A Fully Discrete Galerkin Method
for Integral and Pseudodifferential Equations
on Closed Curves

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Abstract. We propose a cheap fully discrete version of the trigonometric Galerkin method
of optimal accuracy order for integral and pseudodifferential equations on closed curves. A
practical implementation of the method leads to a band system of linear algebraic equations.
The error analysis is based on a thorough study of related integral operators in Sobolev spaces
of periodic functions.

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1. Introduction

This paper is devoted to methods of solving integral and pseudodifferential equations
in Sobolev spaces of periodic functions. These equations arise using a potential type
representation of the solution of boundary value problems on a region $Q \subset \mathbb{R}^2$ with a
smooth Jordan curve $\Gamma = \partial Q$ as the boundary of $Q$.

Let us present some examples of boundary integral equations.

Example 1.1. The Symm’s integral equation

$$-\frac{1}{2\pi} \int_{\Gamma} \log |x - y| v(y) d\Gamma_y = g(x) \quad (x \in \Gamma)$$

arises solving the Dirichlet problem for the Laplace equation in $Q$ (see, e.g., [7: p. 303]).
It is of interest also when a conformal map of $Q$ onto the unit disc is constructed (see,
e.g., [8]). Introducing a smooth 1-periodic parametrization $x : \mathbb{R} \rightarrow \Gamma$ of $\Gamma$ such that
$|x'(t)| > 0$ for all $t \in \mathbb{R}$, the equation reduces to

$$\int_{0}^{1} \log |x(t) - x(s)| u(s) ds = f(t) \quad (t \in [0, 1])$$
or
\[
\int_0^1 \kappa_0(t-s)u(s)\,ds + \int_0^1 a_1(t,s)u(s)\,ds = f(t) \quad (t \in [0,1])
\] (1.1)

where

\[
u(t) = v(x(t))|x'(t)|, \quad f(t) = -2\pi g(x(t)), \quad \kappa_0(t) = \log |\sin \pi t|
\]

and

\[
a_1(t,s) = \begin{cases} 
\log \frac{|x(t) - x(s)|}{|\sin \pi (t-s)|} & \text{for } t \neq s \pmod{1} \\
\log \frac{1}{\pi} |x'(t)| & \text{for } t = s \pmod{1}.
\end{cases}
\]

The kernel \(a_1\) is smooth and 1-biperiodic, the Fourier coefficients of \(\kappa_0\) are known: \(\hat{\kappa}_0(0) = -\log 2\) and \(\hat{\kappa}_0(m) = -\frac{1}{2}|m|^{-1} \quad (0 \neq m \in \mathbb{Z})\).

In [7: p. 326] and [9] one can find also boundary integral equations of the Neumann problem for the Laplace equation.

**Example 1.2.** In the case of the exterior Dirichlet boundary value problem for the Helmholtz equation one has a boundary integral equation of the type (see [11])

\[
\int_0^1 \kappa_0(t-s)u(s)\,ds + \int_0^1 \kappa_1(t-s)a_1(t,s)u(s)\,ds + \int_0^1 a_2(t,s)u(s)\,ds = f(t) 
\] (1.2)

where \(a_1\) and \(a_2\) are smooth 1-biperiodic functions, \(\kappa_0(t) = \log |\sin \pi t|\) and \(\kappa_1(t) = (\sin \pi t)^2 \log |\sin \pi t|\). For the Fourier coefficients of \(\kappa_0\) and \(\kappa_1\) we have \(\hat{\kappa}_0(m) \sim |m|^{-1}\) as in Example 1.1 and \(\hat{\kappa}_1(m) \sim |m|^{-3}\) (cf. Example 1.4).

**Example 1.3.** The exterior Neumann boundary value problem for the Helmholtz equation can be reduced to an equation of the type (see [18])

\[
\frac{\pi}{|x'(t)|} \int_0^1 \kappa_0(t-s)u(s)\,ds + iu(t)
\]

\[
+ \int_0^1 \kappa_2(t-s)a_2(t,s)u(s)\,ds + \int_0^1 a_3(t,s)u(s)\,ds = f(t) 
\] (1.3)

where \(a_2\) and \(a_3\) are smooth 1-biperiodic functions, \(\kappa_0(t) = 1/\sin^2 \pi t\) and \(\kappa_2(t) = \log |\sin \pi t|\) whereby the first integral in (1.3) is understood in the sense of the Hadamard finite part. For the Fourier coefficients of \(\kappa_0\) and \(\kappa_2\) we have \(\hat{\kappa}_0(0) = 0, \hat{\kappa}_0(m) = -2|m| \quad (0 \neq m \in \mathbb{Z})\) and \(|\hat{\kappa}_2(m)| \sim |m|^{-1}\) (see Example 1.1).
Example 1.4. Solving boundary value problems for biharmonic equations the boundary integral equation
\[
\int_{\Gamma} |x - y|^2 \log |x - y| v(y) \, d\Gamma_y = g(x) \quad (x \in \Gamma)
\] (1.4)
is of interest (see [4] and [7: p. 336]). It reduces to the integral equation
\[
\int_{0}^{1} \kappa_0(t - s)a_0(t, s)u(s) \, ds + \int_{0}^{1} a_1(t, s)u(s) \, ds = f(t) \quad (t \in [0, 1])
\]
with \(u(s) = v(x(s))|x'(s)|, f(t) = g(x(t)), 1\)-biperiodic smooth functions

\[
a_0(t, s) = \begin{cases} 
\frac{|x(t) - x(s)|^2}{\sin^2 \pi (t - s)} & \text{for } t \neq s \pmod{1} \\
\frac{1}{\pi^2} |x'(t)|^2 & \text{for } t \equiv s \pmod{1}
\end{cases}
\]

\[
a_1(t, s) = |x(t) - x(s)|^2 \log \frac{|x(t) - x(s)|}{|\sin \pi (t - s)|}
\]
and 1-periodic function

\[
\kappa_0(t) = (\sin \pi t)^2 \log |\sin \pi t|
\]
with known Fourier coefficients:

\[
\hat{\kappa}_0(0) = -\frac{1}{2} \log 2 + \frac{1}{4}, \quad \hat{\kappa}_0(1) = \kappa_0(-1) = \frac{1}{4} \log 2 - \frac{3}{16}
\]
\[
\hat{\kappa}_0(m) = (4|m|(|m^2 - 1|))^{-1} \quad (m \in \mathbb{Z}, |m| \geq 2).
\]
Note that \(a_0(t, t) \neq 0\).

Example 1.5. In this example we identify \(\mathbb{R}^2\) with \(\mathbb{C}\) treating \(x = x_1 + ix_2\) and \(y = y_1 + iy_2\) as complex numbers. Consider the singular integral equation
\[
\frac{b(x)}{\pi i} \int_{\Gamma} \frac{v(y)}{y - x} \, dy + \int_{\Gamma} K(x, y)v(y) \, d\Gamma_y = g(x) \quad (x \in \Gamma)
\]
where \(\Gamma\) is a smooth Jordan curve in the complex plain, and \(b\) and \(K\) are given smooth functions on \(\Gamma\) and \(\Gamma \times \Gamma\), respectively. A smooth 1-periodic parametrization \(x = x_1 + ix_2 : \mathbb{R} \rightarrow \Gamma\) of \(\Gamma\) satisfying \(x'(t) = x'_1(t) + ix'_2(t) \neq 0\) for all \(t \in \mathbb{R}\) yields the equation
\[
\int_{0}^{1} \kappa_0(t - s)a_0(t, s)u(s) \, ds + \int_{0}^{1} a_1(t, s)u(s) \, ds = f(t) \quad (1.5)
\]
where \(a_0(t, t) = \frac{2}{1 - e^{i2\pi t}} = 1 + i \cot \pi t\),

\[
a_0(t, s) = \begin{cases} 
\frac{b(x(t))}{2\pi i} \left(1 - e^{i2\pi (t - s)}\right) \frac{x'(s)}{x(s) - x(t)} & \text{for } t \neq s \pmod{1} \\
b(x(t)) & \text{for } t \equiv s \pmod{1}
\end{cases}
\]
and
\[ a_1(t, s) = K(x(t), x(s))x'(s), \quad u(s) = v(x(s)), \quad f(t) = g(x(t)). \]

The functions \( a_0 \) and \( a_1 \) are 1-biperiodic and smooth (note that \( x(s) \neq x(t) \) for \( s \neq t \) (mod 1) since \( \Gamma \) as a Jordan curve does not cut itself). Further, for the Fourier coefficients of \( \kappa_0 \) we have \( \kappa_0(m) = 1 \) for \( m \geq 0 \) and \( \kappa_0(m) = -1 \) for \( m < 0 \). The first integral in (1.5) is understood in the sense of the Cauchy main value. Note that \( a_0(t, t) \neq 0 \) for \( t \in \mathbb{R} \) if \( b(x) \neq 0 \) for \( x \in \Gamma \).

The integral equations (1.1) - (1.5) are of the form
\[
\sum_{p=0}^{q} \int_{0}^{1} \kappa_p(t - s)a_p(t, s)u(s) \, ds = f(t) \tag{1.6}
\]
where \( a_p \) are 1-biperiodic smooth functions and
\[
\int_{0}^{1} \kappa_p(t - s)e^{im2\pi t} \, ds = \hat{\kappa}_p(m)e^{im2\pi t} \quad (m \in \mathbb{Z}) \tag{1.7}
\]
with known Fourier coefficients \( \hat{\kappa}_p(m) \) \((m \in \mathbb{Z}, p = 0, \ldots, q)\) of the order \( |\hat{\kappa}_p(m)| \sim |m|^{-\alpha_p} \) \((\alpha_0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_q)\). Usually \( \alpha_p > 0 \) but in Example 1.3 \( \alpha_0 = -1 \) and in Example 1.5 \( \alpha_0 = 0 \). Note that \( \kappa_p \) is continuous if \( \alpha_p > 1 \) and may have weak singularity if \( 0 < \alpha_p \leq 1 \). For \( \alpha_p \leq 0 \), the corresponding integral in (1.6) should be understood in the sense of distributions; a more familiar understanding follows from (1.7) using the Fourier expansions of \( a_p \) and \( u \) (see Sections 2 and 3 for more details).

However, if \( \hat{\kappa}_p(m) \) depends on \( m \) in a "smooth" way, namely, if \( \hat{\kappa}_p(m) = |m|^\alpha \) or \( \hat{\kappa}_p(m) = |m|^{\alpha_0} \) \( \alpha, \alpha_0 \in \mathbb{R} \), it is possible to construct a representation
\[
\kappa_p(t - s)a_p(t, s) = \sum_{k=0}^{n} b_{p,k}(t) \kappa_{p,k}(t - s) + a_{p,n+1}(t, s)\kappa_{p,n+1}(t - s) \quad (n \geq 0) \tag{1.8}
\]
where \( |\hat{\kappa}_{p,k}(m)| \sim |m|^{-k} \) \((k = 0, \ldots, n + 1)\), the coefficients \( b_{p,k} \) are smooth and 1-periodic, and \( a_{p,n+1} \) is smooth and 1-biperiodic. Unfortunately, this decomposition is somewhat impractical due to the complexity of the construction and involving high order derivatives of the parametrization function \( x = x(t) \) in Examples 1.2 - 1.5. We shall solve (1.6) directly avoiding possible decompositions of the type (1.8). This also allows to avoid the "smoothness" assumptions for \( \hat{\kappa}_p(m) \).

The trigonometric Galerkin and collocation methods, and fully discretized versions of those have been extensively examined for Symm's integral equation and other integral equation of type (1.1) with \( |\hat{\kappa}_0(m)| \sim |m|^{-\alpha} \) \((\alpha \in \mathbb{R})\) and smooth \( a_1 \) (see, e.g., [2, 10, 13, 21, 23, 24]). More general problems have been treated in [1, 14 - 16, 18, 20, 22]. We refer also to [3] where a fast algorithm is proposed for the generalized airfoil equation with constant coefficients in a non-periodical case. In [11], the product integration was incorporated into the collocation method to solve the integral equation (1.2). This idea
was used also in [4] and [18] for equations (1.3) and (1.4) and in [25] for general integral equations of type (1.6) with $a_0$ depending on both $t$ and $s$.

In this paper we exploit another idea. We start from the trigonometric Galerkin method and obtain a full discretization using trigonometric interpolants of $a_p$ ($p = 0, \ldots, q$). This idea seems to originate from works of P. Henrici (see, e.g., [8]). The order $M$ of interpolants may be essentially smaller than the order $N$ of the approximate solution $u_N$, e.g. $M \sim N^\sigma$ ($0 < \sigma \leq 1$) is acceptable. This leads to a band systems of linear algebraic equations which can be solved in $O(N^{1+2\sigma})$ arithmetical operations. The proposed method is of optimal convergence order in Sobolev norms. Beside (1.6) we discuss the case with the "full" main part $A_0$ (see (2.19)).

The paper is organized as follows. Section 2 contains the description of the method and formulation of main results. In Section 3 we discuss a matrix form of the method. After some preliminaries in Section 4, we examine the properties of integral operators trying to minimize the smoothness assumptions on the coefficients $a_p$. After that, in Section 6, we are ready to present the proofs of the main results formulated in Section 2. Numerical examples are presented in Section 7.

2. Problem, methods, and formulation of the main results

Now we formulate more precisely the conditions posed on the problem

$$A u = f$$

(2.1)

where $f$ is a given 1-periodic function and $u$ is a 1-periodic function which is looked for. The operator has the form

$$A = \sum_{p=0}^{q} A_p, \quad (A_p u)(t) = \int_{0}^{1} \kappa_p(t-s)a_p(t,s)u(s)\,ds$$

(2.2)

where the coefficients $a_p$ are assumed to be $C^\infty$-smooth and 1-biperiodic, and the Fourier coefficients

$$\tilde{\kappa}_p(m) = \int_{0}^{1} \kappa_p(t)e^{-im2\pi t}\,dt = \langle \kappa_p(t), e^{-im2\pi t} \rangle \quad (m \in \mathbb{Z})$$

are assumed to be known and to satisfy the inequalities

$$c_1m^{-\alpha} \leq |\tilde{\kappa}_0(m)| \leq c_2m^{-\alpha} \quad (|m| \geq m_0)$$

$$|\tilde{\kappa}_0(m) - \tilde{\kappa}_0(m-1)| \leq cm^{-\alpha-\beta} \quad (m \in \mathbb{Z})$$

(2.3)

$$|\tilde{\kappa}_p(m)| \leq cm^{-\alpha-\beta} \quad (p = 1, \ldots, q, m \in \mathbb{Z})$$

with some $\alpha \in \mathbb{R}$, $\beta > 0$, $m_0 \in \mathbb{N}$, and positive constants $c_1$, $c_2$ and $c$. Here

$$m^* := \begin{cases} |m| & \text{if } m \neq 0 \\ 1 & \text{if } m = 0 \end{cases} \quad (m \in \mathbb{Z}).$$
Since $\alpha$ may be also negative, problem (2.1) actually includes a variety of pseudodifferential equations, and for smooth $u$, the integral in (2.2) actually means the duality product between elements of appropriate Sobolev spaces $H^{-\lambda}$ and $H^{\lambda}$:

$$(A_p u)(t) = \langle \kappa_p(t-s), a_p(t,s)u(s) \rangle.$$ 

An equivalent understanding was outlined in Section 1 on the basis of (1.7) (see also Section 3).

The Sobolev space $H^\mu$ ($\mu \in \mathbb{R}$) consists of 1-periodic functions (distributions)

$$u(s) = \sum_{m \in \mathbb{Z}} \hat{u}(m)e^{im^2\pi s}, \quad \hat{u}(m) = \int_0^1 u(s)e^{-im^2\pi s}ds,$$

such that

$$\|u\|_\mu = \left(\sum_{m \in \mathbb{Z}} m^{2\mu} |\hat{u}(m)|^2 \right)^{1/2} < \infty.$$ 

Similarly, the Sobolev space $H^{\lambda,\mu}$ ($\lambda, \mu \in \mathbb{R}$) consists of 1-biperiodic functions (distributions)

$$v(t,s) = \sum_{l,m \in \mathbb{Z}} \hat{v}(l,m)e^{il2\pi t}e^{im^2\pi s}, \quad \hat{v}(l,m) = \int_0^1 \int_0^1 v(t,s)e^{-il2\pi t}e^{-im^2\pi s}dtds$$

such that

$$\|v\|_{\lambda,\mu} = \left(\sum_{l,m \in \mathbb{Z}} l^{2\lambda} m^{2\mu} |\hat{v}(l,m)|^2 \right)^{1/2} < \infty.$$ 

**Proposition 2.1.** Under conditions (2.3), for $\lambda \in \mathbb{R}$,

$$A_0 \in \mathcal{L}(H^{\lambda,\mu}, H^{\lambda,\mu+\alpha}) \quad \text{and} \quad A_p \in \mathcal{L}(H^{\lambda}, H^{\lambda+\alpha+\beta}) \quad (1 \leq p \leq m).$$ 

If $a_0(t,t) \neq 0$ for all $t \in \mathbb{R}$, then

$$A = \sum_{p=0}^q A_p \in \mathcal{L}(H^{\lambda}, H^{\lambda+\alpha})$$

is a Fredholm operator of index 0 whereby

$$\mathcal{N}(A) := \{u \in H^{\lambda}: Au = 0\} \subset \cap_\mu H^\mu =: C_1^{\infty}.$$ 

The proof of Proposition 2.1 is given in Section 6.
Our further assumptions about problem (2.1) are as follows:

\[
\begin{align*}
\mathcal{b}(t) := a_0(t, t) &
\neq 0 \quad \text{for all } \ t \in \mathbb{R} \quad (2.4) \\
W(b) := \frac{1}{2\pi} \arg b(t) &\bigg|_{t=0}^{t=1} = 0 \quad (2.5) \\
v \in C_1^\infty, \quad Av = 0 &\implies v = 0. \quad (2.6)
\end{align*}
\]

According to Proposition 2.1 assumptions (2.3), (2.4) and (2.6) guarantee the existence of the bounded inverse \( \mathcal{A}^{-1} \in \mathcal{L}(H^{\lambda+\alpha}, H^\lambda) \) for any \( \lambda \in \mathbb{R} \). Assumption (2.5) is needed for the convergence of the Galerkin method.

For \( N \in \mathbb{N} \) denote

\[
Z_N = \left\{ k \in \mathbb{Z} : -\frac{N}{2} < k \leq \frac{N}{2} \right\} \quad \text{and} \quad T_N = \left\{ u_N = \sum_{k \in Z_N} c_k e^{ik2\pi t} : c_k \in \mathbb{C} \right\}.
\]

In other words, \( T_N \) is the linear span of \( e^{ik2\pi t} \ (k \in Z_N) \). Denote by \( P_N \) the corresponding Fourier projection:

\[
(P_N u)(t) = \sum_{k \in Z_N} \hat{u}(k) e^{ik2\pi t} \quad (u \in H^\lambda).
\]

Obviously, the projection \( P_N \) is orthogonal in \( H^\lambda \) for all \( \lambda \in \mathbb{R} \). It is also clear that

\[
\|u - P_N u\|_\lambda \leq \left( \frac{N}{2} \right)^{\lambda-\mu} \|u\|_\mu \quad (\lambda \leq \mu). \quad (2.7)
\]

First we approximate problem (2.1) by the Galerkin method

\[
u_N \in T_N, \quad P_N Au_N = P_N f. \quad (2.8)
\]

**Theorem 2.2.** Assume (2.3) – (2.6) and \( f \in H^{\mu+\alpha} \ (\mu \in \mathbb{R}) \). Then there is a \( N_0 \) such that for \( N \geq N_0 \) the Galerkin method (2.8) determines a unique polynomial \( u_N \in T_N \) such that

\[
\|u_N - u\|_\lambda \leq c_\lambda \|u - P_N u\|_\lambda \leq c_\lambda \left( \frac{N}{2} \right)^{\lambda-\mu} \|u\|_\mu \quad (\lambda \leq \mu) \quad (2.9)
\]

where \( u = \mathcal{A}^{-1} f \in H^\mu \) is the (unique) solution of problem (2.1), and the constant \( c_\lambda \) is independent of \( N \) and \( u \) (or \( f \)).

The proof of Theorem 2.2 is presented in Section 6.

The Galerkin method performs only a semidiscretization of the problem (2.1). Now we construct a fully discretized version of the method. Let us denote by \( Q_N u \) the interpolation projection of \( u \in H^\mu \ (\mu > \frac{1}{2}) \) to \( T_N \) on the uniform grid:

\[
(Q_N u)(t) = \sum_{k \in Z_N} c_k e^{ik2\pi t}, \quad (Q_N u) \left( \frac{n}{N} \right) = u \left( \frac{n}{N} \right) \quad (n = 0, 1, \ldots, N - 1).
\]
The approximation properties of $Q_N$ are similar to those of $P_N$ (cf. (2.7)) but on a more restricted scale of Sobolev norms [1, 22]:

$$
\|u - Q_N u\|_{\lambda} \leq c_{\lambda,\mu} N^{\lambda - \mu} \|u\|_{\mu} \quad (0 \leq \lambda \leq \mu, \quad \mu > \frac{1}{2}).
$$

(2.10)

Introduce also the interpolation projection $Q_{N,M}$ for functions of two variables:

$$
(Q_{N,M} v)(t, s) = \sum_{k \in \mathbb{Z}_N} \sum_{l \in \mathbb{Z}_M} c_{k,l} e^{i k \pi t} e^{i l \pi s} \in \mathcal{T}_N \otimes \mathcal{T}_M
$$

$$
(Q_{N,M} v) \left( \frac{n}{N}, \frac{m}{M} \right) = v \left( \frac{n}{N}, \frac{m}{M} \right) \quad (n = 0, 1, \ldots, N - 1, \quad m = 0, 1, \ldots, M - 1).
$$

The coefficients $a_p$ in (2.2) may be approximated simply by $Q_{M,M} a_p \in \mathcal{T}_M \otimes \mathcal{T}_M$ but, without a loss of the approximation order, we may drop a part of the Fourier coefficients of $Q_{M,M} a_p$ putting

$$
a_{p,M}(t, s) = \sum_{|k| + |l| \leq M/2} (Q_{M,M} a_p)(k, l) e^{i k \pi t} e^{i l \pi s} \quad (p = 0, \ldots, q).
$$

(2.11)

Due to the $C^\infty$-smoothness of the coefficients $a_p$, we obtain a suitable approximation $a_{p,M}$ already in dimensions $M$ which may be much smaller than $N$. We put

$$
M \sim N^\sigma \quad (0 < \sigma \leq 1)
$$

(2.12)

where $M \sim N^\sigma$ means that $c_1 \leq MN^{-\sigma} \leq c_2$ as $N \rightarrow \infty$ with some positive constants $c_1$ and $c_2$. Introduce the operators (cf. (2.2))

$$
A_M = \sum_{p=0}^q A_{p,M}, \quad (A_{p,M} u)(t) = \int_0^1 \kappa_p(t - s)a_{p,M}(t, s)u(s)ds.
$$

(2.13)

Instead of $P_N f$ we shall use some approximation $f_N \in \mathcal{T}_N$ assuming that $f_N$ is computable. We are ready to introduce a modified Galerkin method which is fully discrete:

$$
u_N \in \mathcal{T}_N, \quad P_N A_M u_N = f_N
$$

(2.14)

(see Section 3 about the implementation of the method).

**Theorem 2.3.** Assume (2.3) -- (2.6) and $f \in H^{\mu + \alpha}$ ($\mu \in \mathbb{R}$). Let the operator $A_M$ be defined by (2.11) -- (2.13). Then there is an $N_0$ such that for $N \geq N_0$ the modified Galerkin method (2.14) determines a unique polynomial $u_N \in \mathcal{T}_N$, and

$$
\|u_N - u\|_{\lambda} \leq c_\lambda \left( \|u - P_N u\|_{\lambda} + \|f_N - P_N f\|_{\lambda + \alpha} + c_{\lambda,r} N^{-r} \|u\|_{\lambda} \right) \quad (\lambda \leq \mu)
$$

(2.15)

where $r > 0$ is arbitrary, $u = A^{-1} f \in H^\mu$ is the solution of problem (2.1) and the constants $c_\lambda$ and $c_{\lambda,r}$ are independent of $N$ and $u$. 
The proof of Theorem 2.3 is given in Section 6.

We complete Theorem 2.3 specifying the error estimate (2.15) for some choices of $f_N \in T_N$.

1. Case: $f_N = Q_Nf$ and $\mu + \alpha > \frac{1}{2}$. Using (2.7) and (2.10) we obtain from (2.15)

$$
\|u_N - u\| \leq c_{\lambda, \mu} N^{\lambda - \mu} \|u\|_{\mu} \quad (-\alpha \leq \lambda \leq \mu).
$$

(2.16)

This is the most standard way to approximate $f$.

2. Case $f_N = \hat{f}(0) + \left(\frac{\mu}{\alpha}\right)^kQ_Nf^{(-k)}$ ($k \in \mathbb{N}$) and $\mu + \alpha > \frac{1}{2}$, where $f^{(0)}(t) = f(t)$ and

$$
c_{-j+1} = \int_0^1 f^{(-j+1)}(s) ds \quad (j = 1, \ldots, k).
$$

$$
f^{(-j)}(t) = \int_0^t \left(f^{(-j+1)}(s) - c_{-j+1}\right) ds
$$

This time (2.15) yields the estimate

$$
\|u_N - u\|_\lambda \leq c_{\lambda, \mu} N^{\lambda - \mu} \|u\|_{\mu} \quad (-\alpha - k \leq \lambda \leq \mu)
$$

(2.17)

which differs from (2.16) by a more wide scale of Sobolev norms towards negative $\lambda$.

Optimal estimates of type (2.17) for strongly negative $\lambda$ are of interest when the solution of a boundary value problem at a point $x \in \Omega$ is determined from the solution of the boundary integral equation on $\Gamma = \partial \Omega$.

Unfortunately, the approximation $f_N$ proposed here is practical only in the case where the values of $f^{(-k)}$ on the grid $\{\{N : j = 0, \ldots, N - 1\}$ are available, e.g. if the integrations to find $f^{(-k)}$ can be performed analytically.

3. Case with known Fourier coefficients of $f$, and $\mu \in \mathbb{R}$ arbitrary. Then one may put $f_N = P_N f$, and (2.15) yields

$$
\|u_N - u\|_\lambda \leq c_{\lambda, \mu} N^{\lambda - \mu} \|u\|_{\mu} \quad (-\infty < \lambda \leq \mu).
$$

(2.18)

Remark 2.4. For simplicity, we have assumed $C^\infty$-smoothness of the coefficients $a_p$ ($p = 0, \ldots, q$). Actually, the arguments of Sections 3 - 6 allow us to point out a finite smoothness of $a_p$ ($p = 0, 1, \ldots, q$) under which the estimates (2.16) - (2.18) remain true for a fixed $\lambda$, or for all $\lambda$ from a finite interval. For instance, in the case $\mu + \alpha > \frac{1}{2}$ and $f_N = Q_N f$, estimate (2.16) remains true for all $\lambda \in [-\alpha, \mu]$ if $a_p \in H^{\lambda_1, \lambda_2}$ ($p = 0, \ldots, q$), with

$$
\lambda_1 = \nu + \frac{\mu + \alpha}{\sigma} \quad \text{and} \quad \lambda_2 = \max(|\alpha|, \nu) + \frac{\mu + \alpha}{\sigma}
$$

where $\nu > \frac{1}{2}$ is arbitrary and $\sigma \in (0, 1]$ is the parameter from condition (2.12).

Remark 2.5. All results can be extended to the case where the operator $A_0$ has a structure

$$
(A_0 u)(t) = \int_0^1 \left(\kappa_+(t - s)a_+(t, s) + \kappa_-(t - s)a_-(t, s)\right) u(s) ds
$$

(2.19)
with smooth 1-biperiodic coefficients $a_{\pm}$ satisfying the conditions (cf. (2.4) and (2.5))

\[ b_1(t) := a_+(t, t) + a_-(t, t) \neq 0 \quad (t \in \mathbb{R}) \]  
\[ b_2(t) := a_+(t, t) - a_-(t, t) \neq 0 \]  

and (see (2.5))

\[ W(b_1) = W(b_2) = 0 \]  

and with 1-periodic functions (distributions) $\kappa_{\pm}$ such that for their Fourier coefficients we have

\[ \hat{\kappa}_+(m) = |m|^{-\alpha} \quad \text{and} \quad \hat{\kappa}_-(m) = |m|^{-\alpha}\text{sign}(m) \quad (0 \neq m \in \mathbb{Z}). \]  

Assumptions (2.3) now reduces to

\[ |\hat{\kappa}_p(m)| \leq cm^{-\alpha - \beta} \quad (p = 1, \ldots, q; \ m \in \mathbb{Z}) \]  

where $\beta > 0$. Of course, we have to assume again (2.6).

Remark 2.6. Every classical elliptic pseudodifferential operator $A$ of negative integer order $r$ on a closed smooth Jordan curve (see, e.g., [1]) can be represented in the form $A = A_0 + B$ where $A_0$ is defined by (2.19), (2.20), (2.22) with $\alpha = |r|$, and $B$ is a smoothing operator, i.e.

\[ (Bu)(t) = \int_0^1 b(t, s)u(s) \, ds \]  

with a smooth 1-biperiodic function $b$. Unfortunately, for a given problem (2.1) and (2.2), an explicit construction of the functions $a_{\pm}$ and $b$ for the representation $A = A_0 + B$ is rather complicated and impractical. We preferred to solve problem (2.1) and (2.2) directly.

3. Matrix form of the method and computational cost

Using the Fourier representations of $u = u(s)$ and $a_p = a_p(t, s)$ it is easy to find the Fourier representation of $Au$ where the operator $A$ is defined by (2.2):

\[ (Au)(t) = \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \sum_{p=0}^q \sum_{m \in \mathbb{Z}} \hat{a}_p(k - m, m - j) \hat{\kappa}_p(m) \hat{u}(j) e^{ik\pi t}. \]  

For $u_N \in T_N$ we have

\[ (P_N Au_N)(t) = \sum_{k \in \mathbb{Z}_N} \sum_{j \in \mathbb{Z}_N} \sum_{p=0}^q \sum_{m \in \mathbb{Z}} \hat{a}_p(k - m, m - j) \hat{\kappa}_p(m) \hat{u}_N(j) e^{ik\pi t}. \]
Thus the (pure) Galerkin method (2.8) is equivalent to the system of linear equations

$$\sum_{j \in Z_N} g_{k,j} \hat{u}_N(j) = \hat{f}(k) \quad (k \in Z_N) \quad (3.1)$$

where

$$g_{k,j} = \sum_{p=0}^{q} \sum_{m \in Z} \hat{a}_p(k-m,m-j)\hat{\kappa}_p(m) \quad (k,j \in Z_N).$$

Similarly, the modified Galerkin methods (2.14) is equivalent to the system of linear equations

$$\sum_{j \in Z_N} g_{k,j}^M \hat{u}_N(j) = \hat{f}_N(k) \quad (k \in Z_N) \quad (3.2)$$

where, for $k,j \in Z_N$,

$$g_{k,j}^M = \begin{cases} \sum_{p=0}^{q} \sum_{m: |m-k|+|m-j| \leq M} (Q_M, M \hat{a}_p)(k-m,m-j)\hat{\kappa}_p(m) & \text{for } |k-j| \leq \frac{M}{2} \\ 0 & \text{for } |k-j| > \frac{M}{2}. \end{cases} \quad (3.3)$$

We obtained a band system of a band width $M+1$. By the standard Gauss elimination method, with pivoting along columns under the main diagonal, system (3.2) can be solved in $O(M^2 N)$ arithmetical operations, or taking into account (2.12), $O(N^{1+2\sigma})$ arithmetical operations. Using the fast Fourier transformation, $(Q_N f)$ or $(Q_N f(-k))$ and $(Q_M, M \hat{a}_p)$ can be found in $O(N \log N)$ and $O(M^2 \log M) = O(N^{2\sigma} \log N)$ arithmetical operations, respectively. Making use of the convolution structure of (3.3) along a fixed diagonal $k-j = \text{const}$, we can use the fast Fourier transformation also to compute the sums over $m$ for $k$ and $j$ on the diagonal. This costs $O(N \log N)$ arithmetical operations for one diagonal and $O(MN \log N) = O(N^{1+\sigma} \log N)$ arithmetical operations for the whole matrix of system (3.2).

Let us summarize.

**Proposition 3.1.** Under the relation $M \sim N^\sigma$ ($0 < \sigma \leq 1$) the computational cost of the fully discretized Galerkin method (2.14) is as follows:

(i) $O(N^{1+\sigma} \log N)$ arithmetical operations for the construction of system (3.2) using the fast Fourier transformation.

(ii) $O(N^{1+2\sigma})$ arithmetical operations for the solution of system (3.2) by the Gauss elimination.

As we saw in Section 2, in the case of sufficiently smooth coefficients $a_p$ ($p = 0, 1, \ldots, q$) the parameter $\sigma$ does not influence on the optimal convergence rate of the method (2.14) but this result is only asymptotical as $N \to \infty$. For moderate $N$ one has to compromise between the accuracy (great $\sigma$) and a cheap implementation of the method (small $\sigma$).
4. Product of biperiodic functions

It is relatively easy to show (see, e.g., [25]) that, for \( \lambda \in \mathbb{R} \) and \( \nu > \frac{1}{2} \),

\[
\|au\|_{\lambda,\nu} \leq c_{\lambda,\nu}\|a\|_{\max(|\lambda|,\nu)}\|u\|_{\lambda,\nu} \quad (a \in H^{\max(|\lambda|,\nu)}, u \in H^{\lambda}). \tag{4.1}
\]

Here we prove a two-dimensional counterpart of (4.1).

**Lemma 4.1.** For any \( \lambda, \mu \in \mathbb{R} \), \( \nu > \frac{1}{2} \), \( u \in H^{\lambda,\mu} \) and \( a \in H^{\max(|\lambda|,\nu), \max(|\mu|,\nu)} \), the inequality

\[
\|au\|_{\lambda,\mu} \leq c_{\lambda,\mu,\nu}\|a\|_{\max(|\lambda|,\nu), \max(|\mu|,\nu)}\|u\|_{\lambda,\mu} \tag{4.2}
\]

holds where the constant \( c_{\lambda,\mu,\nu} \) is independent of \( a \) and \( u \).

**Proof.** Denoting \( a_{l,m} = \hat{a}(l,m) \) and \( u_{p,q} = \hat{u}(p,q) \) we have

\[
(au)(t,s) = \sum_{j,k \in \mathbb{Z}} \left( \sum_{p,q \in \mathbb{Z}} a_{j-p,k-q}u_{p,q} \right)e^{j2\pi tk}e^{k2\pi s}
\]

and

\[
\|au\|_{\lambda,\mu} \leq \left\{ \sum_{j,k \in \mathbb{Z}} \left( \sum_{p,q \in \mathbb{Z}} j^\lambda k^\mu |a_{j-p,k-q}| |u_{p,q}| \right)^2 \right\}^{1/2}. \tag{4.3}
\]

(i) **The case** \( \lambda > \frac{1}{2} \) **and** \( \mu > \frac{1}{2} \). Using the inequalities

\[
j^\lambda \leq 2^\lambda((j-\mu)^\lambda + p^\lambda) \quad \text{and} \quad k^\mu \leq 2^\mu((k-\nu)^\mu + q^\mu) \tag{4.4}
\]
we obtain from (4.3)
\[
\|au\|_{\lambda,\mu} \leq 2^{\lambda+\mu} \left\{ \sum_{j,k \in \mathbb{Z}} \left( \sum_{p,q \in \mathbb{Z}} (j-p)^\lambda (k-q)^\mu |a_{j-p,k-q}| |u_{p,q}| \right) \right. \\
+ \sum_{p,q \in \mathbb{Z}} p^\lambda (k-q)^\mu |a_{j-p,k-q}| |u_{p,q}| \\
+ \sum_{p,q \in \mathbb{Z}} (j-p)^\lambda q^\mu |a_{j-p,k-q}| |u_{p,q}| \\
+ \left. \sum_{p,q \in \mathbb{Z}} p^\lambda q^\mu |a_{j-p,k-q}| |u_{p,q}| \right) \right\}^{1/2} \\
= 2^{\lambda+\mu} \sum_{n=1}^{4} b_n v_n \|_{0,0} \\
\leq 2^{\lambda+\mu} \sum_{n=1}^{4} \|b_n v_n\|_{0,0}
\]

where the functions $b_n$ and $v_n$ are defined by their Fourier coefficients
\[
\hat{b}_1(j,k) = j^{\lambda} k^\mu |a_{j,k}| \quad \hat{v}_1(p,q) = |u_{p,q}| \\
\hat{b}_2(j,k) = k^{\lambda} |a_{j,k}| \quad \hat{v}_2(p,q) = p^\lambda |u_{p,q}| \\
\hat{b}_3(j,k) = j^{\lambda} |a_{j,k}| \quad \hat{v}_3(p,q) = q^\mu |u_{p,q}| \\
\hat{b}_4(j,k) = |a_{j,k}| \quad \hat{v}_4(p,q) = p^\lambda q^\mu |u_{p,q}|.
\]

Let us estimate the norms $\|b_n v_n\|_{0,0}$. Clearly,
\[
\|b_1 v_1\|_{0,0} = \left( \int_0^1 \int_0^1 |b_1(t,s)|^2 |v_1(t,s)|^2 dt ds \right)^{1/2} \\
\leq \|b_1\|_{0,0} \sup_{t,s} |v_1(t,s)| \leq \|a\|_{\lambda,\mu} \sum_{p,q \in \mathbb{Z}} |u_{p,q}| \leq \gamma_\lambda \gamma_\mu \|a\|_{\lambda,\mu} \|u\|_{\lambda,\mu}
\]

where
\[
\gamma_\nu = \left( \sum_{p \in \mathbb{Z}} p^{-2\nu} \right)^{1/2} = \left( 1 + 2 \sum_{p=1}^{\infty} p^{-2\nu} \right)^{1/2} < \infty \quad \text{for} \quad \nu > \frac{1}{2} \quad (4.5)
\]
A similar argument yields also \(\|b_2 v_2\|_{0,0} \leq \gamma \lambda \gamma_\mu \|a\|_{\lambda,\mu} \|u\|_{\lambda,\mu}\). Further,

\[
\|b_2 v_2\|_{0,0} = \left( \int_0^1 \int_0^1 |b_2(t,s)|^2 |v_2(t,s)|^2 \, dt \, ds \right)^{1/2} \\
\leq \left( \int_0^1 \sup_t |b_2(t,s)|^2 \, ds \right)^{1/2} \left( \int_0^1 \sup_s |v_2(t,s)|^2 \, dt \right)^{1/2}
\]

Since

\[
\sup_t |b_2(t,s)| = \sup_t \left| \sum_{j \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} \hat{b}_2(j,k) e^{ik2\pi s} \right) e^{ij2\pi t} \right|
\]

\[
\leq \sum_{j \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} \hat{b}_2(j,k) e^{ik2\pi s} \right| \leq \gamma_\lambda \left( \sum_{j \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} \hat{b}_2(j,k) e^{ik2\pi s} \right|^2 \right)^{1/2}
\]

we have

\[
\int_0^1 \sup_t |b_2(t,s)|^2 \, ds \leq \gamma_\lambda^2 \sum_{j \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} \hat{b}_2(j,k) e^{ik2\pi s} \right|^2 \, ds \\
= \gamma_\lambda^2 \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\hat{b}_2(j,k)|^2 = \gamma_\lambda^2 \|a\|_{\lambda,\mu}^2.
\]

In a similar way, or simply using a symmetry argument, we find

\[
\int_0^1 \sup_s |v_2(t,s)|^2 \, dt \leq \gamma_\mu^2 \|u\|_{\lambda,\mu}^2.
\]

This results to \(\|b_2 v_2\|_{0,0} \leq \gamma \lambda \gamma_\mu \|a\|_{\lambda,\mu} \|u\|_{\lambda,\mu}\). Exploiting the symmetry between \(b_2, v_2\) and \(b_3, v_3\) we immediately obtain also \(\|b_3 v_3\|_{0,0} \leq \gamma \lambda \gamma_\mu \|a\|_{\lambda,\mu} \|u\|_{\lambda,\mu}\). Summing up we obtain (4.2) in the case \(\lambda > \frac{1}{2}\) and \(\mu > \frac{1}{2}\): \(\|au\|_{\lambda,\mu} \leq 2^{\lambda+\mu+2} \gamma \lambda \gamma_\mu \|a\|_{\lambda,\mu} \|u\|_{\lambda,\mu}\).

(ii) The case \(\lambda < -\frac{1}{2}\) and \(\mu > \frac{1}{2}\). Rewrite (4.3) as

\[
\|au\|_{\lambda,\mu} \leq \left\{ \sum_{j,k \in \mathbb{Z}} 2^{2\lambda} \left( \sum_{p,q \in \mathbb{Z}} k^\mu |a_{j-p,k-q}| |p^{\lambda}| |p^{\lambda}| |u_{p,q}| \right) \right\}^{1/2}
\]

Using the inequality \(p^{2\lambda} \leq 2^{2\lambda} ((j-p)^{\lambda} + j^{\lambda})\) and the second one of inequalities (4.4)
we obtain

\[ \|a u\|_{\lambda, \mu} \leq 2^{\lambda+\mu} \left\{ \sum_{j, k \in \mathbb{Z}} \left( \sum_{p, q \in \mathbb{Z}} p^\lambda (k - q)^\mu |a_{j - p, k - q}| |u_{p, q}| \right)^2 \right\}^{1/2} + 2^{\lambda+\mu} \left\{ \sum_{j, k \in \mathbb{Z}} \left( \sum_{p, q \in \mathbb{Z}} q^\mu |a_{j - p, k - q}| |u_{p, q}| \right)^2 \right\}^{1/2} + 2^{\lambda+\mu} \left\{ \sum_{j, k \in \mathbb{Z}} j^{2\lambda} \left( \sum_{p, q \in \mathbb{Z}} p^\lambda (j - p)^\lambda (k - q)^\mu |a_{j - p, k - q}| |u_{p, q}| \right)^2 \right\}^{1/2} + 2^{\lambda+\mu} \left\{ \sum_{j, k \in \mathbb{Z}} j^{2\lambda} \left( \sum_{p, q \in \mathbb{Z}} q^\mu |a_{j - p, k - q}| |u_{p, q}| \right)^2 \right\}^{1/2} \]

\[ = 2^{\lambda+\mu} \left( \|b_2 v_2\|_{0,0} + \|b_4 v_4\|_{0,0} + S_1 + S_2 \right) \]

where \(b_2, v_2, b_4, v_4\) are the same as in case (i), with similar estimates

\[ \|b_2 v_2\|_{0,0} \leq \gamma \|a\|_{\lambda, \mu} \|u\|_{\lambda, \mu} \quad \text{and} \quad \|b_4 v_4\|_{0,0} \leq \gamma \|a\|_{\lambda, \mu} \|u\|_{\lambda, \mu} \]

and

\[ S_1 = \left\{ \sum_{j, k \in \mathbb{Z}} j^{2\lambda} \left( \sum_{q \in \mathbb{Z}} (k - q)^\mu \sum_{p \in \mathbb{Z}} (j - p)^\lambda |a_{j - p, k - q}| p^\lambda |u_{p, q}| \right)^2 \right\}^{1/2} \]

\[ S_2 = \left\{ \sum_{j, k \in \mathbb{Z}} j^{2\lambda} \left( \sum_{q \in \mathbb{Z}} q^\mu \sum_{p \in \mathbb{Z}} (j - p)^\lambda |a_{j - p, k - q}| p^\lambda |u_{p, q}| \right)^2 \right\}^{1/2} \]

The sum over \(p\) in \(S_1\) and \(S_2\) we estimate by the Cauchy inequality:

\[ \sum_{p \in \mathbb{Z}} (j - p)^\lambda |a_{j - p, k - q}| \cdot p^\lambda |u_{p, q}| \]

\[ \leq \left( \sum_{p \in \mathbb{Z}} (j - p)^{2\lambda} |a_{j - p, k - q}|^2 \right)^{1/2} \left( \sum_{p \in \mathbb{Z}} p^{2\lambda} |u_{p, q}|^2 \right)^{1/2} = d_{k-q} w_q \]

where

\[ d_k = \left( \sum_{p \in \mathbb{Z}} p^{2\lambda} |a_{p, k}|^2 \right)^{1/2} \quad \text{and} \quad w_q = \left( \sum_{p \in \mathbb{Z}} p^{2\lambda} |u_{p, q}|^2 \right)^{1/2} \]
are independent of \( j \). Now we can sum up over \( j \):

\[
S_1 \leq \gamma |\lambda| \left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{q \in \mathbb{Z}} (k - q)^\mu d_{k-q} w_q \right)^2 \right\}^{1/2} = \gamma |\lambda| \|g_1 w_1\|_0
\]

\[
S_2 \leq \gamma |\lambda| \left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{q \in \mathbb{Z}} q^\mu d_{k-q} w_q \right)^2 \right\}^{1/2} = \gamma |\lambda| \|g_2 w_2\|_0
\]

where the functions \( g_1, w_1 \) and \( g_2, w_2 \) are determined by their Fourier coefficients

\[
\hat{g}_1(k) = k^\mu d_k, \quad \hat{w}_1(q) = w_q \quad \text{and} \quad \hat{g}_2(k) = d_k, \quad \hat{w}_2(q) = q^\mu w_q.
\]

Clearly,

\[
\|g_1 w_1\|_0 \leq \|g_1\|_0 \sup_t |w_1(t)| \leq \left( \sum_{k \in \mathbb{Z}} k^{2\mu} d_k^2 \right)^{1/2} \sum_{q \in \mathbb{Z}} |w_q| = \|a\|_{|\lambda|,\mu} \gamma_\mu \left( \sum_{q \in \mathbb{Z}} q^{2\mu} w_q^2 \right)^{1/2} = \gamma_\mu \|a\|_{|\lambda|,\mu} \|u\|_{\lambda,\mu}
\]

\[
\|g_2 w_2\|_0 \leq \sup_t \|g_2(t)\|_0 \leq \sum_{k \in \mathbb{Z}} d_k \|u\|_{\lambda,\mu} \leq \gamma_\mu \|a\|_{|\lambda|,\mu} \|u\|_{\lambda,\mu},
\]

therefore \( S_1, S_2 \leq \gamma |\lambda| \gamma_\mu \|a\|_{|\lambda|,\mu} \|u\|_{\lambda,\mu} \). Summing up we obtain inequality (4.2) for the case \( \lambda < -\frac{1}{2} \) and \( \mu > \frac{1}{2} \): \( \|au\|_{\lambda,\mu} \leq 2^{\lambda+\mu+2} \|a\|_{|\lambda|,\mu} \|u\|_{\lambda,\mu} \).

(iii) The case \( \lambda > \frac{1}{2} \) and \( \mu < -\frac{1}{2} \). It is symmetrical to the case (ii), and a symmetry argument yields immediately \( \|au\|_{\lambda,\mu} \leq 2^{\lambda+|\mu|+2} \|a\|_{|\lambda|,|\mu|} \|u\|_{\lambda,\mu} \).

(iv) The case \( \lambda < -\frac{1}{2} \) and \( \mu < -\frac{1}{2} \). It can be treated by a duality argument. Take an element \( v \in H^{-\lambda,-\mu} \) such that \( \|v\|_{-\lambda,-\mu} = 1 \) and \( \|au\|_{\lambda,\mu} = \langle au, v \rangle \). Then, due to the estimate proved in the case (i),

\[
\|au\|_{\lambda,\mu} = \langle au, v \rangle = \langle u, av \rangle \leq \|u\|_{\lambda,\mu} \|av\|_{-\lambda,-\mu}
\]

\[
\leq \|u\|_{\lambda,\mu} 2^{-\lambda-\mu+2} \gamma_{-\lambda,\gamma_{-\mu}} \|a\|_{-\lambda,-\mu} \|v\|_{-\lambda,-\mu}
\]

\[
= 2^{\lambda+|\mu|+2} \gamma_{|\lambda|,|\mu|} \|a\|_{|\lambda|,|\mu|} \|u\|_{\lambda,\mu}.
\]

The summary of the cases (i) - (iv) sounds as follows:

\[
\|au\|_{\lambda,\mu} \leq 2^{\lambda+|\mu|+2} \gamma_{|\lambda|,|\mu|} \|a\|_{|\lambda|,|\mu|} \|u\|_{\lambda,\mu} \quad \text{for} \quad |\lambda| > \frac{1}{2} \text{ and } |\mu| > \frac{1}{2}.
\]
(v) The case $|\lambda| \leq \frac{1}{2}$ and $|\mu| > \frac{1}{2}$. Take a $\nu > \frac{1}{2}$. According to (4.6),

$$
\|au\|_{\nu,\mu} \leq 2^{\nu+|\mu|+2} \gamma_{\nu} \gamma_{|\mu|} \|a\|_{\nu,|\mu|} \|u\|_{\nu,\mu}
$$

$$
\|au\|_{-\nu,\mu} \leq 2^{\nu+|\mu|+2} \gamma_{\nu} \gamma_{|\mu|} \|a\|_{\nu,|\mu|} \|u\|_{\nu,\mu}.
$$

Using the interpolation theorem for an operator in scales of Hilbert spaces (see [12, 27]) we obtain herefrom

$$
\|au\|_{\lambda,\mu} \leq 2^{\nu+|\mu|+2} \gamma_{\nu} \gamma_{|\mu|} \|a\|_{\nu,|\mu|} \|u\|_{\lambda,\mu} \quad (-\nu \leq \lambda \leq \nu). \tag{4.7}
$$

We obtained inequality (4.2) in the case $|\lambda| \leq \frac{1}{2}$ and $|\mu| > \frac{1}{2}$.

(vi) The case $|\lambda| > \frac{1}{2}$ and $|\mu| \leq \frac{1}{2}$. It is symmetrical to the case (v) yielding (4.2):

$$
\|au\|_{\lambda,\mu} \leq 2^{\nu+|\mu|+2} \gamma_{\nu} \gamma_{|\mu|} \|a\|_{\lambda,\nu} \|u\|_{\lambda,\mu}. \tag{4.8}
$$

(vii) The case $|\lambda| \leq \frac{1}{2}$ and $|\mu| \leq \frac{1}{2}$. Take again a $\nu > \frac{1}{2}$. According to (4.7)

$$
\|au\|_{\lambda,\nu} \leq 2^{2(\nu+1)} \gamma_{\nu}^{2} \|a\|_{\nu,\nu} \|u\|_{\lambda,\nu}
$$

$$
\|au\|_{\lambda,-\nu} \leq 2^{2(\nu+1)} \gamma_{\nu}^{2} \|a\|_{\nu,\nu} \|u\|_{\lambda,-\nu}.
$$

The interpolation theorem yields inequality (4.2):

$$
\|au\|_{\lambda,\mu} \leq 2^{2(\nu+1)} \gamma_{\nu}^{2} \|a\|_{\nu,\nu} \|u\|_{\lambda,\mu} \quad (-\nu \leq \mu \leq \nu). \tag{4.9}
$$

Now inequality (4.2) is established in all possible cases.

Inequalities (4.6) – (4.9) give some information about the constant $c_{\lambda,\mu,\nu}$ in inequality (4.2) in different cases. These values can be somewhat reduced.

**Corollary 4.2.** For any $\lambda, \mu \in \mathbb{R}$ the inequality

$$
\|a(t,s)u(s)\|_{\lambda,\mu} \leq c_{\lambda,\mu,\nu} \|a\|_{\max(|\lambda|,\nu),\max(|\mu|,\nu)} \|u\|_{\mu} \tag{4.10}
$$

holds with any $\nu > \frac{1}{2}$.

**Proof.** This follows immediately from inequality (4.2) considering $u = u(s)$ as a biperiodic function (distribution) which is constant with respect to $t$. Note that $\|u(s)\|_{\lambda,\mu} = \|u\|_{\mu}$ for any $\lambda \in \mathbb{R}$.
5. Bounded integral operators in Sobolev spaces

Consider an integral operator

\[(Au)(t) = \int_0^1 \kappa(t - s)a(t, s)u(s) \, ds\]  \hspace{1cm} (5.1)

where the coefficient \(a\) is smooth and 1-biperiodic, and the function (or distribution) \(\kappa\) is 1-periodic and its Fourier coefficients satisfy the inequality

\[|\hat{\kappa}(m)| \leq m^{-\alpha} \hspace{1cm} (m \in \mathbb{Z})\]  \hspace{1cm} (5.2)

with an \(\alpha \in \mathbb{R}\). In the case of a constant coefficient \(a(t, s) = 1\) we have

\[(Au)(t) = \int_0^1 \kappa(t - s)u(s) \, ds = \sum_{m \in \mathbb{Z}} \hat{\kappa}(m)\hat{u}(m)e^{im2\pi t},\]

and due to (5.2) \(A\) is a bounded operator from any \(H^\lambda\) to \(H^{\lambda+\alpha}\) (\(\alpha \in \mathbb{R}\)). Moreover, if \(\kappa\) satisfies also the inverse inequality \(|\hat{\kappa}(m)| \geq c_0m^{-\alpha} \hspace{1cm} (m \in \mathbb{Z})\) with a \(c_0 > 0\), then \(A\) is an isomorphism between \(H^\lambda\) and \(H^{\lambda+\alpha}\) (\(\lambda \in \mathbb{R}\)). Our purpose is to show that the operator \(A\) remains bounded from \(H^\lambda\) to \(H^\lambda\) also in the presence of a smooth 1-biperiodic coefficient \(a\). We deduce this result from Corollary 4.2 and the following lemma.

**Lemma 5.1.** Assume inequality (5.2). Then, for any \(\lambda \in \mathbb{R}\),

\[\left\| \int_0^1 \kappa(t - s)v(t, s) \, ds \right\|_{\lambda+\alpha} \leq \begin{cases} 2^{\lambda+\alpha+1}\gamma_{\lambda+\alpha}\|v\|_{\lambda+\alpha, \lambda}, & \text{if } \lambda + \alpha > \frac{1}{2} \\ 2^{\lambda+\alpha+1}\gamma_{\nu}\|v\|_{\nu, \lambda}, & \text{if } 0 \leq \lambda + \alpha \leq \frac{1}{2} \\ 2^{\lambda+\alpha}\gamma_{\nu}\|v\|_{\lambda+\alpha+\nu, \lambda}, & \text{if } \lambda + \alpha \leq 0 \end{cases}\]  \hspace{1cm} (5.3)

where \(\nu > \frac{1}{2}\) is arbitrary, \(\gamma_\nu\) is defined by (4.5), and \(v\) is any 1-biperiodic function (distribution) of a finite Sobolev norm indicated on the right hand of the inequality.

To prove (5.3) we first formulate an elementary inequality.

**Lemma 5.2.** For any \(\nu_1 \geq 0\) and \(\nu_2 \geq 0\) with \(\nu_1 + \nu_2 > \frac{1}{2}\) and any \(k, l \in \mathbb{Z}\),

\[\left( \sum_{m \in \mathbb{Z}} (m - k)^{-2\nu_1}(m - l)^{-2\nu_2} \right)^{1/2} \leq \gamma_{\nu_1+\nu_2}.\]  \hspace{1cm} (5.4)

**Proof.** In the cases \(\nu_1 = 0\) or \(\nu_2 = 0\) inequality (5.4) is trivial (cf. (4.5)). Let \(\nu_1 > 0\) and \(\nu_2 > 0\). The Hölder inequality with \(p_1 = \frac{\nu_1 + \nu_2}{\nu_1}\) and \(p_2 = \frac{\nu_1 + \nu_2}{\nu_2}\) (obviously,
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\[ \frac{1}{p_1} + \frac{1}{p_2} = 1 \] yields

\[
\sum_{m \in \mathbb{Z}} (m - k)^{-2\nu_1} (m - l)^{-2\nu_2} \leq \left( \sum_{m \in \mathbb{Z}} (m - k)^{-2(\nu_1 + \nu_2)} \right)^{\nu_1/(\nu_1 + \nu_2)} \left( \sum_{m \in \mathbb{Z}} (m - l)^{-2(\nu_1 + \nu_2)} \right)^{\nu_2/(\nu_1 + \nu_2)} = \sum_{m \in \mathbb{Z}} m^{-2(\nu_1 + \nu_2)} = \gamma_{\nu_1 + \nu_2}^2.
\]

Thus the lemma is shown \( \blacksquare \)

**Proof of Lemma 5.1.** We have

\[
\int_0^1 \kappa(t - s)\psi(t, s) \, ds = \sum_{k \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}} \kappa(m) \hat{\psi}(k - m, m) \right) e^{ik2\pi t}
\]

and due to (5.2)

\[
\left\| \int_0^1 \kappa(t - s)\psi(t, s) \, ds \right\|_{\lambda + \alpha} \leq \left\{ \sum_{k \in \mathbb{Z}} k^{2(\lambda + \alpha)} \left( \sum_{m \in \mathbb{Z}} m^{-\alpha} |\hat{\psi}(k - m, m)|^2 \right) \right\}^{1/2}.
\]

(5.5)

\[
\left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}} k^{\lambda + \alpha} m^{-\alpha} |\hat{\psi}(k - m, m)|^2 \right) \right\}^{1/2}.
\]

In the case \( \lambda + \alpha \geq 0 \) we may use the inequality

\[
k^{\lambda + \alpha} \leq 2^{\lambda + \alpha} ((k - m)^{\lambda + \alpha} + m^{\lambda + \alpha}).
\]

Noticing that for \((c_{km})_{k, m \in \mathbb{Z}}\) the expression \( \left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}} |c_{km}| \right)^2 \right\}^{1/2} \) has the norm properties we obtain

\[
\left\| \int_0^1 \kappa(t - s)\psi(t, s) \, ds \right\|_{\lambda + \alpha} \leq 2^{\lambda + \alpha} \left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}} (k - m)^{\lambda + \alpha} m^{-\alpha} |\hat{\psi}(k - m, m)|^2 \right) \right\}^{1/2} + 2^{\lambda + \alpha} \left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}} m^\lambda |\hat{\psi}(k - m, m)|^2 \right) \right\}^{1/2}.
\]

(5.6)
(i) The case $\lambda + \alpha > \frac{1}{2}$. Rewrite (5.6) as follows:

$$\left\| \int_0^1 \kappa(t-s)v(t,s)\,ds \right\|_{\lambda+\alpha} \leq 2^{\lambda+\alpha} \left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}} m^{-(\lambda+\alpha)} \cdot (k-m)^{\lambda+\alpha} m^\lambda |\hat{v}(k-m,m)| \right)^2 \right\}^{1/2}$$

$$+ 2^{\lambda+\alpha} \left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}} (k-m)^{-(\lambda+\alpha)} \cdot (k-m)^{\lambda+\alpha} m^\lambda |\hat{v}(k-m,m)| \right)^2 \right\}^{1/2}.$$ 

Estimating the sums over $m$ by the Cauchy inequality we obtain (5.3).

(ii) The case $0 \leq \lambda + \alpha \leq \frac{1}{2}$. This time we rewrite (5.6) in the form

$$\left\| \int_0^1 \kappa(t-s)v(t,s)\,ds \right\|_{\lambda+\alpha} \leq 2^{\lambda+\alpha} \left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}} m^{-(\lambda+\alpha)} (k-m)^{\lambda+\alpha-\nu} \cdot (k-m)^\nu m^\lambda |\hat{v}(k-m,m)| \right)^2 \right\}^{1/2}$$

$$+ 2^{\lambda+\alpha} \left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}} (k-m)^{-\nu} \cdot (k-m)^\nu m^\lambda |\hat{v}(k-m,m)| \right)^2 \right\}^{1/2}$$

where $\nu > \frac{1}{2}$. Estimating the sums over $m$ again by the Cauchy inequality and taking into account Lemma 5.2 (with $\nu_1 = \lambda + \alpha$ and $\nu_2 = \nu - (\lambda + \alpha)$) we obtain (5.3).

(iii) The case $\lambda + \alpha < 0$. Using Peetre's inequality

$$k^{\lambda+\alpha} \leq 2^{\lambda+\alpha} |m|^{\lambda+\alpha} |k-m|^{\lambda+\alpha}$$

$(k,m \in \mathbb{Z})$

we continue (5.5) as follows:

$$\left\| \int_0^1 \kappa(t-s)v(t,s)\,ds \right\|_{\lambda+\alpha} \leq 2^{\lambda+\alpha+1} \left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}} (k-m)^{\lambda+\alpha} m^\lambda |\hat{v}(k-m,m)| \right)^2 \right\}^{1/2}.$$ 

The Cauchy inequality yields again (5.3).
Proposition 5.3. Assume that $a = a(t, s)$ is $C^\infty$-smooth and 1-biperiodic, and $\kappa = \kappa(t)$ satisfies (5.2). Then the integral operator $A$ defined by (5.1) is bounded from $H^\lambda$ to $H^{\lambda+\alpha}$ for any $\lambda \in \mathbb{R}$. With any $\nu > \frac{1}{2}$, the following estimates of the norm of $A$ hold:

(i) If $\alpha \geq 0$, then for all $\lambda \in \mathbb{R}$

$$\|A\|_{\lambda, \lambda+\alpha} := \|A\|_{\mathcal{L}(H^\lambda, H^{\lambda+\alpha})} \leq c_{\lambda, \alpha, \nu} \|a\|_{\max(|\lambda+\alpha|, \nu), \max(|\lambda|, \nu)}. \quad (5.7)$$

(ii) If $\alpha < 0$, then (5.7) holds for $\lambda \leq 0$ and for $\lambda \geq -\alpha$, whereas for $0 < \lambda < -\alpha$

$$\|A\|_{\lambda, \lambda+\alpha} \leq c_{\lambda, \alpha, \nu} \min \left(\|a\|_{|\lambda+\alpha|+\nu, \max(\lambda, \nu)}, \|a\|_{\max(|\lambda+\alpha|, \nu), \lambda+\nu}\right). \quad (5.8)$$

Proof. Take any $u \in H^\lambda$ and denote $v(t, s) = a(t, s)u(s)$. According to Lemma 5.1 and Corollary 4.2 we have

$$\|Au\|_{\lambda+\alpha} \leq c_{\lambda, \alpha, \nu} \begin{cases} \|a\|_{\lambda+\alpha, \max(|\lambda|, \nu)} & \text{if } \lambda + \alpha > \frac{1}{2} \\ \|a\|_{\nu, \max(|\lambda|, \nu)} & \text{if } 0 \leq \lambda + \alpha \leq \frac{1}{2} \\ \|a\|_{|\lambda+\alpha|+\nu, \max(|\lambda|, \nu)} & \text{if } \lambda + \alpha < 0 \end{cases} \|u\|_\lambda,$$

i.e. the operator $A$ is bounded from $H^\lambda$ to $H^{\lambda+\alpha}$ whereby, for any $\lambda \in \mathbb{R}$,

$$\|A\|_{\lambda, \lambda+\alpha} \leq c_{\lambda, \alpha, \nu} \begin{cases} \|a\|_{\max(|\lambda+\alpha|, \nu), \max(|\lambda|, \nu)} & \text{if } \lambda + \alpha \geq 0 \\ \|a\|_{|\lambda+\alpha|+\nu, \max(|\lambda|, \nu)} & \text{if } \lambda + \alpha \leq 0 \end{cases}. \quad (5.9)$$

The Banach dual (or "transposed") operator $A' \in \mathcal{L}(H^{-\lambda-\alpha}, H^{-\lambda})$ to the operator $A \in \mathcal{L}(H^\lambda, H^{\lambda+\alpha})$ is defined by

$$(A'u)(t) = \int_0^1 \kappa(s-t)a(s,t)u(s) \, ds = \int_0^1 \kappa'(t-s)a'(t,s)u(s) \, ds$$

where $\kappa'(t) = \kappa(-t)$ and $a'(t,s) = a(s,t)$. Since $\dot{\kappa}(m) = -\dot{\kappa}(-m)$, the dual operator $A'$ also satisfies the conditions of the proposition, and (5.9) is true for it:

$$\|A'\|_{-\lambda-\alpha, -\lambda} \leq c_{\lambda, \alpha, \nu} \begin{cases} \|a'\|_{\max(-\lambda, \nu), \max(|\lambda+\alpha|, \nu)} & \text{if } -\lambda \geq 0 \\ \|a'\|_{\lambda+\nu, \max(|\lambda+\alpha|, \nu)} & \text{if } -\lambda \leq 0 \end{cases}.$$}

Since the norms of an operator and its dual are equal we obtain

$$\|A\|_{\lambda, \lambda+\alpha} \leq c_{\lambda, \alpha, \nu} \begin{cases} \|a\|_{\max(|\lambda+\alpha|, \nu), \max(|\lambda|, \nu)} & \text{if } \lambda \leq 0 \\ \|a\|_{\max(|\lambda+\alpha|, \nu), \lambda+\nu} & \text{if } \lambda \geq 0 \end{cases}.$$

Together with (5.9) this covers the assertions of the proposition.
For $\alpha < 0$ and $0 < \lambda < -\alpha$, inequality (5.7) may fail. For instance, for $A = A_N$ with $\hat{a}(m) = |m|$ ($\alpha = -1, m \in \mathbb{Z}$)

$$\hat{a}_N(k,l) = \begin{cases} 1 & \text{for } k + l = 0, \quad 1 \leq l \leq N \\ 0 & \text{for other } k, l \in \mathbb{Z} \end{cases}$$

and for $e_0(t) \equiv 1$ we have

$$A_N e_0 = \left( \sum_{m \in \mathbb{Z}} \hat{a}_N(-m,m)|m| \right) e_0 = \left( \sum_{m=1}^{N} m \right) e_0 \sim N^2 e_0$$

$$\|A_N e_0\|_{\lambda + \alpha} \sim N^2 \quad (\lambda \in \mathbb{R}).$$

For $\lambda = \frac{1}{2}$, inequality (5.7) fails since $\max(|\lambda + \alpha|, \nu) = \nu = \max(|\lambda|, \nu)$ and

$$\|a_N\|_{\nu, \nu} = \left( \sum_{k=1}^{N} k^{4\nu} \right)^{1/2} \sim N^{(4\nu+1)/2} << N^2 \quad \left( \frac{1}{2} < \nu < \frac{3}{4} \right).$$

Of course, we are interested to obtain estimates of the type $\|A\|_{\lambda, \lambda+\alpha} \leq c\|a\|_{\lambda_1, \lambda_2}$ with possibly small $\lambda_1$ and $\lambda_2$. In this sense, estimates (5.7) and (5.8) are the best we know for moderate $|\lambda|$. For great $|\lambda|$, the result can be improved if we allow a sum of two different Sobolev norms in the right-hand side. In the following estimates only one of the indices $\lambda_1$ and $\lambda_2$ increases linearly in $|\lambda|$ as $|\lambda| \to \infty$.

**Proposition 5.4.** Assume the conditions of Proposition 5.3. Then the following estimates for the norm of $A$ hold:

(i) For $\lambda + \alpha > \frac{1}{2}$, with any $\frac{1}{2} < \nu \leq \lambda + \alpha$,

$$\|A\|_{\lambda, \lambda+\alpha} \leq c_{\lambda, \alpha, \nu} \left( \|a\|_{\lambda+\alpha, \max(|\nu-\alpha|, \nu)} + \|a\|_{\nu, \max(|\lambda, \nu|)} \right). \quad (5.10)$$

(ii) For $\lambda < -\frac{1}{2}$, with any $\frac{1}{2} < \nu \leq |\lambda|$,

$$\|A\|_{\lambda, \lambda+\alpha} \leq c_{\lambda, \alpha, \nu} \left( \|a\|_{\max(|\nu-\alpha|, \nu), |\lambda|} + \|a\|_{\max(|\lambda+\alpha|, \nu), \nu} \right). \quad (5.11)$$

**Proof.** Let $\lambda + \alpha > \frac{1}{2}$ with $\frac{1}{2} < \nu \leq \lambda + \alpha$. From (5.6) we obtain

$$\left\| \int_{0}^{1} \kappa(t-s)\nu(t,s) \, ds \right\|_{\lambda + \alpha} \leq 2^{\lambda+\alpha} \gamma_{\nu} (\|v\|_{\lambda+\alpha, \nu-\alpha} + \|v\|_{\nu, \lambda}).$$

Note that $\nu - \alpha \leq \lambda$. Now estimate (5.10) follows with the help of inequality (4.10). Estimate (5.11) can be proved by the dual argument already used in the proof of Proposition 5.3.
Proposition 5.5. Assume that $a = a(t, s)$ is $C^\infty$-smooth, 1-biperiodic, and vanishes on the diagonal: $a(t, t) = 0$ for all $t \in \mathbb{R}$. Let $\kappa = \kappa(t)$ satisfy (cf. (5.2))

$$|\kappa(m) - \kappa(m - 1)| \leq cm^{-\alpha - \gamma} \quad (m \in \mathbb{Z})$$

where $\gamma > 0$. Then the integral operator $A$ defined by (5.1) is bounded from any $H^\lambda$ to $H^{\lambda + \alpha + \gamma}$ ($\lambda \in \mathbb{R}$).

Proof. We represent $A$ in the form

$$(Au)(t) = \int_0^1 \kappa_1(t - s)a_1(t, s)\, ds$$

where

$$\kappa_1(t) = \kappa(t)(1 - e^{i2\pi t}) \quad \text{and} \quad a_1(t, s) = \frac{a(t, s)}{1 - e^{i2\pi(t - s)}}.$$ 

Note that $a_1$ is also 1-biperiodic and $C^\infty$-smooth. Further, we have

$$\kappa_1(m) = \kappa(m) - \kappa(m - 1) \quad \text{and} \quad |\kappa_1(m)| \leq cm^{-\alpha - \gamma} \quad (m \in \mathbb{Z}).$$

Now the assertion follows from Proposition 5.3.

Estimates (5.10) and (5.11) allow to weaken the smoothness assumptions on $a_p$ ($p = 0, \ldots, q$) formulated in Remark 2.4 and based on inequality (5.7). Unfortunately, the formulation becomes more sophisticated.

Estimates (5.10) and (5.11) are rather useful also in the analysis of the numerical stability of the methods described in Section 2 and other approximate methods for problem (2.1). This analysis will be presented in another paper.

6. Proof of Theorems 2.2 and 2.3

Denote by $P_+$ and $P_-$ the projections

$$(P_+ u)(t) = \sum_{k \geq 0} \hat{u}(k)e^{ik2\pi t} \quad \text{and} \quad (P_- u)(t) = \sum_{k < 0} \hat{u}(k)e^{ik2\pi t}.$$ 

Lemma 6.1. Let $b_1$ and $b_2$ be $C^\infty$-smooth 1-periodic functions such that $b_1(t) \neq 0$ and $b_2(t) \neq 0$ for all $t \in \mathbb{R}$ and (see (2.5))

$$W(b_1) = W(b_2) = 0.$$ 

Then there is an $N_0$ such that for all $N \geq N_0$, $v_N \in T_N$, and $\lambda \in \mathbb{R}$ the inequality

$$\|v_N\|_\lambda \leq c_\lambda \|P_N(b_1P_+ + b_2P_-)v_N\|_\lambda$$

holds with a constant $c_\lambda$ independent of $N$ and $v_N$.

Proof. This stability result is well known in the case $\lambda = 0$; a proof can be found in the books [5, 6, 17, 19]. A similar result holds in the case of Hölder spaces (see, e.g., [16, 17]). The proof of [16] can be repeated in our case of Sobolev spaces $H^\lambda$ without changes. We omit the details.
Putting \( b_1 = b_2 = b \), remembering the definition (2.5) of \( W(b) \) and taking into account that \( P_+ + P_- = I \) (the identity operator) we obtain

**Corollary 6.2.** Let \( b \) be a \( C^\infty \)-smooth 1-periodic function such that \( b(t) \neq 0 \) for all \( t \in \mathbb{R} \) and \( W(b) = 0 \). Then there is an \( N_0 \) such that for all \( N \geq N_0, v_N \in T_N \), and \( \lambda \in \mathbb{R} \) the inequality

\[
\|v_n\|_\lambda \leq c_\lambda \|P_N(bv_N)\|_\lambda
\]

holds with a constant \( c_\lambda \) independent of \( N \) and \( v_N \).

**Proof of Proposition 2.1.** The first assertion of Proposition 2.1 follows immediately from Proposition 5.3. Let us assume (2.4) and prove that \( A \) is a Fredholm operator of index 0. Represent it in the form

\[
A = B + K
\]

where

\[
(Bu)(t) = b(t) \int_0^1 \kappa_0(t-s)u(s) \, ds \quad \text{with} \quad b(t) = a_0(t,t)
\]

and

\[
(Ku)(t) = \int_0^1 \kappa_0(t-s)(a(t,s) - a(t,t))u(s) \, ds + \sum_{p=1}^q \int_0^1 \kappa_p(t-s)a_p(t,s)u(s) \, ds.
\]

Propositions 5.3 and 5.5 confirm us that, due to (2.3), \( K \in \mathcal{L}(H^\lambda, H^{\lambda+\alpha+\beta}) \) for all \( \lambda \in \mathbb{R} \). Since the imbedding \( H^\mu \hookrightarrow H^\lambda \) is compact for \( \lambda < \mu \), the operator \( K \in \mathcal{L}(H^\lambda, H^{\lambda+\alpha}) \) is compact. According to (2.3) some of the Fourier coefficients \( \kappa_0(m) \) (\( |m| \leq m_0 \)) may vanish but we do not lose in generality assuming that \( \kappa_0(m) \neq 0 \) for all \( m \in Z \) (in the opposite case we redefine \( \kappa_0 \) that causes a slight change in the structure of the operator \( K \)). In other words, we may assume that the first one of inequalities (2.3) holds for all \( m \in Z \). Thus, we have

\[
B = b(t)\Lambda \quad \text{with} \quad (\Lambda u)(t) = \int_0^1 \kappa_0(t-s)u(s) \, ds.
\]

The operator \( \Lambda \) is an isomorphism between \( H^\lambda \) and \( H^{\lambda+\alpha} \). Since \( b(t) \neq 0 \) for all \( t \neq 0 \), the operator \( B \) itself is also an isomorphism between \( H^\lambda \) and \( H^{\lambda+\alpha} \), and \( A = B + K \in \mathcal{L}(H^\lambda, H^{\lambda+\alpha}) \) is a Fredholm operator of index 0 for all \( \lambda \in \mathbb{R} \).

If \( Au = 0 \), then \( u + B^{-1}Ku = 0 \) and \( u = (-1)^n(B^{-1}K)^nu \) \( (n \in \mathbb{N}) \). Since \( B^{-1}K \in \mathcal{L}(H^\lambda, H^{\lambda+\beta}) \) for all \( \lambda \in \mathbb{R} \), we obtain \( u \in \cap_\lambda H^\lambda \). This proves the last assertion of Proposition 2.1 concerning \( \mathcal{N}(A) \) \( \Box \)

In the case of the operator \( A_0 \) defined by (2.19) - (2.22) we put

\[
(Bu)(t) = a_+(t,t) \int_0^1 \kappa_+(t-s)u(s) \, ds + a_-(t,t) \int_0^1 \kappa_-(t-s)u(s) \, ds
\]

\[
= (b_1(t)P_+ + b_2(t)P_-)^{-\alpha}u
\]
where the functions $b_1$ and $b_2$ are defined by (2.20) and

$$(\Lambda^{-\alpha}u)(t) = \sum_{k \in \mathbb{Z}} k^{-\alpha} \hat{u}(k)e^{ik2\pi t}$$

is an isometry between $H^\lambda$ and $H^{\lambda+\alpha}$ ($\lambda \in \mathbb{R}$). Under conditions (2.20) and (2.21), the operator $b_1P_+ + b_2P_- \in \mathcal{L}(H^\lambda, H^\lambda)$ is known to be an isomorphism for all $\lambda \in \mathbb{R}$: in the case $\lambda = 0$ we refer again, e.g., to [5, 6, 17]; the case $\lambda \neq 0$ can be easily reduced to the case $\lambda = 0$ (see, e.g., [26]). Consequently, $B \in \mathcal{L}(H^\lambda, H^{\lambda+\alpha})$ is an isomorphism for all $\lambda \in \mathbb{R}$, and Proposition 2.1 holds true also in the case (2.19) - (2.23).

**Proof of Theorem 2.2.** We represent equation (2.1) in the form (see (6.1) and (6.2)) $bAu + Ku = f$. The Galerkin method (2.8), in these notations, reads as follows:

$$u_N \in T_N, \quad P_N bAu_N + P_N Ku_N = P_N f.$$ 

Notice that the isomorphism $\Lambda \in \mathcal{L}(H^\lambda, H^{\lambda+\alpha})$ has the property that $\Lambda T_N \subset T_N$ and $\Lambda^{-1}T_N \subset T_N$. Due to Corollary 6.2, for any $u_N \in T_N$

$$||\Lambda u_N||_{\lambda+\alpha} \leq c_\alpha ||P_N bAu_N||_{\lambda+\alpha} \quad (N \geq N_0)$$

or

$$||u_N||_{\lambda} \leq c'_\alpha ||P_N Bu_N||_{\lambda+\alpha}, \quad (N \geq N_0, \lambda \in \mathbb{R}).$$

(6.3)

A standard argument (see, e.g., [28]) enables us to extend this stability inequality for $P_N A = P_NB + P_NK$:

$$||u_N||_{\lambda} \leq c_\lambda ||P_N A u_N||_{\lambda+\alpha} \quad (N \geq N_1, u_N \in T_N, \lambda \in \mathbb{R}).$$

(6.4)

Indeed, assume that (6.4) is violated: there are $u_N \in T_N$ such that $||u_N||_{\lambda} = 1$ and $||P_N Bu_N + P_N Ku_N||_{\lambda+\alpha} \to 0$ as $N \to \infty$ along some subsequence. Due to the compactness of the operator $K \in \mathcal{L}(H^\lambda, H^{\lambda+\alpha})$, we may assume that, along the subsequence, $P_N Ku_N$ converges in $H^{\lambda+\alpha}$ to some limit $w \in H^{\lambda+\alpha}$, and consequently $P_NBu_N$ converges in $H^{\lambda+\alpha}$ to $-w$. Now it follows from the stability inequality (6.3) that the corresponding subsequence of $u_N$ converges in $H^\lambda$ to $v = -B^{-1}w \in H^\lambda$ and $||v||_{\lambda} = 1$. The limiting in the relation $||P_N A u_N||_{\lambda+\alpha} \to 0$ yields $A v = 0$. Thus, $\mathcal{N}(A) \neq \{0\}$ contradicting the condition (2.6) of the theorem and proving the stability inequality (6.4). With the help of the interpolation theorem one can even prove that the minimal constant $c_\lambda$ in (6.4) is uniformly bounded in $\lambda$ on every (finite) interval $[\lambda_1, \lambda_2]$.

Thus, (2.8) determines for $N \geq N_1$ a unique $u_N \in T_N$, the Galerkin approximation to $u = A^{-1}f$. The stability inequality (6.4) yields

$$||u_N - P_N u||_{\lambda} \leq c_\lambda ||P_N A(u_N - P_N u)||_{\lambda+\alpha} = c_\lambda ||P_N f - P_N A P_N u||_{\lambda+\alpha}$$

and

$$||u_N - u||_{\lambda} \leq ||u_N - P_N u||_{\lambda} + ||u - P_N u||_{\lambda} \leq (c_\lambda ||A||_{\lambda, \lambda+\alpha} + 1) ||u - P_N u||_{\lambda}.$$ 

Together with (2.7) this proves the error estimate (2.9).
In the case of an operator $A_0$ of the form (2.19) we use Lemma 6.1 instead of Corollary 6.2. Note that $\Lambda^{-\alpha}v_N \in T_N$ for $v_N \in T_N$.

**Proof of Theorem 2.3.** For the operators $A$ and $A_M$ defined in (2.2) and (2.13), respectively, we have according to Proposition 5.3

$$
\|A_M - A\|_{\lambda,\lambda + \alpha} \leq c_{\lambda,\nu} \sum_{p=0}^{q} \|a_{p,M} - a_p\|_{\lambda + \alpha + \nu, \max(\{\lambda, \nu\})} \quad (\nu > \frac{1}{2})
$$

(we used here the coarser but more universal inequality (5.8); in the cases indicated in Proposition 5.3, the more sharp inequality (5.7) may be used). Now our task is to show that, with any $r > 0$,

$$
\|a_{p,M} - a_p\|_{\lambda,\mu} \leq c_{\lambda,\mu,\nu} M^{-r} \|a_p\|_{\lambda + r,\mu + r} \quad \left(\lambda, \mu > \frac{1}{2}\right)
$$

resulting to

$$
\|A_M - A\|_{\lambda,\lambda + \alpha} \leq c_{\lambda,\alpha,\nu} M^{-\lambda} \sum_{p=0}^{q} \|a_p\|_{\lambda + \alpha + \nu + r, \max(\{\lambda, \nu\}) + r} \quad (\lambda \in \mathbb{R}).
$$

We have (see (2.11))

$$
a_{p,M} - a = \Pi_M Q_{M,M} a - a = \Pi_M (Q_{M,M} a - a) + (\Pi_M a - a)
$$

where $\Pi_m$ is the two-dimensional Fourier projection defined by

$$(\Pi_M a)(t, s) = \sum_{|k| + |l| \leq M/2} \hat{a}(k, l)e^{ik2\pi t}e^{il2\pi s}.
$$

It is clear that $\Pi_m$ is orthogonal in any space $H^\lambda,\mu$ and

$$
\|a - \Pi_M a\|_{\mu} \leq \left(\frac{M}{2} - 1\right)^{-r} \|a\|_{\lambda + r,\mu + r} \quad (r > 0).
$$

Similarly to (2.10) we have (see [26] for a detailed proof)

$$
\|a - Q_{M,M}\|_{\lambda,\nu} \leq c_{\lambda,\mu,\nu} M^{-\lambda} \|a\|_{\lambda + r,\mu + r} \quad \left(\lambda \geq 0, \mu \geq 0, \lambda + r > \frac{1}{2}, \mu + r > \frac{1}{2}\right).
$$

This proves (6.5) and (6.6).

Due to (6.6), the stability property (6.4) extends to the modified Galerkin method (2.14): there is an $N_2$ such that, for any $N \geq N_2$, $v_N \in T_N$ and $\lambda \in \mathbb{R}$,

$$
\|v_n\|_{\lambda} \leq c_\lambda \|P_N A_M v_n\|_{\lambda + \alpha}.
$$
Consequently, for \( N \geq N_2 \) (2.14) provides a unique approximation \( u_N \in T_N \), and with \( u = A^{-1}f \) we have

\[
\|u_N - P_N u\|_\lambda \\
\leq c_\lambda \|P_N A_M (u_N - P_N u)\|_{\lambda + \alpha} \\
= c_\lambda \|f_N - P_N A_M P_N u\|_{\lambda + \alpha} \\
\leq c_\lambda \left( \|f_N - P_N f\|_{\lambda + \alpha} + \|P_N A (u - P_N u)\|_{\lambda + \alpha} + \|A - A_M\|_{\lambda + \alpha} P_N u\|_{\lambda + \alpha} \right) \\
\leq c_\lambda \left( \|u - P_N u\|_{\lambda} + \|f_N - P_N f\|_{\lambda + \alpha} + c_\lambda r M^{-r} \|u\|_{\lambda} \right)
\]

resulting to the error estimate (2.15). Note that since \( r > 0 \) is arbitrary and \( M \sim N^\sigma \) \((0 < \sigma \leq 1)\), we may replace \( M^{-r} \) by \( N^{-r} \) in the last estimate.

No changes in the argument are needed in the case of an operator \( A_0 \) of the form (2.19).

7. Numerical illustration

Let us illustrate the behaviour of the actual error \( u_N - u \) of the method (2.14) with \( f_N = Q_N f \) for different \( M \) and \( N \).

Example 7.1. We took the model problem

\[
\int_0^1 \kappa_0(t-s) a_0(t,s) u(s) \, ds = f(t)
\]

with \( \alpha = 2 \),

\[
\kappa_0(m) = \frac{1}{\pi} \frac{4}{4m^2 - 1} \quad (m \in Z) \\
a_0(t,s) = a(t)a(s), \quad a(t) = 3 + \sum_{0 \neq m \in Z} 2^{-4|m|} e^{im2\pi t}.
\]

We put \( u(t) = \sum_{0 \neq m \in Z} m^{-3} e^{im2\pi t} \) and computed with a high accuracy \( f = f(t) \) from (7.1). After that we applied method (2.14) with \( f_N = Q_N f \) to recover \( u_N \in T_N \). Constructing the system (3.2) we used only grid values

\[
a_0(j_1 M^{-1}, j_2 M^{-1}) \quad (j_1,j_2 = 0,\ldots,M) \quad \text{and} \quad f(jN^{-1}) \quad (j = 0,\ldots,N)
\]

and did not use the Fourier coefficients of \( a_0 \) and \( f \) (which are available in this example). The error \( \|u_N - u\|_0 \) for some \( N \) and \( M \) is presented in Table 7.1. Note that \( u \in H^\mu \) \((\mu < \frac{5}{2})\) and \( u \notin H^{5/2}. \) The estimate (2.16) takes the form \( \|u_N - u\|_0 \leq c_\mu N^{-\mu} \|u\|_\mu \) \((\mu < \frac{5}{2})\). The experimental convergence order with \( M \sim N^{1/2} \), as seen from Table 7.1, is approximately \( cN^{-5/2} \).
\begin{table}
\centering
\begin{tabular}{ccccccc}
N=8 & N=16 & N=32 & N=64 & N=128 & N=256 & N=512 \\
\hline
M = 2 & 1.82 \cdot 10^{-1} & 1.79 \cdot 10^{-1} & 1.80 \cdot 10^{-1} & 1.80 \cdot 10^{-1} & 1.80 \cdot 10^{-1} & 1.80 \cdot 10^{-1} \\
M = 4 & 4.01 \cdot 10^{-2} & 2.44 \cdot 10^{-2} & 2.40 \cdot 10^{-2} & 2.40 \cdot 10^{-2} & 2.40 \cdot 10^{-2} & 2.40 \cdot 10^{-2} \\
M = 8 & 3.04 \cdot 10^{-2} & 4.43 \cdot 10^{-3} & 7.92 \cdot 10^{-4} & 2.96 \cdot 10^{-4} & 2.67 \cdot 10^{-4} & 2.66 \cdot 10^{-4} \\
M = 16 & & 7.46 \cdot 10^{-4} & 1.30 \cdot 10^{-4} & 2.28 \cdot 10^{-5} & 4.01 \cdot 10^{-6} & \\
\hline
\end{tabular}
\caption{The error \(\|u_N - u\|_0\) in Example 7.1}
\end{table}

\textbf{Example 7.2.} This example is similar to Example 7.1 but this time we took more slowly decaying Fourier coefficients of \(a_0(t, s) = a(t)a(s)\), namely

\[ a(t) = 3 + \sum_{0 \neq m \in \mathbb{Z}} 2^{-|m|} e^{im2\pi t}. \]

The theoretical estimate \(\|u_N - u\|_0 \leq c_N N^{-\mu} \|u\|_{\mu} \) \((\mu < \frac{3}{2})\) is the same as in Example 7.1 but now its numerical realization can be seen beginning from greater \(N\) and \(M\). We also present the errors \(\|u_N - u\|_{-2}\). According to (2.16), the theoretical error estimate is \(\|u_N - u\|_{-2} \leq c_N N^{-\mu - 2} \) \((\mu < \frac{5}{2})\).

\begin{table}
\centering
\begin{tabular}{ccccccc}
N=16 & N=32 & N=64 & N=128 & N=256 & N=512 & N=1024 \\
\hline
M = 4 & 1.85 \cdot 10^{0} & 1.80 \cdot 10^{0} & 1.80 \cdot 10^{0} & 1.80 \cdot 10^{0} & 1.80 \cdot 10^{0} & 1.80 \cdot 10^{0} \\
& 3.17 \cdot 10^{-1} & 3.17 \cdot 10^{-1} & 3.17 \cdot 10^{-1} & 3.17 \cdot 10^{-1} & 3.17 \cdot 10^{-1} & 3.17 \cdot 10^{-1} \\
M = 8 & 1.20 \cdot 10^{0} & 1.20 \cdot 10^{0} & 1.20 \cdot 10^{0} & 1.20 \cdot 10^{0} & 1.20 \cdot 10^{0} & 1.20 \cdot 10^{0} \\
& 7.00 \cdot 10^{-2} & 7.00 \cdot 10^{-2} & 7.00 \cdot 10^{-2} & 7.00 \cdot 10^{-2} & 7.00 \cdot 10^{-2} & 7.00 \cdot 10^{-2} \\
M = 16 & 1.26 \cdot 10^{0} & 2.70 \cdot 10^{-1} & 2.70 \cdot 10^{-1} & 2.70 \cdot 10^{-1} & 2.70 \cdot 10^{-1} & 2.70 \cdot 10^{-1} \\
& 7.03 \cdot 10^{-2} & 4.13 \cdot 10^{-3} & 4.13 \cdot 10^{-3} & 4.13 \cdot 10^{-3} & 4.13 \cdot 10^{-3} & 4.13 \cdot 10^{-3} \\
M = 32 & & & & & & \\
& 4.11 \cdot 10^{-3} & 4.11 \cdot 10^{-3} & 4.11 \cdot 10^{-3} & 4.11 \cdot 10^{-3} & 4.11 \cdot 10^{-3} & 4.11 \cdot 10^{-3} \\
& 1.61 \cdot 10^{-5} & 1.61 \cdot 10^{-5} & 1.61 \cdot 10^{-5} & 1.61 \cdot 10^{-5} & 1.61 \cdot 10^{-5} & 1.61 \cdot 10^{-5} \\
M = 64 & & & & & & \\
& 4.33 \cdot 10^{-6} & 7.81 \cdot 10^{-7} & 7.81 \cdot 10^{-7} & 7.81 \cdot 10^{-7} & 7.81 \cdot 10^{-7} & 7.81 \cdot 10^{-7} \\
& 3.47 \cdot 10^{-10} & 2.45 \cdot 10^{-10} & 2.45 \cdot 10^{-10} & 2.45 \cdot 10^{-10} & 2.45 \cdot 10^{-10} & 2.45 \cdot 10^{-10} \\
\hline
\end{tabular}
\caption{The errors \(\|u_N - u\|_0\) and \(\|u_N - u\|_{-2}\) in Example 7.2}
\end{table}
References


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