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On the Discrete Spectrum of Ordinary Differential Operators in Weighted Function Spaces

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Abstract. The investigations deal with the spectrum of ordinary differential equations of order $2n$ where the underlying Hilbert space is weighted. Especially, conditions on the coefficients are given such that the spectra are discrete.

Keywords: *Weighted Hilbert spaces, spectral theory, ordinary differential operators*

AMS subject classification: 34 B 25

1. Introduction

Investigations about their spectrum are of special interest for the study of differential operators. Particular attention has been paid to the situation in which the spectrum of differential operators is discrete. In recent years much work has been done on higher ordinary differential operators

$$Ay = \sum_{k=0}^{n}(-1)^k\big(a_k(x)\,y^{(k)}\big)^{(k)} = \lambda ry \qquad \big(x \in (0,\infty),\, ry \in L_2(0,\infty)\big) \tag{1}$$

in weighted $L_{2,r}$-spaces (see [2, 7 - 10]). To the differential operator on the left-hand side of (1) we associate the form

$$a[u,v] = \sum_{k=0}^{n}\int_0^\infty a_k(x)\,u^{(k)}\,\bar{v}^{(k)}dx, \qquad D[a] = C_0^\infty(0,\infty). \tag{2}$$

It is well-known that if the spectrum of a self-adjoint extension of an operator with deficiency indices (m,m) $(m < \infty)$ is bounded below and discrete, then the same is true for the spectrum of all other self-adjoint extensions, that means the discreteness of the spectrum depends only on the coefficients.

In the present paper we follow the general approach given in [6] and establish conditions on the coefficients to ensure that the spectrum of (1) is discrete. These conditions

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refer to mean values of the functions a_0, a_0^{-1}, a_1^{-1} and r^2 on certain intervals $[x, x+d(x)]$ which cover the semi-axis $(0,\infty)$ and seem to be suitable for studying the spectra of the operators in question. Our result extends the results of [2, 6 - 8]. Some examples show that the obtained criterion is new.

2. Preliminaries

Throughout this paper we take our basic underlying Hilbert space to be $L_{2,r}$, the weighted L_2-space with norm given by $\|u\|_r = \|u\,r\|$ where $\|\cdot\|$ is the usual norm in $L_2(0,\infty)$. Let $r = r(x)$ be continuous and positive on $[0,\infty)$. We also assume that the functions $a_k = a_k(x)$ satisfy the following conditions:

(i) $a_k \in W_2^k(0,X)$ $(0 \le k \le n)$ for all $X > 0$ (see [6: p. 22]).

(ii) $a_n(x) > 0$ and $a_k(x) \ge 0$ $(k = 0,\dots,n-1)$ for all $x \ge 0$.

Lemma 1. *Let p and q be real-valued functions defined on the interval $\omega = [x_1, x_2]$ $(0 \le x_1 < x_2 < \infty)$ with length $|\omega|$ where $p(x) \ge 0$ and $q(x) \ge 0$ for all $x \in \omega$. Assume that p, p^{-1} and q are integrable over ω and let*

$$\mu_\omega = \frac{1}{|\omega|}\int_\omega q(x)\,dx \qquad \text{and} \qquad \tau_\omega = \frac{1}{|\omega|}\int_\omega r^2(x)\,dx$$

denote the mean values of q and r^2, respectively, on ω. Then

$$\int_\omega \Big(p(x)|u'(x)|^2 + q(x)|u(x)|^2\,dx\Big) \ge \tau_\omega^{-1}\left(\mu_\omega^{-1} + |\omega|\int_\omega p^{-1}(x)\,dx\right)^{-1}\|u\|_{\omega,r}^2 \tag{3}$$

for every function $u \in C^1[x_1,x_2]$.

Proof. Let $\rho \in C^1[x_1,x_2]$ be a real-valued function with a zero point $x_0 \in [x_1,x_2]$. Then

$$\begin{aligned}\rho^2(x) = \left(\int_{x_0}^{x}\rho'(t)\,dt\right)^2 &\le \left(\int_{x_0}^{x}p^{-1}(t)\,dt\right)\left(\int_{x_0}^{x}p(t)(\rho'(t))^2dt\right)\\ &\le \left(\int_\omega p^{-1}(t)\,dt\right)\left(\int_\omega p(t)(\rho'(t))^2dt\right)\end{aligned}$$

and

$$\|r\rho\|_\omega^2 \le \left(\int_\omega r^2(x)\,dx\right)\left(\int_\omega p^{-1}(x)\,dx\right)\left(\int_\omega p(x)(\rho'(x))^2dx\right). \tag{4}$$

Let $u_0 = \varphi_0 + i\psi_0$ denote such a value of the complex function $u(x) = \varphi(x) + i\psi(x)$ that $|u(x)|$ ist minimal on ω. It follows from (4) that

$$\int_\omega r^2(x)\,|u(x)-u_0|^2dx \le \left(\int_\omega r^2(x)\,dx\right)\left(\int_\omega p^{-1}(x)\,dx\right)\left(\int_\omega p(x)\,|u'(x)|^2dx\right). \tag{5}$$

Hence,

$$\begin{aligned}
&\int_\omega \Big(p(x)|u'(x)|^2 + q(x)|u(x)|^2\Big)dx \\
&\geq \left(\left(\int_\omega r^2(x)\,dx\right)\left(\int_\omega p^{-1}(x)\,dx\right)\right)^{-1} \|u - u_0\|^2_{\omega,r} + \int_\omega q(x)\,|u_0|^2\,dx \\
&= \left(\left(\int_\omega r^2(x)\,dx\right)\left(\int_\omega p^{-1}(x)\,dx\right)\right)^{-1} \|u - u_0\|^2_{\omega,r} \\
&\quad + \left(\int_\omega r^2(x)\,dx\right)^{-1}\left(\int_\omega q(x)\,dx\right) \|u_0\|^2_{\omega,r} \\
&= \tau_\omega^{-1}\left(\left(|\omega| \int_\omega p^{-1}(x)\,dx\right)^{-1} \|u - u_0\|^2_{\omega,r} + \mu_\omega \|u_0\|^2_{\omega,r}\right) \\
&= \tau_\omega^{-1}\left\{\frac{\mu_\omega}{1+\delta^2}\left(\delta^2\left(|\omega|\mu_\omega \int_\omega p^{-1}(x)\,dx\right)^{-1}\left(1+\frac{1}{\delta^2}\right)\|u - u_0\|^2_{\omega,r}\right.\right. \\
&\quad \left.\left. + \ (1+\delta^2)\,\|u_0\|^2_{\omega,r}\right)\right\}
\end{aligned}$$

for every $\delta > 0$. If we choose

$$\delta^2 = |\omega|\,\mu_\omega \int_\omega p^{-1}(x)\,dx$$

and use the inequality

$$\|u\|^2_{\omega,r} \leq \left(1+\frac{1}{\delta^2}\right) \|u - u_0\|^2_{\omega,r} + (1+\delta^2)\,\|u_0\|^2_{\omega,r}$$

we obtain (3) ∎

Consider the integral

$$d^\lambda \int_x^{x+d} r^2(t)\,dt \qquad (x \geq 0,\ d > 0,\ -\infty < \lambda < +\infty)$$

and define a function d by

$$d(x) = \sup\left\{d > 0 \,\middle|\, d^\lambda \int_x^{x+d} r^2(t)\,dt = 1\right\} \qquad (x \geq 0,\ -\infty < \lambda < +\infty) \qquad (6)$$

where λ is a suitable number. If

$$\int_0^\infty r^2(t)\,dt = \infty,$$

for instance, each $\lambda \geq 0$ can be used in (6). In case that

$$\int_0^\infty r^2(t)\,dt < \infty,$$

each $\lambda > 0$ is usable, but also each $\lambda < -1$. If $r(x) \geq c > 0 \ \ (x \geq 0)$, one can choose some $\lambda > -1$.

Setting

$$I(x) = [x, x + d(x)], \qquad |I(x)| = d(x)$$

we have

$$|I(x)|^\lambda \int_{I(x)} r^2(t)\,dt = 1 \qquad (0 \leq x < \infty,\ \lambda \text{ suitable}). \tag{7}$$

We remark that for certain λ and x the case $d(x) = \infty$ may happen (consider, for instance, $\lambda = -1$ and $r(x) = 1$ for $0 \leq x < \infty$). Then we have $I(x) = [x, \infty)$ or we take a suitable finite number $d(x) \geq 1$ with

$$d^\lambda(x) \int_x^{x+d(x)} r^2(t)\,dt = 1.$$

Define $x_0 = 0$ and the points $x_i \ \ (i \in \mathbb{N})$ by $x_i = x_{i-1} + d(x_{i-1})$.

Lemma 2. *The semi-axis* $[0, \infty)$ *can be covered by the intervals* $I(x_i) \ \ (i \in \mathbb{N})$.

Proof. We have to show that $x_i \to \infty$ for $i \to \infty$. If it is not, then $x_i \to \xi \in (0, \infty)$ for $i \to \infty$. By the mean value theorem of integral calculus it follows from (7) that

$$1 = |I(x_i)|^{\lambda+1}\, r^2(\xi_i) \qquad (x_i \leq \xi_i \leq x_{i+1},\ i \in \mathbb{N}). \tag{8}$$

Since $r^2(\xi_i) \to r^2(\xi) > 0$ and $|I(x_i)|^{\lambda+1} \to 0 \ \ (\lambda > -1)$ or $|I(x_i)|^{\lambda+1} \to \infty \ \ (\lambda < -1)$ for $i \to \infty$, equations (8) lead to a contradiction if $\lambda \neq -1$. In case $\lambda = -1$ we get the contradiction as follows. Let i_0 be so large that $\xi - x_{i_0} < 1$. Then

$$\begin{aligned} \frac{1}{\xi - x_{i_0}} \int_{x_{i_0}}^{\xi} r^2(t)\,dt &= \frac{1}{\xi - x_{i_0}} \left(\sum_{i=i_0}^{\infty} \int_{x_i}^{x_{i+1}} r^2(t)\,dt \right) \\ &= \frac{1}{\xi - x_{i_0}} \left(\sum_{i=i_0}^{\infty} \frac{x_{i+1} - x_i}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} r^2(t)\,dt \right) \\ &= \frac{1}{\xi - x_{i_0}} \sum_{i=i_0}^{\infty} (x_{i+1} - x_i) = 1. \end{aligned}$$

Hence, by construction (6), $x_{i_0+1} = x_{i_0} + d(x_{i_0}) \geq \xi$. Thus, we have a contradiction also in the case $\lambda = -1$ ∎

Since the form $a[u,v]$, $D[a] = C_0^\infty(0,\infty)$, defined by (2) is densely defined, semi-bounded from below, and closable there exists a self-adjoint operator $\bar{A}$, associated with $\bar{a}[u,v]$ $(u,v \in D[\bar{a}])$, the closure of $a[u,v]$, $D[a]$, such that

$$\bar{a}[u,v] = \int_0^\infty \bar{A}u \cdot \bar{v}\,dx \qquad \big(u \in D[\bar{A}],\, v \in D[\bar{a}]\big).$$

Lemma 3. *The spectrum of $\bar{A}$ is discrete if and only if for every $K > 0$ there exists a $c_K > 0$ such that*

$$\int_{c_K}^\infty Au \cdot \bar{u}\,dx \geq K\, \|u\|^2_{(c_K,\infty),r} \qquad \big(u \in C_0^\infty(c_K,\infty)\big).$$

For the proof see [10].

3. The spectrum

Now we come to the main results. Define with $\omega = [a,b]$ and $k = 0,1$

$$\mu_{\omega,k} = \frac{1}{|\omega|}\int_\omega a_k(t)\,dt, \qquad \rho_{\omega,k} = \frac{1}{|\omega|}\int_\omega a_k^{-1}(t)\,dt, \qquad \tau_\omega = \frac{1}{|\omega|}\int_\omega r^2(t)\,dt.$$

Theorem. *Let the functions a_0 and a_1 satisfy the following condition: There exist $X > 0$, $c > 0$ and $\mu \in (-\infty, +\infty)$ such that*

$$\left(\int_{I(x)} a_0(t)\,dt\right)\left(\int_{I(x)} a_1^{-1}(t)\,dt\right) \leq c\,|I(x)|^\mu \tag{9}$$

for all intervals $I(x)$ $(x \geq X)$ of (7).

(i) *If*

$$\sup_{0<x<\infty}\Big(|I(x)|^{-\lambda-1}\big(1 + |I(x)|^\mu\big)\,\rho_{I(x),0}\Big) < \infty,$$

then $\inf \sigma_e[\bar{A}] > 0$ where $\sigma_e[\bar{A}]$ is the set of the essential spectrum of $\bar{A}$.

(ii) *If*

$$\lim_{x\to\infty} |I(x)|^{-\lambda-1}\big(1 + |I(x)|^\mu\big)\,\rho_{I(x),0} = 0, \tag{10}$$

then the spectrum of $\bar{A}$ is discrete.

Proof. (i) By condition (9) and

$$|I(x)|^2 = \left(\int_{I(x)} 1\,dt\right)^2 \leq \left(\int_{I(x)} a_0(t)\,dt\right)\left(\int_{I(x)} a_0^{-1}(t)\,dt\right)$$

we have

$$\mu_{I(x),0}^{-1} \le \rho_{I(x),0} \qquad \text{and} \qquad \rho_{I(x),1} \le c\,|I(x)|^{\mu-2}\rho_{I(x),0}. \tag{11}$$

Let $(x_0, x_1, \ldots)$, $I(x_i) = [x_i, x_{i+1}]$, be the decomposition of $[0,\infty)$ considered in Lemma 2. By Lemma 1, (7), (9) and (11) we get

$$\begin{aligned}
\int_0^\infty Au\cdot\bar{u}\,dx &= \sum_{k=0}^{n}\int_0^\infty a_k(x)\,|u^{(k)}(x)|^2\,dx \\
&\ge \sum_{k=0}^{1}\int_0^\infty a_k(x)\,|u^{(k)}(x)|^2\,dx \\
&= \sum_{k=0}^{1}\sum_{i=0}^{\infty}\int_{I(x_i)} a_k(x)\,|u^{(k)}(x)|^2\,dx \\
&\ge \sum_{i=0}^{\infty}\Big(\tau_{I(x_i)}\big(\mu_{I(x_i),0}^{-1} + |I(x_i)|^2\rho_{I(x_i),1}\big)\Big)^{-1}\|u\|^2_{I(x_i),r} \\
&\ge \sum_{i=0}^{\infty}\Big(\tau_{I(x_i)}\big(\mu_{I(x_i),0}^{-1} + c\,|I(x_i)|^{\mu}\rho_{I(x_i),0}\big)\Big)^{-1}\|u\|^2_{I(x_i),r} \\
&\ge \sum_{i=0}^{\infty}\Big(\tau_{I(x_i)}\big(1 + c\,|I(x_i)|^{\mu}\big)\rho_{I(x_i),0}\Big)^{-1}\|u\|^2_{I(x_i),r} \\
&= \sum_{i=0}^{\infty}\Big(|I(x_i)|^{-\lambda-1}\big(1 + c\,|I(x_i)|^{\mu}\big)\rho_{I(x_i),0}\Big)^{-1}\|u\|^2_{I(x_i),r} \\
&\ge c_1\sum_{i=0}^{\infty}\Big(\sup_i\big(|I(x_i)|^{-\lambda-1}\big(1 + |I(x_i)|^{\mu}\big)\rho_{I(x_i),0}\big)\Big)^{-1}\|u\|^2_{I(x_i),r} \\
&\ge c_1\Big(\sup_{0<x<\infty}\Big(|I(x)|^{-\lambda-1}\big(1 + |I(x)|^{\mu}\big)\rho_{I(x),0}\Big)\Big)^{-1}\|u\|^2_{I(x_i),r}.
\end{aligned}$$

Hence, if

$$\sup_{0<x<\infty}\Big(|I(x)|^{-\lambda-1}\big(1 + |I(x)|^{\mu}\big)\rho_{I(x),0}\Big) < \infty$$

we have

$$\int_0^\infty Au\cdot\bar{u}\,dx \ge c\,\|u\|^2_{(0,\infty),r} \qquad \big(u \in C_0^\infty(0,\infty),\ c > 0\big)$$

that is $\inf \sigma_e[\tilde{A}] > 0$.

(ii) If (10) holds, then for each $K > 0$ there exists a $X > 0$ such that

$$|I(x)|^{-\lambda-1}\big(1 + |I(x)|^{\mu}\big)\rho_{I(x),0} < \frac{c_1}{K} \qquad (x \ge X)$$

which implies

$$\int_X^\infty Au\cdot\bar{u}\,dx \ge K\,\|u\|^2_{(X,\infty),r} \qquad \big(u \in C_0^\infty(X,\infty)\big).$$

By Lemma 3 the spectrum of $\bar{A}$ is discrete ∎

Corollary 1. *Assume* $r(x) \le C$ $(0 \le x < \infty)$ *and choose a positive* λ. *Let the functions* a_0 *and* a_1 *satisfy condition* (9) *where* $\mu \le 2 + \lambda$. *If*

$$\lim_{x\to\infty} \int_{I(x)} a_0^{-1}(t)\,dt = 0, \tag{12}$$

then the spectrum of $\bar{A}$ *is discrete.*

Proof. By (7) we get

$$1 = |I(x)|^{\lambda} \int_{I(x)} r^2(t)\,dt \le C^2 |I(x)|^{\lambda+1}.$$

Hence, there exists a $c_1 > 0$ such that $|I(x)| \ge c_1$ $(0 \le x < \infty)$. Thus, by (12) it follows that (10) is satisfied ∎

Corollary 2. *Assume that there exist positive constants* c_1, c_2 *and* $X > 0$ *such that* $c_1 \le r(x) \le c_2$ $(x \ge X)$, *and let there exist a* $C > 0$ *such that*

$$\left(\int_x^{x+c_1^{-2}} a_0(t)\,dt\right)\left(\int_x^{x+c_1^{-2}} a_1^{-1}(t)\,dt\right) \le C \qquad (X \le x < \infty). \tag{13}$$

If

$$\lim_{x\to\infty} \int_x^{x+\omega} a_0^{-1}(t)\,dt = 0 \tag{14}$$

where ω *is positive and fixed, then the spectrum of* $\bar{A}$ *is discrete.*

Proof. Choose $\lambda = 0$ in (7). By

$$c_1^2\,|I(x)| \le \int_{I(x)} r^2(t)\,dt = 1 \le c_2^2\,|I(x)| \qquad (X \le x < \infty)$$

it follows that $c_2^{-2} \le |I(x)| \le c_1^{-2}$. Therefore, by (13) condition (9) with $\mu = 0$ is satisfied and by (14) condition (10) is fulfilled ∎

Example 1 (to Corollary 1). Consider problem (1) on $[1,\infty)$. Let $r(x) = x^{-1}$, $a_1(x) = x^{-1}$, $I(x) = [x, 2x]$, $\lambda = 1$ and $\mu = 3$. Concerning (7) it follows that

$$|I(x)|^{\lambda} \int_{I(x)} r^2(t)\,dt = \frac{1}{2}.$$

The fact, that in (7) the number 1 on the right-hand side is replaced by an other positive constant is insignificant for our investigations. We have $\mu = 2 + \lambda$, and the conditions (9) and (10) look like

$$\int_x^{2x} a_0(t)\,dt \le cx^3 \quad (c > 0) \qquad \text{and} \qquad \int_x^{2x} a_0^{-1}(t)\,dt \to 0 \quad (t \to \infty),$$

respectively. These conditions are satisfied if, for instance, we choose

$$a_0(x) = \begin{cases} n^{-1/2} & \text{if } n \le x < n + n^{-2},\ n \in \mathbb{N} \\ x^2 & \text{otherwise.} \end{cases}$$

The spectrum of $\bar{A}$ is discrete.

Setting $\mu = \frac{3}{2}$ conditions (9) and (10) look like

$$\int_x^{2x} a_0(t)\, dt \le c x^{3/2} \qquad \text{and} \qquad x^{-3/2} \int_x^{2x} a_0^{-1}(t)\, dt \to 0,$$

respectively. A suitable function a_0 is then

$$a_0(x) = \begin{cases} n^{1/2} & \text{for } n \le x < n + \frac{1}{2} \\ n^{-1/3} & \text{for } n + \frac{1}{2} \le x < n + 1. \end{cases}$$

Corollary 3. *Assume* $\liminf_{x\to\infty} a_0(x) > 0$, $r(x) \to 0$ *for* $x \to \infty$ *and*

$$\int_0^\infty r^2(t)\, dt = \infty.$$

If condition (9) *can be satisfied by a number* μ *with* $\mu < 1$, *then the spectrum of* $\bar{A}$ *is discrete.*

Proof. By choosing $\lambda = 0$ we see that $|I(x)| \to \infty$ as $x \to \infty$. Since

$$\limsup_{x\to\infty} \rho_{I(x),0} < \infty,$$

condition (10) is satisfied ∎

Example 2 (to Corollary 3). Let $r(x) = x^{-1/2}$, $a_0(x) = 1$ and $I(x) = [x, 2x]$ $(x \ge 1)$, $\lambda = 0$ and $\mu < 1$. Then

$$|I(x)|^\lambda \int_{I(x)} r^2(t)\, dt = \ln 2.$$

The spectrum is discrete if (see (9))

$$\int_x^{2x} a_1^{-1}(t)\, dt \le c x^{\mu-1} \qquad (c > 0).$$

In case that $\frac{1}{2} \le \mu < 1$ a possible function a_1 is defined by $a_1^{-1}(x) = x^{-3/2} + \varphi(x)$ $(x \ge 1)$ where

$$\varphi(x) = \begin{cases} 2 & \text{for } 1 \le x < 2 \\ n^4(n - x) + n & \text{for } n \le x < n + n^{-3},\ 2 \ge n \in \mathbb{N} \\ 0 & \text{for } n + n^{-3} \le x < n + 1 - (n+1)^{-3} \\ (n+1)^4(x - n - 1) + n + 1 & \text{for } n + 1 - (n+1)^{-3} \le x < n + 1. \end{cases}$$

Example 3. Choose $r(x) = x^\sigma$ $(-\infty < \sigma < \infty)$, $a_0(x) = x^\alpha$ $(2\sigma < \alpha < \infty,\ x \geq 1)$, $I(x) = [x, 2x]$ and $\lambda = -2\sigma - 1$. Then

$$|I(x)|^\lambda \int_{I(x)} r^2(t)\,dt = \frac{2^{2\sigma+1} - 1}{2\sigma + 1} \qquad \left(\sigma \neq -\frac{1}{2}\right)$$

(in case $\sigma = -\frac{1}{2}$ the constant on the right-hand side equals $\ln 2$). If

$$\int_x^{2x} a_1^{-1}(t)\,dt \leq c x^\rho \qquad (\rho < -2\sigma - 1)$$

holds, then the spectrum of $\bar{A}$ is not discrete.

Proof. By setting $\mu = \alpha + \rho + 1$ condition (9) is fulfilled, and because of $\mu = \alpha + \rho + 1 < \alpha - 2\sigma = \alpha + \lambda + 1$ condition (10) is also satisfied.

Remark. If the inequalities $2\sigma < \alpha$ and $\rho < -2\sigma - 1$ are replaced by $2\sigma = \alpha$ and $\rho = -2\sigma - 1$ the assertion of Example 3 is not true in general. Choose, for instance, $\sigma = \alpha = 0$, $\rho = -1$, and $a_1(t) = t^2$: in the case

$$Ay = -(t^2 y')' + y \qquad (x \in (1, \infty))$$

the spectrum of $\bar{A}$ is not discrete ∎

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References

[1] Edmunds, D. E. and W. D. Evans: *Spectral Theory and Differential Operators.* Oxford: University Press 1987.

[2] Edmunds, D. E. and Sun Jiong: *Embedding theorems and the spectra of certain differential operators.* Proc. Royal Soc. London A 434 (1991), 643 - 656.

[3] Hinton, D. and R. T. Lewis: *Singular differential operators with spectra discrete and bounded below.* Proc. Royal Soc. London A 84 (1979), 117 - 134.

[4] Kamimura, Y.: *A criterion for the complete continuity of the resolvent of a 2nd order differential operator with complex coefficients.* Proc. Royal Soc. Edinburgh B 116 (1990), 161 - 176.

[5] Kwong, M. and A. Zettl: *Discreteness conditions for the spectrum of ordinary differential operators.* J. Diff. Equat. 40 (1981), 53 - 70.

[6] Müller-Pfeiffer, E.: *Spectral Theory of Ordinary Differential Operators.* Chichester: Ellis Horwood 1981.

[7] Oinarov, R. and M. Otelbaev: *A criterion for the discreteness of the spectrum of the general Sturm-Liouville operator and embedding theorems connected with it.* Diff. Uravneniya 24 (1988), 584 - 594.

[8] Sun Jiong: *On the spectrum of a class of differential operators and embedding theorems.* Acta Math. Sinica (New Series) 10 (1994).

[9] Sun Jiong: *The qualitative analysis of the spectrum of ordinary differential operators* (in Chinese). Adv. Math. (China) (to appear).

[10] Read, T. T.: *Positive and discrete spectra for differential operators.* J. Diff. Equat. 43 (1982), 1 - 27.

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