}{2} \sin v(\log \frac{r}{r_0})$, and hence there is a second zero. In this paper we show that in fact the constant $v^2$ may be replaced by a function which tends as weakly as an iterated logarithm to zero, which enters in our definition of some function $A_{k,v} = A_{k,v}(r)$.

Let us first make some notations. For each natural number $k$ we set

$$\log_0(r) = r$$
$$L_k(r) = \log \ldots \log r$$

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whenever is defined, and
\[ A_{k,v}(r) = \frac{1}{4r^2} \left( 1 + \frac{1}{L_1(r)^2} + \cdots + \frac{1}{L_1(r)^2L_2(r)^2 \cdots L_{k-1}(r)^2} \right. \]
\[ \left. + \frac{1 + 4v^2}{L_1(r)^2L_2(r)^2 \cdots L_k(r)^2} \right). \]

For a Riemannian manifold \( M \), we denote by \( Ric(Y) \) the Ricci curvature in the direction \( Y \in T_x(M) \), for a point \( x \in M \) and \( T_x(M) \) being the tangent space to \( M \) in \( x \). The space \( M \) is said to have an *almost positive asymptotic Ricci curvature* (abbreviated to be an AP-Riemannian space) if there exist \( k, v, r_0 > 0 \) and \( p \in M \) such that
\[ Ric(Y) \geq (n - 1) A_{k,v}(r)|Y|^2 \]
holds for all \( x \in M \) whose distance from a fixed point \( p \) is \( r = dist(p, x) \geq r_0 \) and for all vectors \( Y \in T_x(M) \). Also \( | \cdot | \) stands for the norm in the tangent space induced by the metric, and \( n \) is the dimension of \( M \).

Our result can be stated now as follows.

**Theorem 1.** A complete AP-Riemannian manifold is compact, and its diameter \( d(M) \) is bounded by
\[ d(M) \leq \epsilon_{k-1} \left( L_{k-1} \left( \exp^{\pi \max\{r_0, e_k(0)\}} \right) \right) \]
where \( \epsilon_0(x) = x \) and \( \epsilon_{m+1}(x) = \exp \epsilon_m(x) \) for \( m > 0 \).

Notice that the case \( k = 0 \) is discussed in [2] and the case \( k = 1 \) is covered by [3]. Also, Dekster and Kupka [3] proved that the constant \( \frac{1}{4} \) is sharp, i.e. for any function \( A = A(r) \) using in the place of \( A_{k,v} \) so that Theorem 1 holds we must have
\[ \lim_{r \to +\infty} A(r)r^2 \geq \frac{1}{4} \quad \text{and} \quad \lim_{r \to +\infty} \left( A(r)r^2 - \frac{1}{4} \right) (\log r)^2 \geq 1. \]
So our result identifies the higher order terms which might be added in spite to preserve the boundedness of the manifold. We think that the function \( A_{k,v} \) is sharp.

**Proof of Theorem 1.** We write the Jacobi equation associated to the sectional curvature function \( A_{k,v} \), namely
\[ y'' + A_{k,v}(r)y = 0. \]
We claim that this equation admits the basic solutions
\[ \Psi_0 = \Phi_k(r) \cos vL_k(r) \quad \text{and} \quad \Psi_1 = \Phi_k(r) \sin vL_k(r) \]
where
\[ \Phi_k(r) = r^{\frac{k}{2}} L_1(r)^{\frac{1}{2}} \cdots L_{k-1}(r)^{\frac{1}{2}}. \]
For $k = 1$ it is easy to see that $r^\frac{1}{2} \cos(v \log r)$ and $r^\frac{1}{2} \sin(v \log r)$ are solutions for equation (2). By recurrence we prove first that the following relations are fulfilled (for $k = 1$ they are simply to check):

$$\Phi_k' + A_{k,0} \Phi_k = 0 \quad \text{and} \quad 2\Phi_k' L_k' + \Phi_k L_k'' = 0.$$ 

In fact we have

$$\Phi_{k+1} = \Phi_k L_k^\frac{1}{2}$$

and

$$L_{k+1} = \log L_k,$$

hence

$$2\Phi_{k+1}' L_{k+1}' + \Phi_{k+1} k L_{k+1}'' = (2\Phi_k' L_k' + \Phi_k L_k') L_k^\frac{1}{2} = 0.$$

On the other hand

$$\frac{\Phi_{k+1}''}{\Phi_{k+1}'} = -A_{k,0} + (L_k')^2 L_k^{-2} = -A_{k+1,0}$$

holds and the two relations stated above are proved.

Furthermore we verify that

$$\Psi_0' = \Phi_k' \cos(v L_k) - v(2\Phi_k' L_k' + \Phi_k L_k') \sin(v L_k) - v^2 \Phi_k L_k^2 \cos(v L_k).$$

The two relations stated above and the obvious identity

$$L_k' = L_{k-1}' L_{k-1}^{-1} \cdots L_1^{-1}$$

complete the proof of our claim for $\Psi_0$ (the case of $\Psi_1$ is similar).

Both $\Psi_0$ and $\Psi_1$ are defined on the interval $[\varepsilon_k(0), +\infty)$. Set $r_1 = \max\{r_0, \varepsilon_k(0)\}$. Therefore, for each $\lambda \geq \varepsilon_k(0)$ the linear combination

$$Y_{k,v,\lambda}(r) = -\sin(v L_k(\lambda)) \Psi_0(r) + \cos(v L_k(\lambda)) \Psi_1(r)$$

(3)

is a solution for equation (2), which satisfies also $Y_{k,v,\lambda}(\lambda) = 0$. Also, we may write

$$Y_{k,v,\lambda}(r) = \sin(v(L_k(r) - L_k(\lambda))) \Phi_k(r) L_k(r)$$

so that $Y_{k,v,\lambda}$ is positive on the interval $(\lambda, \beta(\lambda))$ where $\beta(\lambda) = \varepsilon_{k-1}(L_{k-1}(\lambda) \exp(\frac{\pi}{v}))$ and vanishes again in $\beta(\lambda)$. This is a consequence of the straightforward formula

$$L_k(\beta(\lambda)) - L_k(\lambda) = \frac{\pi}{v}.$$

A standard argument (see, for instance, [1]) proves that the diameter of the manifold $M$ is less than $\beta(r_1)$. Since $M$ is complete from the Hopf-Rinow theorem it follows that $M$ is in fact compact and this ends the proof of the theorem.

**Remark 2.** The form of the function $A_{k,v}$ is in some sense sharp. In fact, for $v = 0$ the analog result is false: We may choose on $M = \mathbb{R}^n - K$, with $K$ being a sufficiently large compact, the metric with radial symmetry $dr + P_k(r) d\theta$ (in polar coordinates) where $d\theta$ is the metric form on the standard sphere $S^{n-1}$ and

$$P_k(r) = r \left( \sum_{i=1}^k L_i(r)^{-2} \right)^{-\frac{1}{2}}.$$

Then a straightforward computation shows that $\text{Ric}_x(Y) = A_{k,0}(r)|Y|^2$ for all points $x$ outside the compact $K$ and all tangent vectors $Y$.

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References


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