On a Comparison Theorem for Second Order Nonlinear Ordinary Differential Equations

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Abstract. We present a comparison theorem for second order nonlinear differential equations of the form

\[(R(t)w(x(t))x'(t))' + p(t)f(x(t)) = 0 \quad (t \in [t_0, \beta], \beta \leq \infty)\]

where \(p\) is a continuous function on \([t_0, \beta]\) without any restriction on its sign.

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0. Introduction

Consider the second order differential equation

\[(R(t)x'(t))' + p(t)x(t) = 0, \quad (t \in [t_0, \infty)) \tag{D1}_1\]

with given functions \(p\) and \(R\) on \([t_0, \infty)\). A function defined on an interval \([t_0, \beta], \beta \leq +\infty\), is said to be oscillatory at \(\beta\) if for every \(a \in (t_0, \beta)\) it has an infinite number of zeros on the interval \((a, \beta)\), and otherwise it is said to be non-oscillatory at \(\beta\). A differential equation of the form \((D1)_1\) is called oscillatory at \(\beta\) if all its solutions are oscillatory at \(\beta\), and otherwise such an equation is called non-oscillatory at \(\beta\). In the following we set \(I_\infty = [t_0, \infty)\) and \(I_\beta = [t_0, \beta]\).

A fundamental problem concerning the oscillation theory of second order linear or nonlinear ordinary differential equations may be posed as follows. Suppose \(R\) and \(p\) are functions on \(I_\infty\) which make the differential equation \((D1)_1\) oscillatory at \(\infty\). Are there relations between the functions \(R\) and \(r\) as well as functions \(p\) and \(q\) which ensure that the differential equation

\[(r(t)x'(t))' + q(t)x(t) = 0, \quad (t \in I_\infty) \tag{D1}_2\]

is also oscillatory at \(\infty\)? A well-known relation for this linear case is the classical Sturm comparison theorem [6: p. 2]. Hille [3] and Wintner [8] extended Sturm's result in the following way.


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Theorem 1 (Hille-Wintner comparison theorem). Suppose $R \equiv 1$ and $r \equiv 1$ in equations (D1)1 and (D1)2, respectively. Let $p,q \in C(I_\infty)$ be functions such that $P(t) = \int_t^\infty p(s) \, ds$ and $Q(t) = \int_t^\infty q(s) \, ds$ exist with $0 \leq P(t) \leq Q(t)$, for all $t \in I_\infty$. Then if the differential equation (D1)1 is oscillatory at $\infty$, then also the differential equation (D2)2 is oscillatory at $\infty$.

Taam [7] proved the following generalization of the Hille-Wintner theorem.

Theorem 2. Let $p,q,r,R \in C(I_\infty)$ be functions such that $r$ is bounded from above on $I_\infty$, $P(t) = \int_t^\infty p(s) \, ds$ and $Q(t) = \int_t^\infty q(s) \, ds$ exist, $|P(t)| \leq Q(t)$ and $0 < r(t) \leq R(t)$, for all $t \in I_\infty$. Then if the differential equation (D1)1 is oscillatory at $\infty$, so also the differential equation (D2)2 is oscillatory at $\infty$.

These and other Sturm-type comparison theorems hold for very general second order linear and nonlinear differential equations. Butler [1] obtained such a nonlinear extension of Theorem 2 for a certain class of equations.

We consider the differential equations

\begin{align*}
(R(t)w(x(t))x'(t))' + p(t)f(x(t)) &= 0 \quad (t \in I_\beta) \quad (D2)_1 \\
(r(t)w(x(t))x'(t))' + q(t)f(x(t)) &= 0 \quad (t \in I_\beta). \quad (D2)_2
\end{align*}

Theorem 3 (see Butler [1]). Suppose $u \equiv 1$ and $\beta = \infty$ in equations (D2)1 and (D2)2. Let $p,q,r,R \in C(I_\infty)$ be functions such that $P(t) = \int_t^\infty p(s) \, ds$ and $Q(t) = \int_t^\infty q(s) \, ds$ exist, $|P(t)| \leq Q(t)$ and $0 < r(t) \leq R(t)$, for all $t \in I_\infty$. Assume that $f$ is a function on $\mathbb{R}$ satisfying the following conditions:

\begin{enumerate}
\item[(a)_0] $f \in C'(\mathbb{R})$, $uf(u) > 0$ and $f'(u) > 0$, for all $u \neq 0$
\item[(b)_0] $f'$ is non-increasing on $(-\infty,0]$ and non-decreasing on $[0,\infty)$
\item[(c)_0] $\liminf_{|u| \to +\infty} f'(u) > 0$ and $\int_{\pm 1}^{\pm \infty} \frac{du}{f(u)} < \infty$.
\end{enumerate}

Then if the differential equation (D2)1 is oscillatory at $\beta$, so also the differential equation (D2)2 is oscillatory at $\beta$.

Butler also showed that Theorem 2 holds without the restriction that $r$ is bounded.

1. Preliminaries

To obtain the main result of the paper we need the following well-known three theorems.

Theorem 4 (see Rudek [5: Theorem 1]). Consider the nonlinear differential equation

\begin{equation}
(R(t)x'(t))' + a(t)g(x'(t)) + p(t)f(x(t)) = 0 \quad (t \in I_\infty) \quad (D3)
\end{equation}

where $a,p,R$ are functions on $I_\infty$ satisfying the following conditions:
(V1) \(a, p, R \in C(I_\infty), \ a(t) \geq 0 \) and \(R(t) > 0\) for all \(t \in I_\infty\), \(R\) decreasing on \(I_\infty\),
\[
P(t) = \int_t^\infty p(s) \, ds \text{ existing for all } t \in I_\infty \text{ and } \liminf_{T \to \infty} \int_t^T \frac{P(s)}{R(s)} \, ds > -\infty.
\]

(V2) \(f \in C'(\mathbb{R}), uf(u) > 0 \) and \(f'(u) > 0\), for all \(u \neq 0\), and \(0 \leq g \in C(I_\infty)\).

(V3) \(\liminf_{|u| \to \infty} f'(u) > 0\) and \(\int_{\pm 1}^{\pm \infty} \frac{du}{f(u)} < \infty\).

Then the differential equation (D3) is oscillatory at \(\infty\) if the condition

\[
(V4) \int_1^\infty \frac{1}{R(s)} \left( P(s) + \int_s^\infty \frac{(P_+(u))^2}{R(u)} \, du \right) \, ds = \infty
\]
is satisfied.

**Theorem 5** (see Butler [1: Lemma 2.3]). Consider the differential equation

\[
x''(s) + p(s)F(x(s)) = 0 \quad (s \in [0, \infty)).
\]

Let \(P(t) = \int_1^\infty p(s) \, ds\) exists on \(I_\infty\), \(F\) be continuously differentiable with \(uf(u) > 0\) and \(F'(u) > 0\) for all \(u \neq 0\). Suppose further that \(\liminf_{|x| \to \infty} F'(x) > 0\) and \(\int_{\pm 1}^{\pm \infty} \frac{du}{F(u)} < \infty\). Then the relation

\[
\int_1^\infty \left( |P(s)| + \int_s^\infty P^2(u) \, du \right) \, ds = \infty
\]
is a necessary condition for the differential equation (D4) to be oscillatory at \(\infty\).

**Theorem 6** (see Rudek [5: Theorem 2]). Consider the differential equation (D2)_1. Let \(p, R \in C(I_\beta), \ R(t) > 0\) \((t \in I_\beta)\), and assume that \(\int_1^\beta \frac{du}{R(u)}\) and \(P(t) = \int_1^\beta |p(s)| \, ds\) exist on \(I_\beta\). Let further the following conditions be satisfied:

The functions \(f\) and \(w\) are continuous, the product \((fw)(u) = f(u)w(u)\) \((u \in \mathbb{R})\) is continuously differentiable, \(uf(u) > 0\) and \((w(u)f(u))' > 0\) for all \(u \neq 0\). Let there exist \(s_0, S \in \mathbb{R}\) such that \(0 < s_0 \leq w(u) \leq S\) holds for all \(u \in \mathbb{R}\).

Then the differential equation (D2)_1 is non-oscillatory at \(\beta\).
2. Main result

Let $\beta < \infty$ or $\beta = \infty$.

**Theorem 7.** Let $p, q, r, R \in C(I_{\beta})$ be such that there exist $\overline{Q}(t) = \int_t^\beta q(s) \, ds$ and $\overline{P}(t) = \int_t^\beta |p(s)| \, ds$, $\overline{P}(t) \leq \overline{Q}(t)$ and $0 < r(t) \leq R(t)$, for all $t \in I_{\beta}$. Further let $f, w$ be functions on $\mathbb{R}$ satisfying the following conditions:

(a) $f, w \in C^1(\mathbb{R})$, $uf(u) > 0$, $f'(u) > 0$ and $(w(u)f(u))' > 0$, for all $u \neq 0$.

Let there exist $s_0, S \in \mathbb{R}$ such that $0 < s_0 \leq w(u) \leq S$ for all $u \in \mathbb{R}$ and either

(b) \( \frac{f'(u)}{w(u)} \) be non-increasing on $(-\infty, 0]$ and non-decreasing on $[0, \infty)$
or

(c) $\liminf_{|u| \to +\infty} \frac{(w(u)f(u))'}{f(u)} > 0$ and $\int_{-1}^{\pm \infty} \frac{du}{f(u)} < \infty$.

Then if the differential equation $(D2)_1$ is oscillatory at $\beta$, so also the differential equation $(D2)_2$ is oscillatory at $\beta$.

**Proof.** It will be convenient to separate the proof into the following three cases:

(i) Conditions (a), (b) and $\int_0^\beta \frac{du}{r(u)} = \infty$ are fulfilled.

(ii) Conditions (a), (c) and $\int_0^\beta \frac{du}{r(u)} = \infty$ are fulfilled.

(iii) Condition $\int_0^\beta \frac{du}{r(u)} < \infty$ is fulfilled.

Case (i): Let the differential equation $(D2)_1$ be oscillatory at $\beta$. Suppose on the contrary, that the differential equation $(D2)_2$ is not oscillatory at $\beta$. Then there is a solution $x$ of the differential equation $(D2)_2$ which is non-oscillatory at $\beta$. Without loss of generality, we may assume that

\[ x(t) > 0 \quad \text{for all } t \in I_{\beta}. \quad (1) \]

In this case we show all assumptions of a corollary of Tychonov's theorem [2: p. 405] are satisfied. Setting

\[ z(t) = \frac{r(t)w(x(t))x'(t)}{f(x(t))} \quad (t \in I_{\beta}) \quad (2) \]

we obtain from $(D2)_2$

\[ z'(t) = -\frac{f'(x(t))z^2(t)}{r(t)w(x(t))} - q(t) \quad (t \in I_{\beta}). \]
Then we have

\[ z(t) = z(T) + \int_{t}^{T} q(s)ds + \int_{t}^{T} \frac{f'(x(s))z^2(s)}{r(s)w(x(s))} ds \quad (t_0 \leq t \leq T < \beta) \tag{3} \]

where, for \( T \to \beta \),

\[ \int_{t}^{T} q(s)ds \to \overline{Q}(t) \quad \text{and} \quad \int_{t}^{T} \frac{f'(x(s))z^2(s)}{r(s)w(x(s))} ds \to k \]

with \( 0 < k \leq \infty \). Hence we have

\[ \lim_{T \to \beta} z(T) = b \quad \text{where} \quad -\infty \leq b < \infty. \]

Now we show that \( 0 \leq b < \infty \). Suppose that \( -\infty \leq b < 0 \). Then choosing \( T \) sufficiently large, say \( T \geq \overline{T} \), we have \( r(T)x'(T) < 0 \). If there exists an \( \varepsilon > 0 \) such that \( r(T)x'(T) \leq -\varepsilon \) for all \( T \geq \overline{T}_1 \), then we obtain

\[ x(T) - x(T_1) \leq -\varepsilon \int_{T}^{T_1} \frac{ds}{r(s)} \]

The right-hand side tends to \( -\infty \) if \( T \to \beta \). This contradicts \( x(t) > 0 \) \((t \in I_\beta)\). Thus we have \( \limsup_{T \to \beta} r(T)x'(T) = 0 \).

Next we choose a sequence \( (T_n)_{n \geq 1} \subseteq I_\beta \) such that, for sufficiently large \( n \) and for all \( T \in [\overline{T}, T_n) \), we have

\[ -\frac{1}{n} = r(T_n)w(x(T_n))x'(T_n) > r(T)w(x(T))x'(T) \]

and therefore \( \lim_{n \to \infty} T_n = \beta \). Let \( n \) be sufficiently large. Integrating the differential equation \((D2)_2\) from \( T \) to \( T_n \), we have

\[ 0 > \int_{T}^{T_n} q(s)f(x(s))ds. \]

Integration by parts yields

\[ V(T) < \int_{T}^{T_n} H(s)V(s)ds \quad (\overline{T} \leq T < T_n) \]

where

\[ V(T) = f(x(T)) \int_{T}^{T_n} q(u)du \quad \text{and} \quad H(T) = -\frac{x'(T)f'(x(T))}{f(x(T))}. \]
From (1), condition (a) and \( z'(T) < 0 \) we get \( H(t) > 0 \) \((t \in [T, T_n])\). Setting

\[
W(T) = \int_T^{T_n} H(s)V(s)ds \quad (T \in [T, T_n])
\]

we obtain \( W'(T) = -H(T)V(T) > -H(T)W(T) \). Then we have

\[
\frac{d}{dT} \left( W(T) \exp \left( \int_T^T H(s)ds \right) \right) > 0 \quad (T \in [T, T_n])
\]

which implies that

\[
W(T) \exp \left( \int_T^T H(s)ds \right)
\]

is strongly increasing on \([T, T_n]\). Considering \( W(T_n) = 0 \) we obtain that the function \( W \), and hence the function \( V \), is negative for every \( t \in [T, T_n] \), and, consequently, it is easy to see that

\[
\int_T^T q(s)ds < 0,
\]

contradicting the non-negativity of \( Q(t) \) for all \( t \in I_\beta \). Now we have

\[
0 \leq b = \lim_{T \to \beta} z(T) < \infty. \quad (4)
\]

Letting \( T \to \beta \) it follows from (3) that

\[
z(t) = b + Q(t) + \int_T^T \frac{f'(x(s))z^2(s)}{r(s)w(x(s))} ds \quad (t \in I_\beta). \quad (5)
\]

By (2) and the substitution \( x(s) = u \) we have

\[
\int_C^{x(t)} \frac{w(u)}{f(u)} du = \int_{t_0}^t \frac{z(s)}{r(s)} ds \quad (t \in I_\beta)
\]

where \( c = x(t_0) > 0 \). It follows from (1), (4) and (5) that \( z(t) \geq 0 \) for \( t \in I_\beta \). Thus it follows from (2) that the function \( x \) is increasing on \( I_\beta \). We define

\[
\Omega(x) = \int_C^x \frac{w(u)}{f(u)} du \quad (x \geq c). \quad (7)
\]
The function $\Omega$ is strongly increasing on $D(\Omega) = [c, \infty)$. Let $\Gamma$ be the inverse function of $\Omega$ which is also monotone increasing. $D(\Omega)$ is an interval, and therefore the function $\Gamma$ is continuous. From (6) and (7) we get

$$x(t) = \Gamma(\Omega(z(t))) = \Gamma \left( \int_c^t \frac{w(u)}{f(u)} \, du \right) = \Gamma \left( \int_0^t \frac{z(s)}{r(s)} \, ds \right) \quad (t \in I_\beta).$$

Using

$$G(z, r; t) = f' \left( \Gamma \left( \int_0^t \frac{z(\tau)}{r(\tau)} \, d\tau \right) \right) / w \left( \Gamma \left( \int_0^t \frac{z(\tau)}{r(\tau)} \, d\tau \right) \right)$$

it follows by (5) that

$$z(t) = b + \overline{Q}(t) + \int_0^t \frac{z^2(s)}{r(s)} G(z, r; s) \, ds \quad (t \in I_\beta).$$

The right-hand side defines a map $M$ by

$$(Mu)(t) = b + \overline{Q}(t) + \int_0^t \frac{u^2(s)}{r(s)} G(u, r; s) \, ds \quad (t \in I_\beta). \quad (8)$$

Let the domain of this map $M$ be the set $C_z$ of all continuous functions $u$ on $I_\beta$ with the restriction $0 \leq u(t) \leq z(t)$ $(t \in I_\beta)$. Then $M$ is a map of $C_z$ into itself. Since $\Gamma$ and $f'w^{-1}$ (condition (b)) are increasing it is easy to see that $0 \leq (Mu)(t) \leq z(t) = (Mz)(t)$ for every $u \in C_z$.

Now we consider the map $L$ defined by

$$(Lm)(t) = b + \overline{P}(t) + \int_0^t \frac{m^2(s)}{R(s)} G(m, R; s) \, ds \quad (t \in I_\beta). \quad (9)$$

Let the domain of $L$ be the set $D_z$ of all continuous functions $u$ on $I_\beta$ with $\overline{P}(t) \leq u(t) \leq z(t)$ $(t \in I_\beta)$. Then $L$ is a map of $D_z$ into itself.

In the following we need the Fréchet space $C_\rho(I_\beta)$, i.e. the linear, locally convex, compact space $C_\rho(I_\beta)$ of continuous real-valued functions on $I_\beta$. The corresponding topology $\rho$ is defined by the metric

$$\rho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left( \frac{p_i(x - y)}{1 + p_i(x - y)} \right) \quad (x, y \in C_\rho(I_\beta)). \quad (10)$$

Here $\{p_i\}_{i \geq 1}$ is a family of seminorms with

$$p_i(x - y) = \sup_{t \in [t_0, t_1]} |x(t) - y(t)|$$
where

$$t_i = \begin{cases} \beta - \frac{\beta - t_0}{i + 1} & \text{for } \beta < \infty \\ t_0 + i & \text{for } \beta = \infty. \end{cases} \quad (11)$$

From (10) and (11) it follows that $D_z$ is a closed, convex subset of the Fréchet space $C_\beta(I_\beta)$. In the following we show that the functions belonging to $L(D_z)$ are uniformly bounded and equicontinuous on compact subintervals of $I_\beta$. Let $T \in (t_0, \beta)$ and $m \in D_z$. There exists a constant $k \in \mathbb{R}$ with $|z(t)| \leq k$ for all $t \in [t_0, T]$, and therefore we have $|(Lm)(t)| \leq k$ for all $t \in [t_0, T]$ and all $m \in D_z$. Thus there follows the uniform boundedness of $L(D_z)$ on compact subintervals of $I_\beta$. From $m(t) \leq z(t)$ $(t \in I_\beta)$, (8) and (9) it follows for $s, t \in [t_0, T]$ that, for all $m \in D_z$,

$$|(Lm)(t) - (Lm)(s)| \leq \int_s^t |p(u)| \, du + \int_s^t |q(u)| \, du + |z(s) - z(t)|. \quad (12)$$

Let $\varepsilon > 0$ and $|t - s| < \delta(\varepsilon)$. By virtue of the estimations

$$\int_s^t |p(u)| \, du \leq \frac{\varepsilon}{3}, \quad \int_s^t |q(u)| \, du \leq \frac{\varepsilon}{3}, \quad |z(t) - z(s)| \leq \frac{\varepsilon}{3}$$

for $|t - s| < \delta(\varepsilon)$ and (12) we obtain that the functions in $L(D_z)$ are equicontinuous on compact subintervals of $I_\beta$. By the Arzela-Ascoli theorem, it follows that $L(D_z)$ is pre-compact on closed subintervals of $I_\beta$. Now we show that the map $L$ is continuous on $D_z$. Using

$$\Phi(m; s) = \frac{m^2(s)}{R(s)} G(m, R; s) \quad (s \in I_\beta)$$

it follows by (9) that

$$(Lm)(t) = b + \bar{P}(t) + \int_t^\beta \Phi(m; s) \, ds.$$

Let the sequence $(m_n)_{n \geq 1} \subseteq D_z$ be such that $m_n \to m$ $(m \in D_z)$, uniformly on compact subintervals of $I_\beta$. Let $\varepsilon > 0$ be given. There exists a $T \in (t_1, \beta)$ such that

$$\int_T^\beta \frac{z^2(s)}{r(s)} G(z, r; s) \, ds < \frac{\varepsilon}{3}.$$

On the other hand, there exists an $n_0(\varepsilon)$ such that

$$\max \left| \Phi(m_n; t) - \Phi(m; t) \right| \leq \frac{\varepsilon}{3|T - t_0|} \quad (n \geq n_0).$$
Then for all \( t \in [t_0, T] \) and \( n \geq n_0 \) we obtain

\[
\left| (L_{m_n})(t) - (L_m)(t) \right|
\]

\[
= \left| \int_t^\beta \left( \phi(m_n; s) - \phi(m; s) \right) ds \right|
\]

\[
\leq \left| \int_t^\beta \left( \phi(m_n; s) - \phi(m; s) \right) ds \right| + \int_t^\beta \left( \left| \phi(m_n; s) \right| + \left| \phi(m; s) \right| \right) ds
\]

\[
\leq \left| \int_t^T \left( \phi(m_n; s) - \phi(m; s) \right) ds \right| + 2 \int_t^\beta \frac{x^2(s)}{r(s)} G(z, r; s) ds
\]

\[
< \epsilon.
\]

Thus \( L_{m_n} \to L_m \) as \( n \to \infty \), uniformly on compact subintervals of \( I_\beta \).

Finally we need that \( L(D_x) \) is pre-compact on \( I_\beta \). Here we show that every sequence \( \{L(m_j)\}_{j \geq 1} \subseteq L(D_x) \) has a subsequence being a Cauchy sequence. Choosing \( \epsilon_1 > 0 \) we set \( \mu_1 > 0 \) such that

\[
\frac{\mu_1}{1 + \mu_1} \leq \frac{\epsilon_1}{2}.
\]

There exists a number \( t_1 \in I_\beta \) such that, for all \( t \in [t_1, \beta) \),

\[
\left| (L_{m_n}(t) - (L_{m_k})(t) \right| \leq \int_t^\beta \left( \left| \phi(m_n; s) \right| + \left| \phi(m_k; s) \right| \right) ds \leq 2 \int_t^\beta \phi(z; s) ds < \mu_1
\]

independently of \( n \) and \( k \). Therefore we obtain the inequality

\[
\sup_{t \in [t_1, t_1]} \left| (L_{m_n}(t) - (L_{m_k})(t) \right| \leq \mu_1
\]

where

\[
\bar{t}_i = \begin{cases} \beta - \frac{\beta - t_0}{i + j_1} & \text{for } \beta < \infty \\ t_0 + i + j_1 - 1 & \text{for } \beta = \infty \text{ and } j_1 = \min \{ i : t_i > t_1 \} \end{cases}
\]

Thus we have for all \( n, k \)

\[
\sum_{i=1}^{\infty} \frac{1}{2^i} \sup_{t \in [t_1, \bar{t}_i]} \left| (L_{m_n}(t) - (L_{m_k})(t) \right| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\mu_1}{1 + \mu_1} \leq \frac{\epsilon_1}{2}.
\]

Since \( L(D_x) \) is pre-compact on the interval \([t_0, \beta_1]\), the sequence \( \{L_{m_j}\}_{j \geq 1} \) has a convergent subsequence \( \{L_{m_j^{(1)}}\}_{j \geq 1} \) on \([t_0, \beta_1]\) being a Cauchy sequence. It follows that

\[
\sup_{t \in [t_0, \beta_1]} \left| (L_{m_n^{(1)}})(t) - (L_{m_k^{(1)}})(t) \right| \leq \frac{\epsilon_1}{2}
\]
holds if \( n, k \geq n_1 \). We obtain

\[
\rho \left( L_{m_n^{(1)}}, L_{m_k^{(1)}} \right) = \sum_{i=1}^{\infty} \frac{1}{2^i} \sup_{t \in [t_0, \bar{t}_i]} \left| (L_{m_n^{(1)}}(t) - (L_{m_k^{(1)}}(t)) \right| \\
\leq \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{1}{1 + \sup_{t \in [t_0, \bar{t}_i]} \left| (L_{m_n^{(1)}}(t) - (L_{m_k^{(1)}}(t)) \right|} \\
+ \sum_{i=1}^{\infty} \frac{1}{2^i} \sup_{t \in [\bar{t}_1, \bar{t}_i]} \left| (L_{m_n^{(1)}}(t) - (L_{m_k^{(1)}}(t)) \right| \\
\leq \epsilon_1
\]

for all \( n, k \geq n_1 \). This process can be repeated infinitely often often setting \( \epsilon_m = \epsilon_{m-1}/2 \) and using successively \( \mu_m, T_m, j_m \) and \( n_m \) \((n = 2, 3, \ldots)\). Then the diagonal sequence

\[
\left( L(m^{(j)}_j) \right)_{j \geq 1} \subseteq \left( L(m_j) \right)_{j \geq 1}
\]

is a Cauchy sequence. Since

\[
\left( L(m^{(j)}_j) \right)_{j \geq 1} \subseteq C_p(I_\beta)
\]

this Cauchy sequence is convergent. Therefore \( L(D_\beta) \) has a compact closure on \( I_\beta \). By the corollary of Tychnov’s Theorem (see [2: p. 405]), \( L \) has a fixed point \( \bar{n} \in D_\beta \), \( L\bar{n} = \bar{n} \). Then

\[
x(t) = \Gamma \left( \int_{t_0}^{t} \frac{\bar{n}(s)}{R(s)} \, ds \right) > 0 \quad (t \in I_\beta)
\]

is a non-oscillatory solution of the differential Equation \( (D2)_1 \) which contradicts the supposition.

Case (ii): Let the differential equation \( (D2)_1 \) be oscillatory at \( \beta \) and \( x \) a solution of the differential equation \( (D2)_2 \). By using

\[
s = \int_{t_0}^{t} \frac{d\tau}{r(\tau)w(x(\tau))}, \quad y(s) = x(t(s)) \quad (t \in I_\beta)
\]

we transform the differential equation \( (D2)_2 \) into the equation

\[
\frac{d^2 y}{ds^2} + r(t(s))y(t(s))w(y)f(y) = 0 \quad (s \in [0, \infty)).
\]
We set
\[ a(s) = r(t(s))q(t(s)) \quad \text{and} \quad F(y(s)) = w(y(s))f(y(s)) \]
and obtain the differential equation
\[ \frac{d^2 y(s)}{ds^2} + a(s)F(y(s)) = 0 \quad (s \in [0, \infty)). \quad (14) \]

By (13) we get
\[ A(s) = \int_a^\infty a(\tau) d\tau \geq 0 \quad (s \in [0, \infty)). \]

Hence we have
\[ \int_T^\infty A(s) ds \geq \frac{1}{S^2} \int_{t(T)}^\beta \frac{Q(u)}{r(u)} du \]
\[ \int_T^\infty \left( \int_s^\infty A^2(\tau) d\tau \right) ds \geq \frac{1}{S^4} \int_{t(T)}^\beta \int_u^\beta \frac{Q^2(v)}{r(v)} dv du. \quad (15) \]

Using
\[ s = \int_{t_0}^t \frac{d\tau}{R(\tau)w(x(\tau))} \quad \text{and} \quad y(s) = x(t(s)) \quad (t \in I_\beta) \]
we transform the differential equation (D2)_1 into the equation
\[ \frac{d^2 y(s)}{ds^2} + \bar{a}(s)F(y(s)) = 0 \quad (s \in [0, \infty)) \quad (14)' \]
where \( \bar{a}(s) = R(t(s))p(t(s)) \). The oscillation properties of the differential equations (D2)_1 and (D2)_2 are invariant under these transformations. Analogously, we have the inequalities
\[ |\bar{A}(s)| = \left| \int_s^\infty \bar{a}(\tau) d\tau \right| \leq \frac{1}{s_0} \bar{P}(t(s)) \]
\[ \int_T^\infty |\bar{A}(s)| ds \leq \frac{1}{S^6} \int_{t(T)}^\beta \frac{\bar{P}(u)}{R(u)} du \quad (15)' \]
\[ \int_T^\infty \left( \int_s^\infty (\bar{A}(\tau))^2 d\tau \right) ds \leq \frac{1}{S^8} \int_{t(T)}^\beta \int_u^\beta \frac{(\bar{P}(v))^2}{R(v)} dv du. \]
Concerning the differential equation (14)' it follows from Theorem 5 that

\[ \int_{\gamma}^{\infty} \left( |\tilde{A}(s)| + \int_{\gamma}^{\infty} (\tilde{A}(u))^2 \, du \right) \, ds = \infty. \]

On the other hand, by relations (15)' and, without loss of generality, \( s_0 \leq 1 \) and \( S \geq 1 \), we have

\[ \infty = \int_{\gamma}^{\infty} \left( |\tilde{A}(s)| + \int_{s}^{\infty} (\tilde{A}(u))^2 \, du \right) \, ds \leq \frac{1}{s_0} \int_{(T)}^{\beta} \frac{1}{R(u)} \left( \widetilde{P}(u) + \int_{u}^{\beta} (\widetilde{R}(v))^2 \, dv \right) \, du. \]

Correspondingly, in view of inequalities \( 0 < r(t) \leq R(t) \) and \( |P(t)| \leq Q(t) \) for all \( t \in I_\beta \) we obtain

\[ \frac{1}{S^4} \int_{(T)}^{\beta} \frac{1}{r(u)} \left( S^2 \widetilde{Q}(u) + \int_{u}^{\beta} \frac{Q^2(v)}{r(v)} \, dv \right) \, du = \infty. \quad (16) \]

Hence, by (15) and (16) we obtain

\[ \int_{\gamma}^{\infty} \left( A(s) + \int_{s}^{\infty} A^2(u) \, du \right) \, ds = \infty. \]

Thus, in view of Theorem 4, the differential equation (14) is oscillatory at \( \beta \) and hence also the differential equation (D2)$_2$ is oscillatory at \( \beta \).

**Case (iii):** The inequalities \( 0 < r(t) \leq R(t) \) (\( t \in I_\beta \)) and condition (iii) imply the existence of the integral \( \int_{(T)}^{\beta} \frac{du}{R(u)} \). All suppositions of Theorem 6 are fulfilled for the differential equations (D2)$_1$ and (D2)$_2$. Both equations are non-oscillatory. Thus the proof of Theorem 7 is complete.

Under the supposition \( p(t) \geq 0 \) for all \( t \in [t_0, \infty) \), Theorem 7 generalizes a result of Butler [1] (Theorem 3 in that paper).

### 3. A corollary

Consider the nonlinear differential equations

\[ \begin{align*}
 x''(t) + A(t)x'(t) + B(t)f(x(t)) &= 0 \quad (t \in I_\beta) \quad (D3)_1 \\
 x''(t) + a(t)x'(t) + b(t)f(x(t)) &= 0 \quad (t \in I_\beta). \quad (D3)_2
\end{align*} \]

**Corollary.** Let \( a, A, b, B \in C(I_\beta) \) be functions such that

\[ a(t) \leq A(t) \quad \text{and} \quad |B(t)| \leq b(t) \exp \left( \int_{t_0}^{t} (a(s) - A(s)) \, ds \right) \quad (t \in I_\beta). \]
Suppose that there exist

\[ P^*(t) = \int_0^\beta |B(s)| \exp \left( \int_0^s A(u) \, du \right) \, ds \quad (t \in I_\beta). \]

\[ Q^*(t) = \int_0^\beta b(s) \exp \left( \int_0^s a(u) \, du \right) \, ds \]

Assume further that \( f \) is a function on \( \mathbb{R} \) satisfying the condition (a) and either the condition (b) or (c) of Theorem 7, where \( w = 1 \) is supposed.

Then if the differential equation \((D3)_1\) is oscillatory at \( \beta \), so also the differential equation \((D3)_2\) is oscillatory at \( \beta \).

**Proof.** By setting

\[ r(t) = \exp \left( \int_0^t a(s) \, ds \right) \quad \text{and} \quad R(t) = \exp \left( \int_0^t A(s) \, ds \right) \quad (t \in I_\beta), \]

the corollary immediately follows from Theorem 7.  

4. An example

Let \( \beta < \infty \) and \( t_0 > 0 \). In view of Theorem 2 in [4] we can see that the differential equation

\[ \left( (\beta - t)^5 \left( \frac{a}{e^{x(t)} + e^{-x(t)}} + b \right) x'(t) \right)' + \frac{1}{t} \left( |x(t)|^\alpha \text{sign} \, x(t) + cx(t) \right) = 0 \]

for \( t \in I_\beta \) with

\[ c > 0, \quad 0 < a \leq b, \quad 1 < \alpha = \frac{2\beta + 1}{2\gamma + 1} \quad (\beta, \gamma \in \mathbb{N}) \]

is oscillatory at \( \beta \). Hence, by Theorem 7 it follows that the differential equation

\[ \left( r(t) \left( \frac{a}{e^{x(t)} + e^{-x(t)}} + b \right) x'(t) \right)' + q(t) \left( |x(t)|^\alpha \text{sign} \, x(t) + cx(t) \right) = 0 \]

is oscillatory at \( \beta \), if the functions \( r \) and \( q \) satisfy the conditions

\[ r, q \in C(I_\beta), \quad 0 < r(t) \leq (\beta - t)^5, \quad \ln \frac{\beta}{t} \leq \int_0^\beta q(s) \, ds < \infty \quad (t \in I_\beta). \]
References


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