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Initial-Mixed Boundary Value Problems for Parabolic Complex Equations of Second Order with Measurable Coefficients

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Abstract. In [2], the authors introduced some results on initial-boundary value problems for divergence parabolic equations of second order with measurable coefficients. In [1], the authors considered the first boundary value problem for non-divergence parabolic equations of second order with discontinuous coefficients. In this paper, we consider initial-mixed boundary value problems for non-divergence parabolic complex equations of second order in a multiply connected domain. Firstly, we give a priori estimates of solutions of the above initial-boundary value problems by the method of symmetric extension, and then by using these estimates, the methods of auxiliary functions and parameter extension, we prove the solvability for the foregoing problems. Here the condition (1.3) is weaker than the corresponding one in [1] and [3], i.e. the constant 4/3 in [1] and [3] is replaced by 3/2 in (1.3).

Keywords: *Initial-mixed problems, parabolic equations, measurable coefficients*

AMS subject classification: 35 K 15, 35 K 20

1. Formulation of initial-mixed boundary value problems

Let D be an $(N+1)$-connected bounded domain in the $z = (x+iy)$-plane $\mathbb{C}$ with the boundary $\Gamma = \sum_{j=0}^{N} \Gamma_j \in C_\mu^2$ $(0 < \mu < 1)$. Without loss of generality, we may consider that D is a circular domain in $|z| < 1$ with the boundary $\Gamma = \sum_{j=0}^{N} \Gamma_j$, where $\Gamma_j = \{|z - z_j| = \gamma_j\}$ $(j = 1, \ldots, N)$, $\Gamma_0 = \Gamma_{N+1} = \{|z| = 1\}$ and $z = 0 \in D$. Denote $G = D \times (0, T]$ with a constant $T > 0$, and $\partial G = \partial G_1 \cup \partial G_2$ with ∂G_1 the bottom $\{(z, 0) : z \in D\}$ and ∂G_2 the lateral boundary $\{(z, t) : z \in \Gamma, t \in [0, 1]\}$ of the domain G.

We discuss the linear uniformly parabolic equation of second order

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu + g = Hu_t \qquad \text{in } G \tag{1.1}$$

where the coefficients a, b, c, d, e, f, g are real-valued measurable functions of $(z, t) \in G$ and H is a real constant satisfying the condition $0 < H \leq 1$. It is easy to see that the real equation (1.1) can be rewritten in the complex form

$$\mathcal{L}u := A_0 u_{z\bar{z}} - \operatorname{Re}(Q u_{zz} + A_1 u_z) - A_2 u - H u_t = A_3 \qquad \text{in } G \tag{1.2}$$

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in which

$$Q = 2(-a + c - 2bi)$$
$$A_0 = 2(a + c), \quad A_1 = -2(d + ei), \quad A_2 = -f, \quad A_3 = -g.$$

Suppose that equation (1.2) satisfies the following

Condition (C): The functions $A_j = A_j(z,t)$ $(j = 0,1,2,3)$ and $Q = Q(z,t)$ are measurable in G and satisfy the conditions

$$\begin{gathered} 0 < \delta \le A_0 \le \frac{1}{\delta}, \quad \left|\frac{Q}{A_0}\right| \le q_0 < 1, \quad \frac{\sup_G(A_0^2 + |Q|^2)}{\inf_G A_0^2} < \frac{3}{2} \\ |A_j| \le k_0 \quad (j = 1,2), \quad L_p(A_3, \overline{G}) = \left(\iiint_G |A_3|^p dxdydt\right)^{1/p} \le k_0 \quad (p > 4) \end{gathered} \tag{1.3}$$

where δ, q_0, k_0 and p are non-negative constants.

The so-called *initial-mixed boundary value problem* is formulated as follows.

Problem (M): Find a solution $u \in C^{1,0}(\overline{G})$ satisfying the initial condition

$$u(z,0) = g(z) \qquad \text{on } D \tag{1.4}$$

and the boundary condition $a_1(z,t)\frac{\partial u}{\partial \nu} + a_2(z,t)u = a_3(z,t)$ on ∂G_2, i.e.

$$2\mathrm{Re}\,(a_1 \bar{\lambda} u_z) + a_2 u = a_3 \qquad \text{on } \partial G_2 \tag{1.5}$$

where $\nu(z,t)$ is a unit vector at every point on ∂G_2 (there is no harm in assuming that $\nu(z,t)$ is parallel to the $(t = 0)$-plane), g, a_j $(j = 1,2,3)$ and λ with $\lambda(z,t) = \cos(\nu, x) - i\cos(\nu, y)$ are known functions satisfying the conditions

$$\begin{gathered} C_\alpha^2[g, \bar{D}] \le k_0 \\ C^{1,0}_{\alpha,\alpha/2}[a_j, \partial G_2] = C^{00}_{\alpha,\alpha/2}[a_j, \partial G_2] + C^{00}_{\alpha,\alpha/2}[a_{jz}, \partial G_2] \le k_0 \quad (j = 1,2,3) \\ C^{1,0}_{\alpha,\alpha/2}[\lambda, \partial G_2] = C^{0,0}_{\alpha,\alpha/2}[\lambda, \partial G_2] + C^{0,0}_{\alpha,\alpha/2}[\lambda_z, \partial G_2] \le k_0 \end{gathered}$$

with

$$C^{00}_{\alpha,\alpha/2}[\lambda, \partial G_2] = \sup_{(z_1,t_1),(z_2,t_2)\in\partial G_2} \frac{|\lambda(z_1,t_1) - \lambda(z_2,t_2)|}{[|z_1 - z_2|^2 + |t_1 - t_2|^2]^\alpha}$$

and

$$\cos(\nu, n) \ge \eta \quad \text{on } \partial G_2 \qquad \text{and} \qquad a_1\frac{\partial g}{\partial \nu} + a_2 g = a_3 \quad \text{on } \Gamma \times \{t = 0\}$$
$$a_j(z,t) \ge 0 \quad (j = 1,2) \qquad \text{and} \qquad a_1(z,t) + a_2(z,t) \ge 1 \qquad \text{on } \partial G_2$$

in which α $(\frac{1}{2} < \alpha < 1)$, k_0, η $(0 < \eta \le 1)$ are non-negative constants and n is the outward normal at every point $(z,t) \in \partial G_2$.

Remark that when $a_1(z,t) = 0$ on ∂G_2, then Problem (M) is the Dirichlet problem.

Problem (O): When $a_1(z,t) \neq 0$ for every point $(z,t) \in \partial G_2$, then Problem (M) is the initial-regular oblique derivative problem, which will be called *Problem* (O).

Now, we prove the uniqueness of solutions of Problem (M) for equation (1.2).

Theorem 1.1: *Suppose that the complex equation* (1.2) *satisfies Condition* (C). *Then the solution of Problem* (M) *for equation* (1.2) *is unique.*

Proof: Denote by u_1 and u_2 two solutions of Problem (M) for equation (1.2). It is clear that the function $u = u_1 - u_2$ is a solution of the following homogeneous initial-boundary value problem (M_0):

$$\begin{aligned} A_0 u_{z\bar z} - \operatorname{Re}[Q u_{zz} + A_1 u_z] - A_2 u - H u_t = 0 \quad &\text{in} \quad G \\ u(z,0) = 0 \quad &\text{on} \quad D \\ a_1(z,t)\frac{\partial u}{\partial \nu} + a_2(z,t) u = 0 \quad &\text{on} \quad \partial G_2. \end{aligned} \tag{1.6}$$

Making a transformation of the unknown function $U(z,t) = u(z,t)e^{-Bt}$ where B is a real constant such that $HB + \inf_{\overline{G}} A_2 > 0$, obviously, U is a solution of the initial-boundary value problem

$$A_0 U_{z\bar z} - \operatorname{Re}(Q U_{zz} + A_1 U_z) - (HB + A_2)U = H U_t \quad \text{in} \quad G \tag{1.7}$$

$$U(z,0) = 0 \quad \text{on} \quad D \tag{1.8}$$

$$a_1(z,t)\frac{\partial U}{\partial \nu} + a_2(z,t) U = 0 \quad \text{on} \quad \partial G_2. \tag{1.9}$$

On the basis of the maximin principle of solutions for equation (1.7), if $U \not\equiv 0$ on $\overline{G}$, then U takes its positive maximum or negative minimum on the lateral boundary ∂G_2. Suppose that U takes a positive maximum at a point $p_0 = (z_0, t_0) \in \partial G_2$. It can be derived that $U(p_0) > 0$ and $\frac{\partial U}{\partial \nu}\big|_{p=p_0} > 0$. Thus $\big(a_1(z,t)\frac{\partial U}{\partial \nu} + a_2(z,t)U\big)\big|_{p=p_0} > 0$. This contradicts (1.9). Similarly, we can prove that U does not attain a negative minimum at a point $p_0 \in \partial G_2$. This shows that $U = 0$ on $\overline{G}$, i.e. $u_1 = u_2$ on $\overline{G}$ ∎

2. A priori estimates of solutions of the initial-mixed problem

First of all, we shall give a boundness estimate of solutions of Problem (M).

Theorem 2.1: *Let equation* (1.2) *satisfy Condition* (C). *Then any solution u of Problem* (M) *for equation* (1.2) *satisfies the estimate*

$$C[u, \overline{G}] \leq M_1 \tag{2.1}$$

where $M_1 = M_1(\delta, q_0, \alpha, k_0, p, G)$ is a non-negative constant only dependent on $\delta, q_0, \alpha, k_0, p, G$.

Proof: We first find a solution ψ of equation (1.2) with the initial-boundary condition $\psi = 0$ on ∂G. On the basis of the results in [1], [2: Chapter 3] and [3], we can obtain for ψ the estimates

$$C^{1,0}_{\beta,\beta/2}[\psi,\overline{G}] \le M_2 \qquad \text{and} \qquad \|\psi\|_{W^{2,1}_2(G)} \le M_2 \tag{2.2}$$

in which β $(0 < \beta \le \alpha)$ and $M_2 = M_2(\delta, q_0, \alpha, k_0, p, G)$ are non-negative constants. Secondly, we find a solution Ψ of the homogeneous equation

$$\mathcal{L}u = A_0 u_{z\bar z} - \operatorname{Re}(Q u_{zz} + A_1 u_z) - A_2 u - H u_t = 0 \qquad \text{in } G$$

with the initial-boundary condition $\Psi = 1$ on ∂G. Similarly, it can be proved that Ψ satisfies the estimates

$$C^{1,0}_{\beta,\beta/2}[\Psi,\bar G] \le M_3, \qquad \|\Psi\|_{W^{2,1}_2(G)} \le M_3, \qquad 0 < M_4 \le \Psi \le 1 \tag{2.3}$$

where $M_j = M_j(\delta, q_0, \alpha, k_0, p, G) \ge 0$ $(j = 3,4)$. According to the method in [3] and [4: Chapter 3/Theorem 3.3], we see that the function $U = \frac{u-\psi}{\Psi}$ is a solution of the initial-boundary value problem

$$A_0 U_{z\bar z} - \operatorname{Re}[Q U_{zz} + A U_z] - H U_t = 0 \quad \text{in } G \tag{2.4}$$

$$U(z,0) = \frac{u(z,0) - \psi(z,0)}{\Psi(z,0)} = U_0(z) \quad \text{on } D$$

$$a_1 \frac{\partial U}{\partial \nu} + a_4 U = a_5, \quad a_4 = a_2 + a_1 \frac{\partial \ln \Psi}{\partial \nu}, \quad a_5 = \frac{a_3 - a_1 \frac{\partial \psi}{\partial \nu} - a_2 \psi}{\Psi} \quad \text{on } \partial G_2$$

where $A = A_1 - 2A_0(\ln \Psi)_{\bar z} + 2Q(\ln\Psi)_z$ in G and $a_4 > 0$ on ∂G_2. By means of the maximin principle of solutions of equation (2.4), we know that U attains its maximum and minimum at points $p^* = (z^*, t^*)$ and $p_* = (z_*, t_*)$ in ∂G_2, respectively. It can be derived

$$C[U,\overline{G}] = \max\left\{U(z^*,t^*), |U(z_*,t_*)|\right\} \le \max\left\{\max_D |U_0(z)|, \frac{\max_{\partial G_2}|a_5|}{\min_{\partial G_2} a_4}\right\}. \tag{2.5}$$

Combining (2.2), (2.3) and (2.5), the estimate (2.1) is derived ∎

Next, we shall prove

Theorem 2.2: *Let equation* (1.2) *satisfy condition* (C). *Then any solution u of Problem* (M) *for equation* (1.2) *satisfies the estimates*

$$C^{1,0}_{\beta,\beta/2}[u,G^*] \le M_5 \qquad \textit{and} \qquad \|u\|_{W^{2,1}_2(G^*)} \le M_5 \tag{2.6}$$

where

$$G^* = \overline{G} \cap \left\{ \cap_{(z^*,t^*)\in\partial G_2^*} \{|z - z^*|^2 + |t - t^*| \ge \varepsilon\}\right\}$$

ε is a small positive number, $G_2^* = \{(z,t) \in \partial G_2 : a_1(z,t) = 0\}$, ∂G_2^* is the boundary of G_2^*, β $(0 < \beta \le \alpha)$ and $M_5 = M_5(\delta, q_0, \alpha, k_0, p, G)$ are non-negative constants.

Proof: According to the method in [3], we can obtain that any solution u of Problem (M) for equaton (1.2) satisfies the estimates

$$C^{1,0}_{\beta,\beta/2}[u,G_m] \le M_6 \qquad \text{and} \qquad \|u\|_{W_2^{2,1}(G_m)} \le M_6 \tag{2.7}$$

in which $G_m = \{(z,t) \in \overline{G} : \operatorname{dist}(z,\Gamma) \ge \frac{1}{m}\}$ for a positive integer m, β $(0 < \beta \le \alpha)$ and $M_6 = M_6(\delta, q_0, \alpha, k_0, p, \varepsilon, m, G)$ are non-negative constants. In the following, we shall give the estimates of the solution near the lateral ∂G_2 of G. We arbitrarily choose an inner point $p^* = (z^*, t^*)$ of G^* and denote $\widetilde{G}_2 = \{(z,t) \in G_2^* : |z-z^*|^2+|t-t^*| < \varepsilon^*\}$, where ε^* $(< \varepsilon)$ is an appropriately small positive number such that $\widetilde{G}_2 \cap \partial G_2 \subset G_2^* \backslash \partial G_2^*$. There is no harm in assuming that $|z^*| = 1$, otherwise through a linear fractional transformation this requirement can be realized. Now we find a solution u_1 of equation (1.2) satisfying the boundary condition $u_1 = \frac{a_3}{a_2}$ on $\widetilde{G}_2$. Thus the function $U = u - u_1$ satisfies the equation and boundary condition

$$\begin{aligned} \mathcal{L}U := A_0 U_{z\bar z} - \operatorname{Re}[QU_{zz} + A_1 U_z] - A_2 U - HU_t = A \quad &\text{in} \quad G \\ U(z,0) = 0 \quad &\text{on} \quad \widetilde{G}_2 \end{aligned}$$

where $A = A_3 - \mathcal{L}u_1$. In this case, the solution U can be continuously extended along $\widetilde{G}_2$ from G into $\widetilde{G}$ (the symmetric domain of G). In fact, it is sufficient to introduce the function $\widetilde{U}$ by

$$\widetilde{U}(z,t) = \begin{cases} U(z,t) & \text{if } (z,t) \in G \cup \widetilde{G}_2 \\ -U(1/\bar z, t) & \text{if } (z,t) \in \widetilde{G}. \end{cases}$$

Due to $\widetilde{U} = 0$ on $\widetilde{G}_2$, it is clear that $\operatorname{Re}(iz\widetilde{U}_z) = 0$, i.e. $zU_z = \bar z U_{\bar z}$ for $(z,t) \in \widetilde{G}_2$, and when $z = 1/\bar\zeta \in \widetilde{G} + \widetilde{G}_2$, then $\widetilde{U}_z = -U_{\bar\zeta}(-1/z^2)$, hence $z\widetilde{U}_z = \bar z U_{\bar z}$ for $(z,t) \in \widetilde{G}_2$. This shows that $\widetilde{U}$ and $\widetilde{U}_{\bar z}$ are continuous in $G \cup \widetilde{G} \cup \widetilde{G}_2$. Noting that

$$\widetilde{U}_{z\bar z} = -|z|^{-4} U_{\zeta\bar\zeta} \qquad \text{and} \qquad \widetilde{U}_{\bar z\bar z} = -\bar z^{-4} U_{\zeta\zeta} - 2\bar z^{-3} U_\zeta \qquad ((z,t) \in \widetilde{G} \cup \widetilde{G}_2)$$

it is seen that $\widetilde{U}$ is a solution of the equation

$$\widetilde{A}_0 \widetilde{U}_{z\bar z} - \operatorname{Re}(\widetilde{Q}\widetilde{U}_{zz} + \widetilde{A}_1 \widetilde{U}) - \widetilde{A}_2 \widetilde{U} - H\widetilde{U}_t = \widetilde{A} \qquad \text{in } G \tag{2.8}$$

where

$$\widetilde{A}_0 = \begin{cases} A_0(z,t) \\ A_0(1/\bar z, t)|z|^4 \end{cases} \qquad \widetilde{A}_1 = \begin{cases} A_1(z,t) \\ -\overline{A_1(1/\bar z,t)} z^2 + 2\overline{Q(1/\bar z,t)} z^3 \end{cases}$$

$$\widetilde{Q} = \begin{cases} Q(z,t) \\ \overline{Q(1/\bar z,t)} z^4 \end{cases} \qquad \widetilde{A}_j = \begin{cases} A_j(z,t) \\ (-1)^j A_j(1/\bar z, t) \end{cases} \qquad (j = 2,3)$$

for $(z,t) \in G$ and $(z,t) \in \widetilde{G}$, respectively. It is not difficult to see that equation (2.8) satisfies conditions similar to Condition (C). Therefore similarly to (2.7), we can derive estimates of $\widetilde{U}$ and u in the neighbourhood

$$\widehat{G}_2 = \left\{(z,t) \in G \cup \widetilde{G}_2 : \operatorname{dist}\left((z,t), \partial G_2 \setminus \widetilde{G}_2\right) \ge \varepsilon^*\right\}$$

of $\widetilde{G}_2$, namely

$$C^{1,0}_{\beta,\beta/2}[u,\widehat{G}_2] \le M_7 \quad \text{and} \quad \|u\|_{W^{2,1}_2(\widehat{G}_2)} \le M_7 \tag{2.9}$$

where $M_7 = M_7(\delta, q_0, \alpha, k_0, p, \varepsilon, \varepsilon^*, G)$.

Secondly, if there exists a surface

$$G_3 = \Big\{(z,t) \in \partial G_2 : a_1(z,t) > 0,\ \nu = n\Big\} \supset \partial G_2,$$

then we can find a harmonic function $\sigma = \sigma(z,t)$ of z in G such that σ satisfies the boundary condition $\frac{\partial \sigma}{\partial n} = \frac{a_2}{a_1}$ on G_3. Thus the function V defined by $V(z,t) = u(z,t)e^{\sigma(z,t)}$ satisfies the boundary condition $\frac{\partial V}{\partial n} = \frac{a_3}{a_1}e^{\sigma}$, i.e.

$$a_1\frac{\partial u}{\partial n} + a_2 u = a_3 \qquad \text{on } G_3. \tag{2.10}$$

Noting that

$$\begin{aligned} u_z &= e^{-\sigma}[V_z - \sigma_z V], \qquad u_t = e^{-\sigma}(V_t - \sigma_t V) \\ u_{zz} &= e^{-\sigma}\big(V_{zz} - 2\sigma_z V_z - (\sigma_{zz} - \sigma_z^2)V\big) \\ u_{z\bar z} &= e^{-\sigma}\big(V_{z\bar z} - \operatorname{Re}(\sigma_{\bar z} V_z) + |\sigma_{\bar z}|^2 V\big) \end{aligned}$$

it is easy to see that the function V satisfies the equation

$$A_0 V_{z\bar z} - \operatorname{Re}\big(QV_{zz} + B_1 V_z\big) - B_2 V - HV_t = B_3 \qquad \text{in } G \tag{2.11}$$

where the coefficients satisfy conditions similar to those in Condition (C). Now, we find a harmonic function $V_0 = V_0(z,t)$ of z in G satisfying the boundary condition (2.10). Moreover, the function $\widetilde{V}$ defined by

$$\widetilde{V}(z,t) = \begin{cases} V(z,t) - V_0(z,t) & \text{for } (z,t) \in G \cup G_3 \\ V(1/\bar z,t) - V_0(1/\bar z,t) & \text{for } (z,t) \in \widetilde{G} \end{cases}$$

satisfies the equation

$$\widetilde{A}_0\widetilde{V}_{z\bar z} - \operatorname{Re}\big(\widetilde{Q}\widetilde{V}_{zz} + \widetilde{B}_1\widetilde{V}_z\big) - \widetilde{B}_2\widetilde{V} - H\widetilde{V}_t = \widetilde{B}_3 \qquad \text{in } G \cup G_3 \cup \widetilde{G}$$

where the coefficients satisfy conditions similar to Condition (C). Hence by using the method as stated in [1], [3] or [4: Chapter 3/Theorem 4.8], it can be derived that u satisfies the estimates

$$C^{1,0}_{\beta,\beta/2}[u,\widehat{G}_3] \le M_8 \quad \text{and} \quad \|u\|_{W^{2,1}_2(\widehat{G})} \le M_8 \tag{2.12}$$

where

$$\widehat{G}_3 = \Big\{(z,t) \in G \cup G_3 : \operatorname{dist}\big((z,t), \partial G_2 \setminus G_3\big) \ge \varepsilon^*\Big\}$$

and $M_8 = M_8(\delta, q_0, \alpha, k_0, p, \varepsilon, \varepsilon^*, G)$.

Finally, we discuss the surface

$$G_4 = \left\{(z,t) \in \partial G_2 : a_1(z,t) > 0,\ \nu \not\equiv n\right\}.$$

By using a similar method as before, we can transform the boundary condition (1.5) into a homogeneous boundary condition $\frac{\partial V^*}{\partial \nu} = 0$ on G_4 where the function $V^* = V^*(z,t)$ satisfies the equation

$$A_0 V^*_{z\bar{z}} - \operatorname{Re}\left(Q V^*_{zz} + B_1 V^*_z\right) - B_2 V^* - H V^*_t = B_3 \qquad \text{in } G.$$

Without loss of generality, we may assume that G lies in the lower half-plane $\mathrm{Im} < 0$ and $(0,t) \in G_4$, because through a conformal mapping this requirement can be realized. Setting $b_1 = \cos(\nu, x)$ and $b_2 = \cos(\nu, y)$, and making a transformation

$$z = \frac{1}{2}(1 + b_1 + ib_2)\zeta + \frac{1}{2}(-1 + b_1 + ib_2)\bar{\zeta} \qquad (\zeta = \xi + i\eta) \tag{2.13}$$

it is obvious that (2.13) is a homeomorphism $\zeta = \zeta(z,t)$ in a neighbourhood of $\zeta = 0$, which maps the surface G_4 in the ζ-plane onto a surface H_4 on the imaginary axis in the z-plane. Denote by $z = z(\zeta,t)$ the inverse function of $\zeta = \zeta(z,t)$. Thus the function $\widetilde{V}(\zeta,t) = V^*\big(z(\zeta,t),t\big)$ satisfies the equation and boundary condition

$$\begin{aligned} \widetilde{A}_0 \widetilde{V}_{\zeta\bar{\zeta}} - \operatorname{Re}\left(\widetilde{Q}\widetilde{V}_{\zeta\zeta} + \widetilde{B}_1 \widetilde{V}_\zeta\right) - \widetilde{B}_2 \widetilde{V} - H\widetilde{V}_t = \widetilde{B}_3 &\quad \text{in } \zeta(G) \\ \frac{\partial \widetilde{V}}{\partial n} = 0 &\quad \text{in } H_4 = \zeta(G_4). \end{aligned}$$

Furthermore, by applying the method used for deriving the estimate (2.12), we can obtain the estimates of $\widetilde{V}$, V^* and u, i.e.

$$C^{1,0}_{\beta,\beta/2}[u, \widetilde{G}_4] \le M_9 \qquad \text{and} \qquad \|u\|_{W_2^{2,1}(\widetilde{G}_4)} \le M_9 \tag{2.14}$$

where

$$\widetilde{G}_4 = \left\{(z,t) \in G \cup G_4 : \operatorname{dist}\big((z,t), \partial G_2 \setminus G_4\big) \ge \varepsilon^*\right\}$$

and $M_9 = M_9(\delta, q_0, \alpha, k_0, p, \varepsilon, \varepsilon^*, G)$. Combining (2.7), (2.9), (2.12) and (2.14), the estimates in (2.6) are derived ∎

3. The solvability of the initial-mixed boundary value problem

In this section, we use the estimates of solutions in Theorem 2.2 and the compactness principle of solutions to prove the solvability of Problem (M) for equation (1.2).

Theorem 3.1: *If equation* (1.2) *satisfies Condition* (C), *then Problem* (M) *for equation* (1.2) *has a solution* $u = u(z,t)$.

Proof: We are free to choose a positive integer m and to consider Problem (M_m) for equation (1.2) with the initial and boundary conditions

$$u(z,0) = g(z) + g_m(z,0) \quad \text{on } D$$
$$\left(a_1 + \frac{1}{m}\right)\frac{\partial u}{\partial \nu} + a_2 u = a_3 \quad \text{on } \partial G_2$$

where $g_m = g_m(z,t)$ is an appropriate solution of the oblique derivative problem for the homogeneous equation (1.6) with the boundary condition

$$\left(a_1 + \frac{1}{m}\right)\frac{\partial g_m}{\partial \nu} + a_2 g_m = -\frac{1}{m}\frac{\partial g}{\partial \nu} \quad \text{on } \partial G_2$$

such that g_m satisfies estimates similar to (2.6) and $g_m \to 0$ as $m \to \infty$. According to the following Theorem 3.2, we know that g_m exists and Problem (M_m) has a solution u_m $(m = 1, 2, \ldots)$ which satisfies the estimates (2.6). Hence from the sequence $\{u_m\}$ we can select a subsequence $\{u_{m_k}\}$, which uniformly converges to a solution $u_0 = u_0(z,t)$ of equation (1.2) in any closed subset $G_* \subset \overline{G}\setminus\partial G_2^*$, and u_0 satisfies the initial condition (1.4) and boundary condition

$$a_1 \frac{\partial u_0}{\partial \nu} + a_2 u_0 = a_3 \quad \text{on } \partial G_2 \setminus \partial G_2^*.$$

It remains to prove that u_0 is continuous on $\overline{G}$ and satisfies the boundary condition (1.5). We select an arbitrary point $p^* = (z^*, t^*) \in \partial G_2^*$ (it can be replaced by any surface $S \subset G_2^*$), and denote by G_β the point set $\{|z - z^*|^2 + |t - t^*| < \beta\} \cap \partial G_2$ where β is a sufficiently small positive number. We construct a real continuous function $f = f(z,t)$ as follows:

$$f(z,t) = \begin{cases} M_{10} + 1 & \text{for } (z,t) \in \partial G_2 \setminus G_{\beta/2} \\ \eta > 0 & \text{for } (z,t) \in G_{\beta/4} \end{cases}$$

$$\eta \le f(z,t) \le M_{10} + 1 \quad \text{for } (z,t) \in G_{\beta/2} \setminus G_{\beta/4}$$

where M_{10} is an undetermined positive number. The function f satisfies the estimate

$$C^{1,0}_{\mu-\varepsilon}[f, \partial G_2] \le \frac{M_{11}}{\beta^{1+\mu-\varepsilon}} \qquad \left(\frac{1}{2} < \mu - \varepsilon < 1\right)$$

where ε is a sufficintly small positive constant and $M_{11} = M_{11}(M_{10}, \partial G_2)$. Let $\hat{u}_m$ be a solution of the homogeneous equation (1.6) satisfying the boundary condition $\hat{u}_m = f$

on ∂G_2. It is not difficult to see that $\hat{u}_m$ satisfies the estimate $C^{1,0}[\hat{u}_m(z,t),\bar{G}] \le M_{12}/\beta^{1+\mu-\epsilon}$, where $M_{12} = M_{12}(\delta, q_0, \alpha, k_0, G)$. Now, we extend a_2 from G_2^* to ∂G_2, such that we have a new function $a^* \in C^{1,0}_{\alpha,\alpha/2}(\partial G_2)$, $a_2^*(z,t) > 0$ and $a^* = a_2$ on $G_{\beta/2}$. Moreover, a solution u^* of equation (1.2) can be found satisfying the boundary condition $u_m^* = a_3/a_2^*$ on ∂G_2. Setting $\tilde{u}_\pm = \pm\hat{u}_m - u_m + u_m^*$ it is easily seen that $\tilde{u}_\pm$ are solutions of the equation $\mathcal{L}u = 0$. In the following, we verity that

$$\tilde{u}_+(z,t) \ge 0 \qquad \text{and} \qquad \tilde{u}_-(z,t) \le 0 \qquad ((z,t) \in \bar{G}).$$

In fact, obviously $\tilde{u}_+(z,t) \ge M_{10} + 1 - M_{10} > 0$ on $\partial G_2 \setminus G_{\beta/2}$ where

$$M_{10} = M + \max_{\partial G_2} \frac{a_3(z,t)}{a_2^*(z,t)} > 0 \qquad \text{with } M = \max_{\overline{G}} |u_m(z,t)|$$

with a constant M being similar to M_1 in (2.1). If $\tilde{u}_+$ takes a negative minimum in $\overline{G}$, then there exists a point $p' = (z',t') \in G_{\beta/2}$ such that $\tilde{u}_+(z',t') \le \min_{\overline{G}} \tilde{u}_+(z,t)$. However we have

$$\begin{aligned}
&\left(a_1(z,t) + \frac{1}{m}\right)\frac{\partial \tilde{u}_+}{\partial \nu} + a_2(z,t)\tilde{u}_+(z,t) \\
&= \left(a_1(z,t) + \frac{1}{m}\right)\frac{\partial \hat{u}_m}{\partial \nu} + a_2(z,t)\hat{u}_m(z,t) + \left(a_1(z,t) + \frac{1}{m}\right)\frac{\partial u_m^*}{\partial \nu} \\
&> M_{13}\eta - \max_{G_{\beta/2}} a_1(z,t)\frac{M_{14}}{\beta^{1+\mu-\epsilon}} - \frac{M_{14}}{m\beta^{1+\mu-\epsilon}} + \left(a_1(z,t) + \frac{1}{m}\right)\frac{\partial u_m^*}{\partial \nu}
\end{aligned}$$

for all $(z,t) \in G_{\beta/2}$, where $M_{13} = \min_{G_{\beta/2}} a_2(z,t)$ and $M_{14} = M_{14}(M_{12}, \partial G_2)$. Due to $a_1 \le M_{15}\beta^{1+\mu}$ on $G_{\beta/2}$ with a positive constant M_{15}, we first choose β small enough, and then select m large enough, such that

$$\beta^\epsilon M_{14} M_{15}, \qquad \frac{M_{14}}{m\beta^{1+\mu-\epsilon}}, \qquad \left|a_1(z,t)\frac{\partial u_m^*}{\partial \nu}\right|, \qquad \left|\frac{1}{m}\frac{\partial u_m^*}{\partial \nu}\right|$$

are less than $\frac{1}{4}M_{13}\eta$ for all $(z,t) \in G_{\beta/2}$. So

$$\left(a_1 + \frac{1}{m}\right)\frac{\partial \tilde{u}_+}{\partial \nu} + a_2\tilde{u}_+ > 0 \qquad \text{on } G_{\beta/2}.$$

This shows that $\tilde{u}_+$ cannot take a negative minimum on $G_{\beta/2}$. On the basis of the maximin principle of solutions of equation $\mathcal{L}u_n = 0$, $\tilde{u}_+(z,t) = \hat{u}_m(z,t) - u_m(z,t) + u_m^*(z,t) \ge 0$, i.e. $u_m - u_m^* \le \hat{u}_m$ in $\overline{G}$ can be obtained. By the same reasoning, we have $\tilde{u}_-(z,t) \le 0$, i.e. $u_m - u_m^* \ge -\hat{u}_m$ in $\overline{G}$. From $|\hat{u}_m| \le \eta$ on $G_{\beta/4}$ it follows that $|u_m - u_m^*| < \eta$ on $G_{\beta/4}$. By the equicontinuity of the sequence $\{\hat{u}_m\}$ in $\overline{G}$,

$$|u_m(z,t) - u_m^*(z,t)| \le |\hat{u}_m(z,t)| \le 2\eta$$

is seen for all (z,t) of a neighbourhood of (z^*,t^*) in $\overline{G}$. Denote by $\tilde{u}_0$ the limit function of $\{u_m - u_m^*\}$ in G. It is clear that $|\tilde{u}_0(z,t)| \le 2\eta$. Noting that η is an arbitrary positive number, it is seen that $\tilde{u}_0$ at $p^* = (z^*,t^*)$ is continuous, and $\tilde{u}_0(z^*,t^*) = 0$. Hence $u_0 = \tilde{u}_0 + u_0^*$ at (z^*,t^*) is also continuous, where u_0^* is a limit function of a subsequence of $\{u_m^*\}$. This completes the proof ∎

Theorem 3.2: *Under the Condition* (C), *Problem* (O) *for equation* (1.2) *has a solution* $u = u(z,t)$ *satisfying the estimates*

$$C^{1,0}_{\beta,\beta/2}[u,\bar{G}] \le M_{16} \qquad \text{and} \qquad \|u\|_{W^{2,1}_2(G)} \le M_{16} \tag{3.1}$$

where β $(0 < \beta < \alpha)$ *and* $M_{16} = M_{16}(\delta, q_0, \alpha, k_0, p, G)$ *are non-negative constants.*

Proof: The estimate (3.1) can be derived from Theorem 2.1. In the following, we shall prove the existence of solutions of Problem (O) for equation (1.2). The boundary condition of Problem (O) can be written in the form $\frac{\partial u}{\partial \nu} + a_2(z,t)u = a_3(z,t)$, i.e.

$$\cos(\nu,n)\frac{\partial u}{\partial n} + \cos(\nu,s)\frac{\partial u}{\partial s} + a_2 u = a_3 \qquad \text{on } \partial G_2$$

where s is the tangent vector at every point $(z,t) \in \partial G_2$. In order to use the method of parameter extension, we consider the initial and boundary conditions with the parameter $\tau \in [0,1]$:

$$\begin{aligned} u(z,0) &= g(z) + g_\tau(z,0) && \text{on } D \\ \cos(\nu,n)\frac{\partial u}{\partial n} + \tau\cos(\nu,s)\frac{\partial u}{\partial s} + a_2 u &= a && \text{on } \partial G_2 \end{aligned}$$

where a is any function in the space $C^{1,0}_{\beta,\beta/2}(\overline{G})$ and g_τ is an appropriate solution of the oblique derivative problem for equation (1.6) with the boundary condition

$$\cos(\nu,n)\frac{\partial g_\tau}{\partial n} + \tau\cos(\nu,s)\frac{\partial g_\tau}{\partial s} + a_2 g_\tau = a - a_3 + (1-\tau)\cos(\nu,s)\frac{\partial g}{\partial s} \qquad \text{on } \partial G_2$$

where $g_1(z,t) = 0$ on $\overline{G}$ if $a = a_3$ on ∂G_z. By using the method as stated in [4: Chapter 2/Proof of Theorem 3.3] and the result in [3], there exists a solution u_0 of Problem (O) with $\tau = 0$ for equation (1.2) and $u_0 \in C^{1,0}_{\beta,\beta/2}(\overline{G})$. By the method in [3] and [4: Chapter 1/Proof of Theorem 2.5], we can prove that there exists a number $\varepsilon > 0$ such that for $\tau = \varepsilon, 2\varepsilon, \ldots, \left[\frac{1}{\varepsilon}\right]\varepsilon, 1$ Problem (O) for equation (1.2) is solvable. In particular, when $\tau = 1$ and $a = a_3$, then Problem (O) for equation (1.2), i.e. Problem (M) for equation (1.2) has a solution u ∎

References

[1] Alkhutov, Yu. A. and I. T. Mamedov: *The first boundary value problem for nondivergence second order parabolic equations with discontinuous coefficients.* Math. USSR Sbornik 59 (1988), 471 - 495.

[2] Ladyzhenskaya, O. A., Solonnikov, V. A. and N. N. Ural'ceva: *Linear and Quasilinear Equations of Parabolic Type.* Providence (R.I.): Amer. Math. Soc. 1968.

[3] Wen, Guo Chun: *Two boundary value problems for second order nonlinear parabolic equations with measurable coefficients* (in Chinese). J. Yantai Univ. (Natur. Sci. Engin.) (1993)1, 1 - 8.

[4] Wen, Guo Chun and H. Begehr: *Boundary Value Problems for Elliptic Equations and Systems.* Harlow: Longman Sci. & Techn. 1986.

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